Random Matrices and Analyticity of the Planar Limit

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Wick's Formula

Assume that $\mu \sim N(0, C)$ on $\mathbb{R}^n$.

- $\langle f \rangle$ is the expectation with respect to μ
- $\langle x_i \rangle = 0$, $\langle x_i x_j \rangle = c_{ij}$.
- For any linear functions $f_1, f_2, \dots, f_{2k} : \mathbb{R}^n \to \mathbb{C}$,

$$\langle f_1 f_2 \dots f_{2k} \rangle = \sum \langle f_{p_1} f_{q_1} \rangle \langle f_{p_2} f_{q_2} \rangle \dots \langle f_{p_k} f_{q_k} \rangle$$

  $p_1 q_1 p_2 q_2 \dots p_k q_k$ permutation of $\{1, 2, \dots, 2k\}$ with $p_1 < p_2 < \dots < p_k$ and $p_1 < q_1$, $p_2 < q_2$, $\dots$, $p_k < q_k$.

For $\mu \sim N(0, 1)$ and $f_1 = f_2 = f_3 = f_4 = x$,

$$\langle x^4 \rangle = \langle f_1 f_2 f_3 f_4 \rangle = \langle f_1 f_2 \rangle \langle f_3 f_4 \rangle + \langle f_1 f_3 \rangle \langle f_2 f_4 \rangle + \langle f_1 f_4 \rangle \langle f_2 f_3 \rangle = 3.$$

Random Matrices and Moments

$\mathcal{M}_N^{sa}(\mathbb{C})$ are $N \times N$ Hermitian matrices.

$$\mathrm{Tr} M^2 = \sum_i x_{ii}^2 + 2\sum_{i<j} Re^2(x_{ij}) + 2\sum_{i<j} Im^2(x_{ij})$$

The distribution is

$$\frac{1}{C_N} e^{-\frac{N}{2}\mathrm{Tr} M^2} dM$$

dM the Lebesgue measure on $\mathcal{M}_N^{sa}(\mathbb{C})$.

$$\langle x_{ij} \rangle = 0$$

$$\langle x_{ij} x_{kl} \rangle = \frac{1}{N} \text{ iff } i = l, j = k.$$

$$\mathrm{Tr}(M^{2k}) = \sum x_{i_1 i_2} x_{i_2 i_3} \dots x_{i_{2k-1} i_1}$$

$$\langle \mathrm{Tr}(M^{2k}) \rangle = \sum \langle x_{i_1 i_2} x_{i_2 i_3} \dots x_{i_{2k-1} i_1} \rangle$$

Example:

$$\langle N\mathrm{Tr}(M^4)\rangle = N\sum\langle x_{i_1 i_2} x_{i_2 i_3} x_{i_3 i_4} x_{i_4 i_1}\rangle$$

Use Wick's formula to compute this.

Example: Couple $i_1 i_2$ with $i_2 i_3$ and $i_3 i_4$ with $i_4 i_1$, thus

$$i_1 = i_3, i_2 = i_2, i_3 = i_1, i_4 = i_4.$$

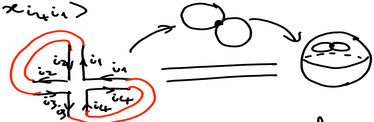
There are three free indices, thus the contribution from such a choice to $\langle Tr(M^4)\rangle$ is N^2.
These indices define a "diagram" or a "map" on a sphere (a genus 0 surface).

$N\langle \mathrm{Tr}\, M^4\rangle = N\sum \langle x_{i_1i_2}x_{i_2i_3}x_{i_3i_4}x_{i_4i_1}\rangle$

Case 1 $\quad i_1i_2 \rightsquigarrow i_2i_3$, $\; i_3i_4 \rightsquigarrow i_4i_1$ $\quad\Big|\quad i_1=i_3$, $i_2=i_2$, $i_4=i_4$

First index, exit index

Second index, entry index



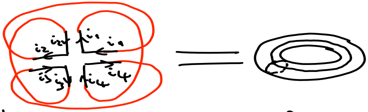
3 free indices = 3 faces

Contribution $N\times\frac{1}{N^2}\times N^3 = N^{2-2g}$ to $N\langle \mathrm{Tr}\, M^4\rangle$.

