Limit Theorems for Linear Eigenvalue Statistics of Random Matrices with Independent Entries

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- We are interested in the limiting laws of $\mathcal{N}_n[\varphi]$ as $n \to \infty$.
  possibly after putting a normalization factor in front (LLN and CLT type)

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  Marchenko, P 67; P 72; Girko 70-80; Bai-Silverstein 80-90, Costin-Lebowitz 95; Khorunzhy-Khoruzhenko-P. 96; Spohn 97; Johansson 98; Sinai-Soshnikov 98; Soshnikov 98, 00; Keating-Snaith 00; Cabanal-Duvillard 01; Diaconis-Evans 01; Guionnet 02; Bai-Silverstein 04; Anderson-Zeitouni 05; P. 06; Lytova-P. 09

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- Noblesse oblige (L.P.): *Lyapunov* (first modern proof of CLT), *S. Bernstein* (first CLT for dependent r.v.'s), both from Kharkov

Definition: $M_n = n^{-1/2} W_n, \; W_n = \{W_{jk}\}_{j,k=1}^n$

$$P(dW) = Z_n^{-1} e^{-\mathrm{Tr} W^2/4w^2} \prod_{1 \le j \le k \le n} dW_{jk}.$$

Since

$$\mathrm{Tr} W_n^2 = \sum_{1 \le j \le n} W_{jj}^2 + 2 \sum_{1 \le j \le k \le n} W_{jk}^2,$$

the above implies that $\{W_{jk}\}_{1 \le j \le k \le n}$ are independent Gaussian random variables such that

$$\mathbf{E}\{W_{jk}\} = 0, \; \mathbf{E}\{W_{jk}^2\} = w^2(1 + \delta_{jk}).$$

Gaussian Orthogonal Ensemble (GOE)

Theorem

Let M_n be the GOE) and $\mathcal{N}_n[\varphi]$ be a linear eigenvalue statistics of its eigenvalues. Then we have for any bounded and continuous $\varphi : R \to C$ with probability 1:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{l=1}^{n} \varphi\left(\lambda_l^{(n)}\right) = \int \varphi(\lambda) N_{scl}(d\lambda),$$

where the measure

$$N_{sc}(d\lambda) = \rho_{sc}(\lambda)d\lambda,\ \rho_{sc}(\lambda) = (2\pi w^2)^{-1}\sqrt{4w^2-\lambda^2}\mathbf{1}_{|\lambda|\le 2w}$$

is known as the Wigner or the semicircle law.

Wigner 52 and many others.

It suffices to consider the Normalized Counting Measure of eigenvalues (NCM)

$$N_n(\Delta) = \sharp\{\lambda_l^{(n)} \in \Delta\}/n, \ \forall \Delta \subset \mathbb{R}$$

and its Stieltjes transform

$$g_n(z) = \int \frac{N_n(d\lambda)\}}{\lambda - z}, \ \mathrm{Im}\, z \neq 0,$$

determining N_n. Use now

(i) Gaussian differentiation formula:

$$\mathbf{E}\{\xi_l \Phi(\xi)\} = \mathbf{E}\{\xi_l^2\}\mathbf{E}\{\Phi_l'(\xi)\}, \ l = 1, ..., p;$$

(ii) Poincaré-Nash-Chernoff inequality:

$$\mathbf{Var}\{\Phi\} \leq \sum_{l=1}^{p} \mathbf{E}\{\xi_l^2\}\mathbf{E}\left\{|\Phi_l'|^2\right\}.$$

By spectral theorem $g_n(z) = n^{-1}\mathrm{Tr}(M_n - z)^{-1}$, by resolvent identity for $f_n(z) = \mathbf{E}\{g_n(z)\}$

$$f_n(z) = z^{-1} + (zn)^{-1} \sum_{j,k=1}^{n} \mathbf{E}\{M_{jk} G_{kj}(z)\},$$

by (i) $f_n(z) = z^{-1} + z^{-1}\mathbf{E}\{g_n^2(z)\}$, and by (ii) *(Bose-Chatterjee 04; P. 05)*

$$\mathbf{Var}\{g_n(z)\} \leq 2w^2/n^2|\operatorname{Im} z|^4$$

while $\mathbf{Var}\{g_n(z)\} \leq w^2/n|\operatorname{Im} z|^4$ for random Schrodinger.
This leads to

$$f_{sc}(z) = z^{-1} + z^{-1} f_{sc}^2(z)$$

for $\lim_{n\to\infty} f_n = f_{sc}$ uniformly on any compact set of $\mathbb{C}\setminus\mathbb{R}$, thus $f_{sc}(z) = (\sqrt{z^2 - 4w^2} - z)/2w^2$ $(\operatorname{Im} f(z) \operatorname{Im} z > 0)$. Convergence of g_n to f_{sc}, hence N_n to N_{sc} by Borel-Cantelli.

