The iterations of intersection body operator.

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Workshop on Probability and Geometry in High Dimensions.
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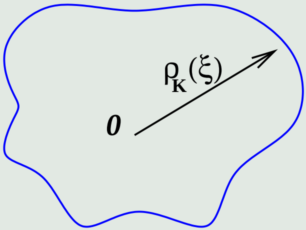
Radial function: $\rho_K(\xi) = \sup\{a : a\xi \in K\}$, for $\xi \in S^{n-1}$.



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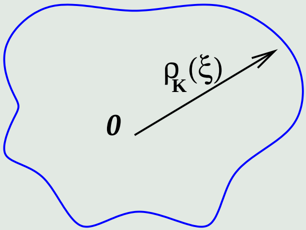

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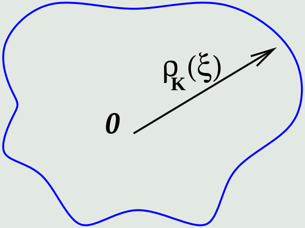

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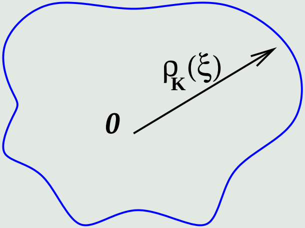

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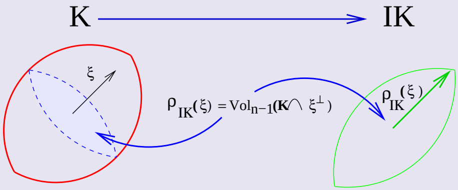
- $\xi^{\perp} = \{x \in \mathbb{R}^n : x \cdot \xi = 0\}$.

E. Lutwak: Intersection body, of a body K



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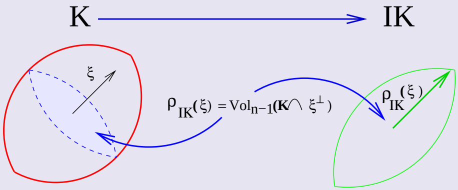

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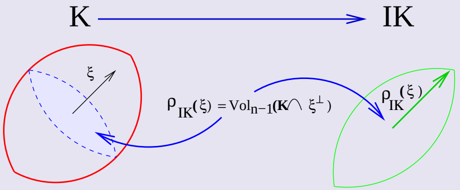

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Solution of Busemann-Petty problem. Definition of L_{-1}. Very nice questions in Harmonic Analysis & just for fun.

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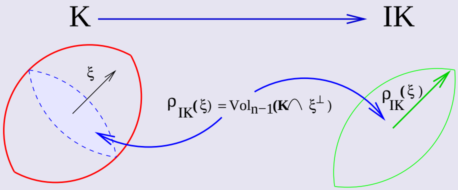

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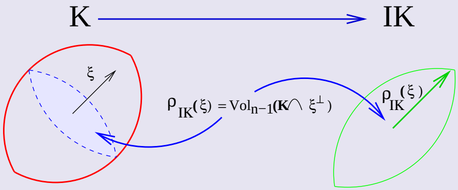

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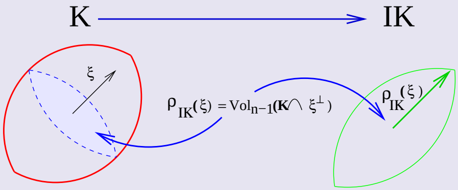

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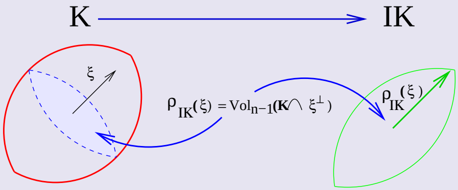

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- R. Gardner, A. Koldobsky, T. Schlumprecht: All convex symmetric bodies are intersection bodies in $\mathbb{R}^n$, $n \leq 4$. NOT true for $n \geq 5$.

Connection to Spherical Radon Transform

Spherical coordinates in $\xi^\perp$

$$\rho_{\mathrm{I}K}(\xi) = \mathrm{Vol}_{n-1}(K \cap \xi^\perp) = \frac{1}{n-1} \int\limits_{S^{n-1}\cap\xi^\perp} \rho_K^{n-1}(\theta)d\theta = \frac{1}{n-1} R\rho_K^{n-1}(\xi).$$

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Many geometric questions about intersection bodies can be rewritten as questions about R.

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More general definition of Intersection Body (C^∞-case).

A symmetric star body L is an intersection body if $R^{-1}\rho_L \geq 0$.

Intersection Bodies: Fix $\varepsilon \in (0, 1/10)$

Consider body K such that for every $u \in S_{n-1}$ there exits an intersection body K_u, which coincide with K on a ε-neighborhood of u. Is it true that K must be an intersection body itself?

Example: Local/Equatorial characterization.