$\chi = V-E+F = 1-2+3 = 2-2g$

Case 2 $\quad i_1i_2 \rightsquigarrow i_3i_4$, $\; i_2i_3 \rightsquigarrow i_4i_1$ $\quad\Big|\quad i_1=i_4$, $i_2=i_3$, $i_2=i_1$, $i_3=i_4$

$i_1=i_2=i_3=i_4$

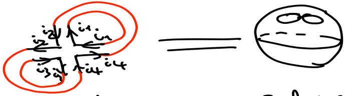


1 free index = 1 face

$\chi = 1-2+1 = 0 = 2-2g$

Contribution $N\times\frac{1}{N^2}\times N = N^{2-2g}$

Case 3 $\quad i_1i_2 \rightsquigarrow i_4i_1$, $\; i_2i_3 \rightsquigarrow i_3i_4$ $\quad\Big|\quad i_1=i_1$, $i_2=i_4$, $i_3=i_3$



3 free indices = 3 faces

$\chi = 1-2+3 = 2-2g$.

Contribution $N\times\frac{1}{N^2}\times N^3 = N^{2-2g}$

$$N^2\langle (\mathrm{Tr}\,M^4)^2\rangle = \sum_{\substack{i_1,i_2,i_3,i_4\\ j_1,j_2,j_3,j_4}} \langle x_{i_1 i_2} x_{i_2 i_3} x_{i_3 i_4} x_{i_4 i_1} x_{j_1 j_2} x_{j_2 j_3} x_{j_3 j_4} x_{j_4 j_1}\rangle$$

$$\langle x_{i_1 i_2} x_{i_2 i_3} x_{i_3 i_4} x_{i_4 i_1} x_{j_1 j_2} x_{j_2 j_3} x_{j_3 j_4} x_{j_4 j_1}\rangle$$

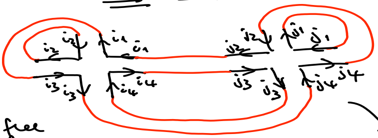
$i_1 i_2 \to i_2 i_3$

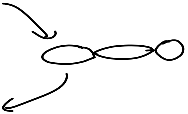
$i_3 i_4 \to j_3 j_4$

$i_4 i_1 \to j_2 j_3$

$j_1 j_2 \to j_4 j_1$

$$\begin{cases} i_1 = i_3 \\ i_2 = i_2 \\ i_4 = j_3 = j_4 \\ j_1 = j_1 \end{cases}$$





4 free indices

||

4 faces

$\chi = V - E + F = 2 - 4 + 4$

$= 2 - 2g$

Contribution $N^2 \times \frac{1}{N^4} \times N^4 = N^{2-2g}$.

The Formal Matrix Models: An Example

$$
\begin{aligned}
\mathcal{U}(t,N) &:= \langle e^{Na_4 \mathrm{Tr} M^4} \rangle \\
&= \sum_{n \geq 1} \frac{(a_4 N)^n}{n!} \langle (\mathrm{Tr} M^4)^n \rangle \\
&= \sum_{n \geq 1} \frac{a_4^n}{n!} \sum N^{\chi}
\end{aligned}
$$

where the sum is over all diagrams with n 4-valent vertices. χ is the Euler characteristic of the diagram.
If the diagram is on a surface, then

$$\chi = V - E + F = 2 - 2g$$

where g is the genus of the surface.

$$\mathcal{Z}(t,N) := \frac{1}{N^2}\log\langle e^{Na_4 \mathrm{Tr} M^4}\rangle$$
$$= \sum_{g\geq 0} \mathcal{F}_g(a_4) N^{-2g}$$

where

$$\mathcal{F}_g(a_4) = \sum_{n\geq 0} \frac{a_4^n}{n!} K_n(g)$$

where $K_n(g)$ is the number of *connected* diagrams with n 4-valent vertices and genus g.
This can be thought of as the generating function of these diagrams.
We will be mainly interested in the first of these, namely the planar limit $\mathcal{F}_0$.