Theorem

Let M_n be the GOE matrix, $\varphi : \mathbb{R} \to \mathbb{R}$ be a differentiable function with a polynomially bounded derivative. Then $\mathcal{N}_n^\circ[\varphi] = \mathcal{N}_n[\varphi] - \mathbf{E}\{\mathcal{N}_n[\varphi]\}$ converges in distribution to the Gaussian random variable with zero mean and the variance

$$\begin{aligned} V_{GOE}[\varphi] &= \frac{1}{2\pi^2} \int_{-2w}^{2w} \int_{-2w}^{2w} \left(\frac{\varphi(\lambda_1) - \varphi(\lambda_2)}{\lambda_1 - \lambda_2} \right)^2 \\ &\quad \times \frac{4w^2 - \lambda_1 \lambda_2}{\sqrt{4w^2 - \lambda_1^2}\sqrt{4w^2 - \lambda_2^2}} d\lambda_1 d\lambda_2. \end{aligned}$$

A MIRACLE!

Proof is again based on the Gaussian differentiation formula and the bound

$$\mathbf{Var}\{\mathcal{N}_n[\varphi]\} \le \frac{2w^2}{n}\mathbf{E}\{\mathrm{Tr}\varphi'(M_n)(\varphi'(M_n)^*)\} \le 2w^2(\sup_{\lambda\in\mathbb{R}}|\varphi'(\lambda)|)^2$$

for $\mathcal{N}_n[\varphi] = \mathrm{Tr}\varphi(M)$ by Poincaré. We have to prove that

$$\lim_{n\to\infty} Z_n(x) = \exp\left\{-x^2 V_{GOE}[\varphi]/2\right\},\ Z_n(x) = \mathbf{E}\left\{e^{ix\overset{\circ}{\mathcal{N}}_n[\varphi]}\right\}$$

uniformly in x, varying on a finite interval of $\mathbb{R}$. Assume first that φ admits the Fourier transform $\widehat{\varphi}$ and $(1+|t|)|\widehat{\varphi}(t) \in L^1(\mathbb{R})$. Then

$$Z_n(x) = 1 + \int_0^x Z_n'(y)dy,\ Z_n'(x) = i\int \widehat{\varphi}(t)Y_n(x,t)dt,$$

where

$$Y_n(x,t) = \mathbf{E}\left\{\overset{\circ}{u}_n(t)e_n(x)\right\},\quad e_n(x) = e^{ix\overset{\circ}{\mathcal{N}}_n[\varphi]},\ u_n(t) = \mathrm{Tr}e^{itM}.$$

Use $U_n(t) = e^{itM_n}$ (instead of $G_n(z) = (M_n - z)^{-1}$) and the Duhamel formula

$$u_n(t) = n + i \int_0^t \sum_{j,k=1}^{n} M_{jk} U_{jk}(t_1) dt_1,$$

the differentiation formula, the Poincaré, and the Schwarz to obtain

$$Y_n(x,t) + 2w^2 \int_0^t dt_1 \int_0^{t_1} dt_2 \overline{v}_n(t_1 - t_2) Y_n(x, t_2) = xZ_n(x)A_n(t) + r_n(x,t),$$

$$A_n(t) = -2w^2 \int_0^t dt_1 \int e^{it_1\lambda} \varphi'(\lambda) \mathbf{E}\{N_n(d\lambda)\}, \ \overline{v}_n(t) = \mathbf{E}\{n^{-1}\mathrm{Tr}U(t)\}$$

$$\begin{aligned} |Y_n| &\leq \sqrt{2}w|t| \sup_{t\in\mathbb{R}} |\varphi'(\lambda)|, \\ |(Y_n)'_t| &\leq \sqrt{2}w(1+w^2t^2)^{1/2}, \ |(Y_n)'_x| \leq 2w^2 t \sup_{\lambda\in\mathbb{R}} |\varphi'|. \end{aligned}$$