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F. Nazarov, D. Ryabogin, A. Z., 2008:

- NO!

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- NO!
- If we instead of regular neighborhoods around points would take neighborhood around equators then YES for even n and NO for odd n!!!

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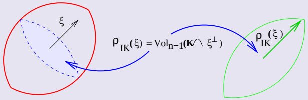
$$K \longrightarrow IK$$



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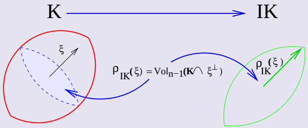

Interesting facts:

- Take $T \in GL(n)$, then $\mathrm{I}(TK) = |\det T|(T^*)^{-1}\mathrm{I}K$.

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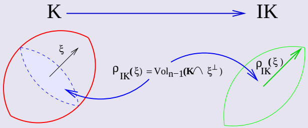

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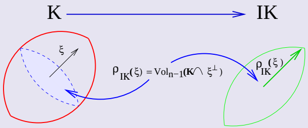

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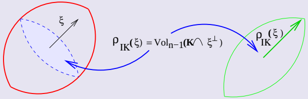

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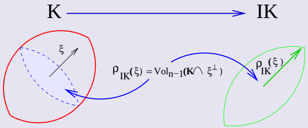

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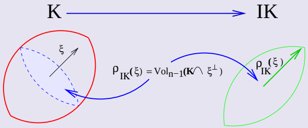

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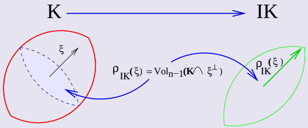

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- $d_{BM}(E, \mathrm{I}E) = 1$.
- $d_{BM}(K, \mathrm{I}K) = 1$, $K \subset \mathbb{R}^2$, K-symmetric.

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A. Fish, F. Nazarov, D. Ryabogin, A.Z.:

Consider a star body $K \subset \mathbb{R}^n$, $n \geq 3$, is it true that

$$d_{BM}(I^m K, B_2^n) \to 1, \text{ as } m \to \infty,$$

i.e. iterations of intersection body operator of a star body K will converge to B_2^n in d_{BM}?

Support function: $h_L(\theta) = \sup\{x \cdot \theta,\ x \in L\}$.

Dual story – Projection body (convex, sets only!)

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ΠL – projection body of L:

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- $\Pi B_\infty^n = c_n B_\infty^n$, where $B_\infty^n = [-1,1]^n$.

Fixed point is NOT unique! W. Weil (71) described polytopes that satisfy this property. General case is still open.

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$\exists \varepsilon_n > 0$ such that $\forall K \subset \mathbb{R}^n$ such that K-start body, $d_{BM}(K, B_2^n) < 1 + \varepsilon_n$, we get

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- Big hope: $d_{BM}(\mathrm{I}K, B_2^n) < d_{BM}(K, B_2^n)$?

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Is it true that $d_{BM}(I^m K, B_2^n) \to 1$, as $m \to \infty$?

A. Fish, F, Nazarov, D. Ryabogin, A.Z., (2009)

$\exists \varepsilon_n > 0$ such that $\forall K \subset \mathbb{R}^n$ such that K-start body, $d_{BM}(K, B_2^n) < 1 + \varepsilon_n$, we get

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Remarks:

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- (J. Kim, V. Yaskin, A.Z.) Wrong without assumption of convexity! there is a star body (p-convex) K such that $d_{BM}(\mathrm{I}K, B_2^n) >> d_{BM}(K, B_2^n)$.

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Denote by $\mathcal{R} = \frac{1}{\mathrm{Vol}_{n-2}(S^{n-2})}R$, i.e. $\mathcal{R}1 = 1$.

Question: $(n \geq 3)$

Consider symmetric function $f : S^{n-1} \to \mathbb{R}^+$, such that $f = \mathcal{R}f^{n-1}$, is it true that then $f = 1$?

$\mathcal{H}_k-$ space of Spherical Harmonics of degree k.

Main Idea: Spherical Radon Transform and Spherical Harmonics

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Formulas Exists: Clebsch–Gordan coefficients — but they are hard, not clear (to me!) how to use for this problem.

Back to our result: $f \approx 1$

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Thus we need to KILL H_2^{ϕ}.

Linear Transform $T \in GL(n)$ applied to function f on S^{n-1}

$$\rho_{T^{-1}K}(\xi) = \|T\xi\|_K^{-1} = \left\| \frac{T\xi}{|T\xi|} \right\|_K^{-1} |T\xi|^{-1} = \rho_K\left(\frac{T\xi}{|T\xi|}\right)|T\xi|^{-1}.$$

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Applying this to the function ρ_K, we conclude that if K is sufficiently close to B_n, then, after proper normalization, $\rho_{\mathrm{I}^k K}$ can be written as $1 + \varphi$ with $\|\varphi\|_{\mathcal{U}_\alpha}$ as small as we want,