Take

$$\mathcal{V}(x) = \sum_{k\geq 1} \frac{a_k x^k}{k}.$$

$$\mathcal{Z}(t,N) := \frac{1}{N^2} \log\langle e^{N\mathrm{Tr}\mathcal{V}(\sqrt{t}M))}\rangle$$

Then

$$\mathcal{Z}(t,N) = \mathcal{F}_0(t) + N^{-2}\mathcal{F}_1(t) + N^{-4}\mathcal{F}_2(t) + \dots$$

where $\mathcal{F}_g(t)$ is the generating function of the numbers $K_n(g)$ of *connected genus g-diagrams with n edges and weights a_k on the vertices of valence k*.
We are interested in the planar limit $\mathcal{F}_0(t)$.

Conjecture (T'Hooft's)

If $\mathcal{V}$ is analytic at 0, then $\mathcal{F}_0$ is also analytic at 0.

In particular, the numbers $K_n(0)$ grow at most exponentially.
Extreme potentials: $\mathcal{V}(x) = \sum_{k\geq 1} x^{2k}/(2k) = -\frac{1}{2}\log(1-x^2)$ and $\mathcal{V}(x) = \sum_{k\geq 1} x^k/k = -\log(1-x)$.

The Analytical Matrix Models

For a potential which grows sufficiently fast at infinity,

$$I_V = -\lim_{N\to\infty} \frac{1}{N^2} \log \int \exp(-N\mathrm{Tr}(V(M)))dM$$
$$= \inf_{\mu\in\mathcal{P}(\mathbb{R})} \left\{ \int V(x)\mu(dx) - \iint \log|x-y|\mu(dx)\mu(dy) \right\},$$

If $V(x) = x^2/2 - \mathcal{V}(tx)$ and this makes sense, then, formally

$$I_V(t) = 3/4 - \mathcal{F}_0(t)$$

This is one of the ways of computing the planar limit in the physics literature.

The Minimization

$$I_V = \inf_{\mu \in \mathcal{P}(\mathbb{R})} \left\{ \int V(x)\mu(dx) - \iint \log|x-y|\mu(dx)\mu(dy) \right\}$$

There is a unique equilibrium measure and it has compact support.

Step 1 Find the support of the equilibrium measure.

In the physics literature, for planar maps calculations, this is assumed to be a single interval.

Step 2 For the case of a single interval support of the equilibrium measure, find a “nice” formula for I_V (eventually one which does not involve the equilibrium measure.)

Haagerup's Formula

Lemma

For any real $x, y \in [-2,2]$, $x \neq y$, we have

$$\log|x-y| = -\sum_{n=1}^{\infty} \frac{2}{n} T_n\left(\frac{x}{2}\right) T_n\left(\frac{y}{2}\right)$$

where the series here is convergent on $x \neq y$.
If $x > 2$ and $y \in [-2,2]$, we have

$$\log|x-y| = \log\left|\frac{x+\sqrt{x^2-4}}{2}\right| - \sum_{n=1}^{\infty} \frac{2}{n}\left(\frac{x-\sqrt{x^2-4}}{2}\right)^n T_n\left(\frac{y}{2}\right)$$

where the series is absolutely convergent.

Here T_n is the n^{th} Chebysev polynomial: $T_n(\cos x) = \cos(nx)$.
Orthogonal polynomials w.r.t. $\omega(dx) = \mathbb{1}_{[-2,2]}(x)\frac{dx}{\pi\sqrt{4-x^2}}$.