Hence, there exists $\{Y_{n_j}\}$ converging uniformly on any compact set of $\mathbb{R}^2$ to Y, satysfying

$$Y(x,t) + 2w^2 \int_0^t dt_1 \int_0^{t_1} dt_2 v(t_1 - t_2) Y(x, t_2) = xZ(x)A(t),$$

$$A(t) = -2w^2 \int_0^t dt_1 \int e^{it_1\lambda} \varphi'(\lambda) N_{sc}(d\lambda), \;\; v(t) = \int e^{i\lambda t} N_{sc}(d\lambda).$$

This leads (by the Laplace transformation) to

$$Z(x) = 1 - V_{GOE} \int_0^x yZ(y)dy.$$

The equation is uniquely soluble and yield the result for $(1+|t|)|\widehat{\varphi}(t) \in L^1(\mathbb{R})$. General case of C^1 (even *Lip* 1) test functions is obtained by Poincaré and approximations.
The scheme dates back to *Khorunzhy-Khoruzhenko-P. 96*, where the Stieljtes transform (the resolvent) was used, thus real analytic test functions. Here we use the Fourier transform and obtain C^1 test functions.

$$M_n = n^{-1/2} W_n,\ \ W_n = \{W_{jk}^{(n)}\}_{j,k=1}^{n}$$

with $W_{jk}^{(n)} = W_{kj}^{(n)} \in \mathbb{R}$, $1 \le j \le k \le n$ independent and

$$\mathbf{E}\{W_{jk}^{(n)}\} = 0, \quad \mathbf{E}\{(W_{jk}^{(n)})^2\} = (1+\delta_{jk})w^2,$$

i.e. the two first moments of the entries coincide with those of the GOE or

$$\mathbf{P}(dW_n) = \prod_{1\le j\le k\le n} F_{jk}^{(n)}(dW_{jk}),$$

where $F_{jk}^{(n)}$ has above moments. The GOE corresponds to

$$F_{jk}^{(n)}(dW) = \frac{1}{(2\pi\sigma_{jk}^2)^{1/2}} e^{-W^2/2\sigma_{jk}^2} dW, \quad \sigma_{jk}^2 = (1+\delta_{jk})w^2.$$

Theorem

Let $M_n = n^{-1/2} W_n$ be the Wigner matrix, satisfying the L2 (à la Lindeberg)

$$\lim_{n\to\infty} n^{-2} \sum_{1\le j\le k\le n} \int_{|w|\ge\tau\sqrt{n}} W^2 F_{jk}^{(n)}(dW), \quad \forall \tau > 0.$$

and N_n be the Normalized Counting Measure of its eigenvalues. Then with p.1: $\lim_{n\to\infty} N_n(\Delta) = N_{sc}(\Delta)$, $\forall \Delta \subset \mathbb{R}$ (macroscopic universality).

P. 72; Girko 75. No Poincaré but the martingale-type bounds $\mathbf{E}\{|N_n^\circ(\Delta)|^4\} = O(n^{-2})$. Thus, it suffices to prove that if M_n is the Wigner matrix and $\widehat{M}_n$ is the corresponding GOE, then

$$R_n(z) := \mathbf{E}\{n^{-1}\mathrm{Tr}(M_n - z)^{-1}\} - \mathbf{E}\{n^{-1}\mathrm{Tr}(\widehat{M}_n - z)^{-1}\} \to 0, \ n \to \infty$$

uniformly on a compact set of $\mathbb{C} \setminus \mathbb{R}$, cf recent results by *Erdos et al 09* for $\mathrm{Im}\, z = O(n^{-1})$

Proof is based on

- General differentiation formula (*Khorunzhy-Khoruzhenko-P. 95*):
If $\mathbf{E}\{|\xi|^{p+2}\} < \infty$, $p \in \mathbb{N}$, $\Phi : \mathbb{R} \to \mathbb{C}$ of C^{p+1} with bounded derivatives, then

$$\begin{aligned} \mathbf{E}\{\xi\Phi(\xi)\} &= \sum_{j=0}^{p} \frac{\kappa_{l+1}}{l!}\mathbf{E}\{\Phi^{(l)}(\xi)\} + \varepsilon_p, \\ |\varepsilon_p| &\leq C_p \mathbf{E}\{|\xi|^{p+2}\} \sup_{t\in\mathbb{R}} |\Phi^{(p+1)}(t)|, \end{aligned}$$

where $\{\kappa_l\}_{l=1}^{\infty}$ are the cumulants of W_{12}. Note that the $l = 1$ term is "Gaussian".