The Proof of Haagerup's Formula

$x = 2\cos u$ and $y = 2\cos v$ with $u, v \in (0, \pi)$, $u \neq v$

$$x - y = 2(\cos u - \cos v) = 4\sin\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right),$$

$$\begin{aligned}
\log|x-y| &= \log\left|2\sin\left(\frac{u+v}{2}\right)\right| + \log\left|2\sin\left(\frac{u-v}{2}\right)\right| \\
&= \log|1-e^{i(u+v)}| + \log|1-e^{i(u-v)}| \\
&= Re\left(\log(1-e^{i(u+v)}) + \log(1-e^{i(u-v)})\right) \\
&\underset{\log(1-z)=-\sum_{n=1}^{\infty}\frac{z^n}{n}}{\overset{|z|=1, z\neq 1}{=}} -\sum_{n=1}^{\infty}\frac{1}{n}Re\left(e^{in(u+v)} + e^{in(u-v)}\right) \\
&= -\sum_{n=1}^{\infty}\frac{1}{n}\left(\cos(n(u+v)) + \cos(n(u-v))\right) \\
&= -\sum_{n=1}^{\infty}\frac{2}{n}\cos(nu)\cos(nv) = -\sum_{n=1}^{\infty}\frac{2}{n}T_n\left(\frac{x}{2}\right)T_n\left(\frac{y}{2}\right).
\end{aligned}$$

For $x > 2$, $y \in [-2, 2]$, write $x = 2\cosh u = e^u + e^{-u}$, where $u = \log \frac{x+\sqrt{x^2-4}}{2}$ and $y = 2\cos v$

$$\begin{aligned}
\log|x-y| &= \log\Big(e^u(1-e^{-u+iv})(1-e^{-u-iv})\Big) \\
&= u + \log(1-e^{-u+iv}) + \log(1-e^{-u-iv}) \\
&= u - \sum_{n=1}^{\infty} \frac{2}{n} e^{-nu}\cos(nv).
\end{aligned}$$

Corollary

If $\omega(dx) = \mathbb{1}_{[-2,2]}(x)\frac{dx}{\pi\sqrt{4-x^2}}$ *is the arcsine law of the interval* $[-2, 2]$, *then*

$$\int \log|x-y|\omega(dy) = \begin{cases} 0, & |x| \le 2 \\ \log\frac{|x|+\sqrt{x^2-4}}{2}, & |x| > 2. \end{cases}$$

Working with measures on $[-2, 2]$

Corollary

The logarithmic potential of a measure on $[-2, 2]$ is given by

$$\int \log|x-y|\mu(dx) = -\sum \frac{2}{n} T_n\left(\frac{x}{2}\right) \int T_n\left(\frac{y}{2}\right)\mu(dy)$$

where this series makes sense pointwise, and the logaritmic energy of the measure μ is given by

$$\iint \log|x-y|\mu(dx)\mu(dy) = -\sum_{n=1}^{\infty} \frac{2}{n}\left(\int T_n\left(\frac{x}{2}\right)\mu(dx)\right)^2.$$

In particular $\iint \log|x-y|\mu(dx)\mu(dy)$ is finite if and only if $\sum_{n=1}^{\infty} \frac{2}{n}\left(\int T_n\left(\frac{x}{2}\right)\mu(dx)\right)^2$ is finite.

Working with measures on $[-2,2]$

Corollary

If $\mu \in \mathcal{P}([-2,2])$ *and* V *is a* C^3 *potential on* $[-2,2]$, *then*

$$I_V(\mu) = \int V d\mu - \iint \log|x-y| \mu(dx)\mu(dy)$$
$$= \beta_0(V) + 2\sum_{n=1}^{\infty}\left(\beta_n(V)\alpha_n + \frac{\alpha_n^2}{n}\right)$$

where

$$\alpha_n = \int T_n\left(\frac{x}{2}\right)\mu(dx) \text{ and } \beta_n(V) = \int_{-2}^{2} \frac{V(x) T_n\left(\frac{x}{2}\right) dx}{\pi\sqrt{4-x^2}}.$$

Working with measures on $[-2, 2]$

$$
\begin{aligned}
I_V(\mu) &= \beta_0(V) - \frac{1}{2}\sum_{n=1}^{\infty} n\beta_n(V)^2 + 2\sum_{n=1}^{\infty}\frac{1}{n}\left(\alpha_n + \frac{n\beta_n(V)}{2}\right)^2 \\
&\geq \beta_0(V) - \frac{1}{2}\sum_{n=1}^{\infty} n\beta_n(V)^2
\end{aligned}
$$

with equality if and only if

$$
1 - \sum_{n=1}^{\infty} n\beta_n(V) T_n\left(\frac{x}{2}\right) \geq 0 \quad \text{for any} \quad x \in [-2, 2],
$$

in which case

$$
\mu(dx) = \left(1 - \sum_{n=1}^{\infty} n\beta_n(V) T_n\left(\frac{x}{2}\right)\right)\frac{dx}{\pi\sqrt{4 - x^2}}.
$$

Working with measures on $[-2,2]$

Theorem

The equilibrium measure on $[-2,2]$ *has full support if and only if*

$$1 - \sum_{n=1}^{\infty} n\beta_n(V) T_n\left(\frac{x}{2}\right) > 0 \quad \textit{for } x \textit{ on a dense subset of } [-2,2].$$

in which case

$$I_V = \inf_{\mu \in \mathcal{P}([-2,2])} I_V(\mu) = \beta_0(V) - \frac{1}{2}\sum_{n=1}^{\infty} n\beta_n(V)^2$$

$$= \int_{-2}^{2} \frac{V(x)dx}{\pi\sqrt{4-x^2}} - \int_0^1 s\left[\left(\int_{-2}^{2} \frac{xV'(sx)dx}{2\pi\sqrt{4-x^2}}\right)^2 + \left(\int_{-2}^{2} \frac{V'(sx)dx}{\pi\sqrt{4-x^2}}\right)^2\right] ds.$$

A More Familiar Look

$$1 - \sum_{n=1}^{\infty} n\beta_n(V) T_n\left(\frac{x}{2}\right) = \int_{-2}^{2} \frac{V'(x) - V'(y)}{x - y} \frac{dy}{\pi\sqrt{4 - y^2}}.$$

Proof: Linearity + check for $V(x) = T_k\left(\frac{x}{2}\right)$.

$$\beta_0(V) - \frac{1}{2}\sum_{n=1}^{\infty} n\beta_n(V)^2$$

$$= \int_{-2}^{2} \frac{V(x)dx}{\pi\sqrt{4-x^2}} - \int_0^1 s\left[\left(\int_{-2}^{2} \frac{xV'(sx)dx}{2\pi\sqrt{4-x^2}}\right)^2 + \left(\int_{-2}^{2} \frac{V'(sx)dx}{\pi\sqrt{4-x^2}}\right)^2\right] ds.$$

Proof: Polarization reduces this to

$$\frac{1}{2}\sum_{n\geq 1} n\beta_n(V_1)\beta_n(V_2)$$

$$= \int_0^1 s\left(\int_{-2}^{2} xV_1'(sx)\frac{dx}{2\pi\sqrt{4-x^2}}\right)\left(\int_{-2}^{2} xV_2'(sx)\frac{dx}{2\pi\sqrt{4-x^2}}\right) ds$$
$$+ \int_0^1 s\left(\int_{-2}^{2} V_1'(sx)\frac{dx}{\pi\sqrt{4-x^2}}\right)\left(\int_{-2}^{2} V_2'(sx)\frac{dx}{\pi\sqrt{4-x^2}}\right) ds.$$