- "Interpolation trick" (P. 2000): use the product space of the Wigner M_n and the GOE $\widehat{M}_n$ with the same first and second moments and set

$$M_n(s) = s^{1/2} M_n + (1-s)^{1/2} \widehat{M}_n, \quad 0 \leq s \leq 1,$$

Assume first $w_3 := \sup_n \max_{1\le j\le k\le n} \mathbf{E}\Big\{\Big|W_{jk}^{(n)}\Big|^3\Big\} < \infty$ and write

$$\begin{aligned} R_n(z) &= \int_0^1 \frac{d}{ds}\mathbf{E}\{n^{-1}\mathrm{Tr}\,(M_n(s)-z)^{-1}ds = \frac{1}{2}\int_0^1 (T_1 - T_2)ds \\ T_1 &= (n^3 s)^{-1/2}\sum_{1\le j,k\le n}\mathbf{E}\{W_{jk}^{(n)}(G^2)_{jk}\}, \\ T_2 &= (n^3(1-s))^{-1/2}\sum_{1\le j,k\le n}\mathbf{E}\{\widehat{W}_{jk}(G^2)_{jk}\}. \end{aligned}$$

Apply to T_1 the general differentiation formula with $p=1$ and $\Phi = (G_n^2)_{jk}$ and to T_2 the Gaussian differentiation formula. We have the cancelation, resulting only in ε_1:

$$|\varepsilon_1| \le \frac{C_1 w_3}{n^{5/2}}\sum_{1\le j\le k\le n}\sup_{M\in\mathcal{S}_n}|D_{jk}(G_n^2)_{jk}|\} \le \frac{C_1' w_3}{n^{1/2}|\Im z|^4},\ D_{jk} = \frac{\partial}{\partial M_{jk}}.$$

S_n is the set of $n\times n$ real symmetric matrices.

Theorem

Let $M_n = n^{-1/2} W_n$, $W_n = \{W_{jk}^{(n)}\}_{j,k=1}^n$ *be the real symmetric Wigner random matrix. Assume that* $\mu_4 = \mathbf{E}\{(W_{jk}^{(n)})^4\}$ *does not depend on* j, k *and* n, $\kappa_4 = \mu_4 - 3w^4 = 0$, *and the L4:*

$$\lim_{n\to\infty} n^{-2} \sum_{j,k=1}^{n} \int_{|W|\geq\tau\sqrt{n}} W^4 F_{jk}^{(n)}(dW) = 0, \ \forall\tau > 0,$$

If φ *possesses the Fourier transform* $\widehat{\varphi}$ *and* $(1 + |t|^5)|\widehat{\varphi}(t)| \in L^1(\mathbb{R}$, *then* $\overset{\circ}{\mathcal{N}}_n[\varphi]$ *converges in distribution to the Gaussian random variable with zero mean the GOE variance (again the macroscopic universality, even a bit more).*

Proof by the "interpolation" trick from the GOE. For "Lindeberg-4 " see *KKP, 95*.

Theorem

Let $M_n = n^{-1/2} W_n$ *be the real symmetric Wigner random matrix,* $\mu_4 = \mathbf{E}\{(W_{jk}^{(n)})^4\}$ *do not depend on* j, k *and* n *and*

$$w_6 := \sup_n \max_{1 \le j \le k \le n} \mathbf{E}\{(W_{jk}^{(n)})^6\} < \infty.$$

If $(1 + |t|^5)|\widehat{\varphi}(t)| \in L^1(\mathbb{R}$, *then* $\overset{\circ}{\mathcal{N}}_n[\varphi] = \mathcal{N}_n[\varphi] - \mathbf{E}\{\mathcal{N}_n[\varphi]\}$ *converges in distribution to the Gaussian random variable of zero mean and of variance*

$$\begin{aligned} V_{Wig}[\varphi] &= V_{GOE}[\varphi] + \frac{\kappa_4}{2\pi^2 w^8} I_{Wig}^2, \\ I_{Wig} &= \int_{-2w}^{2w} \varphi(\mu) \frac{2w^2 - \mu^2}{\sqrt{4w^2 - \mu^2}} d\mu, \end{aligned}$$

Assume that $\kappa_4 \neq 0$, then:

$I_{Wig} = 0$: the GOE CLT, e.g. for an ODD φ.

$I_{Wig} \neq 0$: a modified CLT, generically and, in particular, for an EVEN φ such that

$$\int_0^{2w} \varphi(\mu) \frac{2w^2 - \mu^2}{\sqrt{4w^2 - \mu^2}} d\mu \neq 0.$$

Proof: by combining the schemes of proof of the CLT for the GOE and the "interpolation" trick, in particular, by proving and using

Theorem

Let $M_n = n^{-1/2} W_n$ be the real symmetric Wigner random matrix and $\mathcal{N}_n[\varphi]$ be the linear eigenvalue statistic of its eigenvalues. Assume that

$$w_6 := \sup_n \max_{1 \le j,k \le n} \mathbf{E}\left\{ \left| W_{jk}^{(n)} \right|^6 \right\} < \infty.$$

Then

$$\mathbf{Var}\{\mathcal{N}_n[\varphi]\} \le C(w_6) \left(\int (1 + |t|^{5/2}) |\widehat{\varphi}(t)| dt \right)^{1/2},$$

where $C(w_6)$ depends only on w_6.

The bound replaces the Poincaré one in the case of Wigner ensembles.

$$Y_n(x,t) = \sum_{a=1}^{3} T_a + \varepsilon_3,$$

where now

$$T_a = \frac{i}{a!n^{(a+1)/2}} \int_0^t \sum_{j,k=1}^{n} \kappa_{a+1,jk} \mathbf{E}\left\{ D_{jk}^a (U_{jk}(s) e_n^\circ(x)) \right\} ds, \ D_{jk} = \frac{\partial}{\partial M_{jk}}$$

and

$$|\varepsilon_3| \le C(x) w_6^{5/6} (1 + |t|^4)/n^{1/2}.$$

The term T_3 contains $U_{jj}(t_1)U_{jj}(t_2)U_{kk}(t_3)U_{kk}(t_4)$ Because of

$$D_{jk} U_{ab}(t) = i\beta_{jk} \int_0^t ds \ \left[U_{aj}(t-s)U_{bk}(s) + U_{bj}(t-s)U_{ak}(s) \right].$$

These are only combinations of U's that contribute.

Universality class w.r.t. to the CLT: the set of random matrices, having the same CLT (variance) for linear eigenvalue statistics.
Universality classes of the Wigner matrices w.r.t. the CLT are indexed by the first two even moments of their off-diagonal entries:

$$w^2 = \mathbf{E}\{(W_{jk}^{(n)})^2\}, \ \mu_4 = \mathbf{E}\{(W_{jk}^{(n)})^4\}, \ 1 \le j < k \le n$$

(two dimensional moduli space).
An example of "collective theorem", *Linnik 70's*.
The Gaussian universality classes: $\kappa_4 := \mu_4 - 3w^4 = 0$.
In the conventional probability setting for the CLT of independent random variables $\{\xi_l^{(n)}\}_{l=1}^n$ the universality classes w.r.t. the CLT of linear statistics are indexed by a single parameter, the variance $\sigma^2 = \mathbf{E}\{(\xi_l^{(n)})^2\}$.
All classes are Gaussian.

$M_{m,n}$ is a $n \times n$ real symmetric matrix of the form (matrix χ^2)

$$M_{m,n} = n^{-1} A^T_{m,n} A_{m,n},$$

with $A_{m,n} = \{A_{\alpha j}\}_{\alpha,j=1}^{m,n}$ having i.i.d. entries (m observation on n parameters)

$$\mathbf{P}(dA_{m,n}) = \prod_{\alpha=1}^{m} \prod_{j=1}^{n} F_{\alpha j}^{(n)}(dA_{\alpha j})$$

such that

$$\mathbf{E}\{A_{\alpha j}\} = 0, \quad \mathbf{E}\{A_{\alpha j}\} = a^2.$$

The case of i.i.d. Gaussian $\{A_{\alpha j}\}_{\alpha,j=1}^{m,n}$ is known since the early 30's as the (white or null) Wishart Ensemble.