Check this for $V_1(x) = x^m$, $V_2(x) = x^n$.
The rest of the proof reduces to the following:

Lemma

$$\sum_p p\binom{2m}{m-p}\binom{2n}{n-p} = \frac{mn}{2(m+n)}\binom{2m}{m}\binom{2n}{n}$$

$$\sum_p (2p+1)\binom{2m+1}{m-p}\binom{2n+1}{n-p} = \frac{(2m+1)(2n+1)}{m+n+1}\binom{2m}{m}\binom{2n}{n}$$

with the convention that $\binom{j}{q} = 0$ *for* $q < 0$ *or* $q > j$*.*

Proved using the *wzb* method implemented for Mathematica.

The real line case

$V \in C^3(\mathbb{R})$. Then the equilibrium measure on $\mathbb{R}$ has support the interval $[-2c+b, 2c+b]$ if and only if (c,b) is the unique absolute maximizer of

$$H(c,b) := \log c - \frac{1}{2}\int_{-2}^{2} \frac{V(cx+b)dx}{\pi\sqrt{4-x^2}}$$

and for x in a dense set of $[-2,2]$,

$$\psi_{b,c}(x) = \int_{-2}^{2} \frac{V'(cx+b) - V'(cy+b)}{x-y} \frac{dy}{\pi\sqrt{4-y^2}} > 0.$$

In addition,

$$I_V = -\log c + \int_{-2}^{2} \frac{V(cx+b)dx}{\pi\sqrt{4-x^2}} - \int_0^c s\left[\left(\int_{-2}^{2} \frac{xV'(sx+b)dx}{2\pi\sqrt{4-x^2}}\right)^2 + \left(\int_{-2}^{2} \frac{V'(sx+b)dx}{\pi\sqrt{4-x^2}}\right)^2\right]ds.$$

Critical Point System

$$\begin{cases} \int_{-2}^{2} cxV'(cx+b)\frac{dx}{\pi\sqrt{4-x^2}} = 2 \\ \int_{-2}^{2} V'(cx+b)\frac{dx}{\pi\sqrt{4-x^2}} = 0. \end{cases}$$

Critical Point System

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In the physics literature, $R = c^2$, $S = b$.

$$\begin{cases} \int_{-2}^{2} \sqrt{R}xV'(\sqrt{R}x+S)\frac{dx}{\pi\sqrt{4-x^2}} = 2 \\ \int_{-2}^{2} V'(\sqrt{R}x+S)\frac{dx}{\pi\sqrt{4-x^2}} = 0. \end{cases}$$

Analytic Perturbations

- Let V be analytic on a neighborhood of the one interval support $[b - 2c, b + 2c]$.
- $u \to V_u$ “analytic” with u in a “good” Banach space B, $V_0 = V$.

Theorem

Near 0*,*

- *the equilibrium measure of V_u has support one interval $[b_u - 2c_u, b_u + 2c_u]$*
- *$u \to (b_u, c_u)$ is analytic.*
- *$I(u)$ depends analytically on u.*

Example: $V = x^2/2$ and $V_t(x) = \frac{x^2}{2} + \sum_{n\geq 1} a_n t^n x^n$ at least on a neighborhood of $[b - 2c, b + 2c]$.

Example: $V_u(x) = \frac{x^2}{2} + \sum_{n\geq 1} a_n \rho^n x^n$ with $\rho > 0$ fixed and $u = (a_1, a_2, \ldots, a_n, \ldots) \in \ell^1$.

Going Formal

$$\mathcal{V}(x) = \frac{x^2}{2} - \sum_{n\geq 1} \frac{a_n x^n}{n}.$$

$\mathcal{R}, \mathcal{S}$, uniquely solves as power series in $a_1, a_2, \ldots$

$$\begin{cases} \int_{-2}^{2} \sqrt{\mathcal{R}} x \mathcal{V}'(\sqrt{\mathcal{R}} x + \mathcal{S}) \frac{dx}{\pi\sqrt{4-x^2}} = 2 \\ \int_{-2}^{2} \mathcal{V}'(\sqrt{\mathcal{R}} x + \mathcal{S}) \frac{dx}{\pi\sqrt{4-x^2}} = 0 \\ \mathcal{R} = 1 + O(a_i), \mathcal{S} = O(a_i). \end{cases}$$

Equivalently,

$$\begin{cases} \mathcal{R} = 1 + \sum_{n=1}^{\infty} a_n \sum_{j=0}^{\frac{n-2}{2}} \binom{n-1}{j}\binom{n-1-j}{j+1} \mathcal{R}^{j+1} \mathcal{S}^{n-2-2j} \\ \mathcal{S} = \sum_{n=1}^{\infty} a_n \sum_{j=0}^{\frac{n-1}{2}} \binom{n-1}{j}\binom{n-1-j}{j} \mathcal{R}^{j} \mathcal{S}^{n-1-2j} \\ \mathcal{R} = 1 + O(a_i), \mathcal{S} = O(a_i) \end{cases}$$

Going Formal

$$\mathcal{V}(x) = \frac{x^2}{2} - \sum_{n\geq 1} \frac{a_n x^n}{n}.$$

$\mathcal{R}, \mathcal{S}$, uniquely solves as power series in $a_1, a_2, \dots$

$$\begin{cases} \int_{-2}^{2} \sqrt{\mathcal{R}} x \mathcal{V}'(\sqrt{\mathcal{R}} x + \mathcal{S}) \frac{dx}{\pi\sqrt{4-x^2}} = 2 \\ \int_{-2}^{2} \mathcal{V}'(\sqrt{\mathcal{R}} x + \mathcal{S}) \frac{dx}{\pi\sqrt{4-x^2}} = 0 \\ \mathcal{R} = 1 + O(a_i), \mathcal{S} = O(a_i). \end{cases}$$

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It is known (J. Bouttier, P. Di Francesco, E. Guitter in 2002) that $\mathcal{R}$ and $\mathcal{S}$ are the generating functions of connected planar diagrams with one/two distinguished one/two vertices.

$$\mathcal{V}(x) = \frac{x^2}{2} - \sum_{n\geq 1} \frac{a_n x^n}{n}.$$

Define as a formal power series in $a_1, a_2, \ldots$

$$\mathcal{I}_{\mathcal{V}} = -\frac{1}{2}\log\mathcal{R} + \int_{-2}^{2} \frac{\mathcal{V}(\sqrt{\mathcal{R}}x + \mathcal{S})dx}{\pi\sqrt{4-x^2}}$$
$$-\frac{1}{2}\int_0^{\mathcal{R}} \left[\left(\int_{-2}^{2} \frac{x\mathcal{V}'(\sqrt{s}x + \mathcal{S})dx}{2\pi\sqrt{4-x^2}} \right)^2 + \left(\int_{-2}^{2} \frac{\mathcal{V}'(\sqrt{s}x + \mathcal{S})dx}{\pi\sqrt{4-x^2}} \right)^2 \right] ds.$$

Theorem

$$\mathcal{F}_0 = \frac{3}{4} - \mathcal{I}_{\mathcal{V}}$$

Counting Planar Diagrams with a Fixed Number of Edges

For

$$\mathcal{V}(x) = -\frac{x^2}{2} + \sum_{n\geq 1} \frac{a_n t^{n/2} x^n}{n},$$

$$\mathcal{F}_0(t) = 3/4 - \mathcal{I}(t)$$

$$\mathcal{F}_0(t) = \sum_{n\geq 1} f_n t^n.$$

We are interested in the growth of f_n when n is large.

Theorem

S.G and I.P

$$\mathcal{F}_0(t) = \frac{1}{t}\int_0^t \frac{(t-s)(2\mathcal{R}(s)\mathcal{S}(s)^2 + \mathcal{R}(s)^2 - 1)}{2s} ds,$$

Alternatively

$$t^2(t^2\mathcal{F}_0'(t))' = 2\mathcal{R}(t)\mathcal{S}^2(t) + \mathcal{R}^2(t).$$

Corollary

If $\mathcal{V}$ is a polynomial, then $t^2(t^2\mathcal{F}_0'(t))'$ is an algebraic function. In particular the coefficients f_n of $\mathcal{F}_0(t)$ satisfy a linear recurrence relation with polynomial coefficients in n (f_n are holonomic).

Extreme Potentials, Counting All Diagrams

Counting diagrams with even valent vertices.

$$\mathcal{V}(x) = \frac{x^2}{2} - \sum_{n\geq 1} \frac{t^n x^{2n}}{2n} = \frac{x^2}{2} + \log(1 - tx^2)$$

$$\begin{aligned}
\mathcal{S}(t) &= 0 \\
\mathcal{R}(t) &= \frac{1 + 4t - \sqrt{1 - 8t}}{8t} \\
\mathcal{F}_0(t) &= \frac{1 - 24t + 72t^2 - (1 - 20t)\sqrt{1 - 8t}}{128t^2} \\
&\quad -\frac{3}{8}\log\frac{1 - 4t + \sqrt{1 - 8t}}{2} \\
&= \frac{1}{2}t^2\,{}_3F_2(1, 1, 3/2; 2, 4; 8t) = \sum_{n\geq 1} \frac{3(2n-1)!2^{n-1}}{n!(n+2)!} t^n \\
f_n &= \frac{3}{4\sqrt{\pi}} \frac{8^n}{n^{7/2}} \left(1 - \frac{25}{8n} + \frac{945}{128n^2} - \frac{16275}{1024n^3} + O\left(\frac{1}{n^4}\right)\right).
\end{aligned}$$

Extreme Potentials, Counting All Diagrams

Counting planar diagrams with arbitrary valency.

$$\mathcal{V}(x) = \frac{x^2}{2} - \sum_{n\geq 1} \frac{t^{n/2}x^n}{n} = \frac{x^2}{2} + \log(1 - \sqrt{t}x)$$

$$\begin{aligned}
\mathcal{S}(t) &= \frac{1-\sqrt{1-12t}}{6\sqrt{t}} \\
\mathcal{R}(t) &= \frac{1+12t-\sqrt{1-12t}}{18t} && (1) \\
F_0(t) &= \frac{1-36t+162t^2-(1-30t)\sqrt{1-12t}}{216t^2} && (2) \\
&\quad -\frac{1}{2}\log\frac{1-6t+\sqrt{1-12t}}{2} \\
f_n &= \frac{2}{\sqrt{\pi}}\frac{12^n}{n^{7/2}}\left(1-\frac{25}{16n}+\frac{945}{256n^2}-\frac{16275}{2048n^3}+O\left(\frac{1}{n^4}\right)\right).
\end{aligned}$$

t'Hooft's conjecture in sharp form

Theorem

Assume that

$$\mathcal{V}(t) = \frac{x^2}{2} - \sum_{n\geq 1} \frac{a_n x^n}{n}$$

and

$$\alpha(V) = \sup_{n\geq 1} |a_n|^{1/n}.$$

Then $\mathcal{F}_0(t)$ *has radius of convergence at least* $\frac{1}{12\sqrt{\alpha(V)}}$ *with equality for the extreme potential* $\mathcal{V} = \frac{x^2}{2} - \sum_{n\geq 1} \frac{x^n}{n}$.

More

Already done

- Study the generating functions for the case of counting planar diagrams with a certain number of faces.
- Study the growth of the number of diagrams with certain particular structure. For example the valency is only 3 or 4.

To think about

- Genus one extension.
- Extend this to multitrace matrix models.
- Multiple matrix models.