Theorem

Let $M_{m,n}$ be the sample covariance matrix such that $\tau > 0$

$$\lim_{n\to\infty, m\to\infty, m/n\to c\in(0,\infty)} \frac{1}{mn}\sum_{\alpha=1}^{m}\sum_{j=1}^{n}\int_{|y|>\tau\sqrt{n}} y^2 F_{\alpha j}^{(n)}(dy) \to 0, \ \forall \tau > 0$$

Then its Normalized Counting Measures N_n converges with probability 1 to the non-random measure: $N_W(d\lambda) = \rho_W(\lambda)d\lambda$

$$\rho_W(\lambda) = (1-c)_+\delta_0 + \sqrt{((\lambda - a_-)(a_+ - \lambda))_+} \Big/ 2\pi a^2\lambda,$$

where $a_\pm = a^2(1\pm\sqrt{c})^2$ *(macroscopic universality again)*

Marchenko, P. 67; Girko 70's.

Proof: Wishart by the resolvent identity, Gaussian differentiation formula, and Poincaré. General case as for the Wigner (i.e. the interpolation again).

Theorem

Let $M_{m,n}$ be the Wishart random matrix. If φ is C^1, then $\overset{\circ}{\mathcal{N}}_n[\varphi]$ converges in distribution as $m, n \to \infty$, $m/n \to c > 0$ to the Gaussian random variable with zero mean and the variance

$$
\begin{aligned}
V_{Wish}[\varphi] &= \frac{1}{2\pi^2} \int_{a_-}^{a_+} \int_{a_-}^{a_+} \left(\frac{\varphi(\lambda_1) - \varphi(\lambda_2)}{\lambda_1 - \lambda_2} \right)^2 \\
&\quad \times \frac{4a^4 c - (\lambda_1 - \overline{a})(\lambda_2 - \overline{a})}{\sqrt{4a^4 c - (\lambda_1 - \overline{a})^2} \sqrt{4a^4 c - (\lambda_2 - \overline{a})^2}} \, d\lambda_1 d\lambda_2,
\end{aligned}
$$

where $\overline{a} = 1/2(a_- + a_+) = a^2(c+1)$.

Proof: By mimicking the proof for the GOE, i.e. by the Gaussian differentiation formula and Poincaré.

Theorem

Let $M_{m,n}$ be the sample covariance matrix such that:

(i) $w_5 := \sup_{m,n} \max_{1\le\alpha\le m,\,1\le j\le n} \mathbf{E}\left\{\left|A_{\alpha j}\right|^5\right\} < \infty$

(ii) $\mu_4 = \mathbf{E}\left\{\left|A_{\alpha j}\right|^4\right\}$ *do not depend on* α, j, m, *and* n, *and*

$$\kappa_4 := \mu_4 - 3a^4 = 0.$$

If $(1+|t|^5)|\widehat{\varphi}(t)| \in L^1(\mathbb{R})$, *then* $\overset{\circ}{\mathcal{N}}_n[\varphi]$ *converges in distribution as* $m, n \to \infty$, $m/n \to c > 0$ *to the Gaussian random variable with zero mean and the variance* $V_{Wish}[\varphi]$.

Proof: by interpolation from Wishart.

Bai, Silverstein, 04: Stieltjes transform, real analytic test functions, direct and rather long proof.

Theorem

Let $M_{m,n}$ be the sample covariance matrix such that:

(i) $w_6 := \sup_{m,n} \max_{1\le\alpha\le m,\, 1\le j\le n} \mathbf{E}\left\{\left|A_{\alpha j}\right|^6\right\} < \infty$

(ii) $\mu_4 = \mathbf{E}\left\{\left|A_{\alpha j}\right|^4\right\}$ *do not depend on* α, j, m, *and* n.

If $(1+|t|^4)|\widehat{\varphi}(t)| \in L^1(\mathbb{R})$, *then* $\overset{\circ}{\mathcal{N}}_n[\varphi]$ *converges in distribution as* $m, n \to \infty$, $m/n \to c > 0$ *to the Gaussian random variable with zero mean and the variance*

$$V_{Wish}[\varphi] + \frac{\kappa_4}{4c\pi^2 a^8}\left(\int_{a_-}^{a_+} \varphi(\mu)\frac{\mu-\overline{a}}{\sqrt{4a^4c-(\mu-\overline{a})^2}}d\mu\right)^2.$$

Proof: by the same scheme as in the Wigner case, i.e., by combining the schemes of proof of the CLT for the Wishart case and the "interpolation" trick

$\varphi : \mathbb{R}^p \to \mathbb{R}$ symmetric and

$$\mathcal{U}_{p,n}[\varphi] = \sum_{0 \le l_1 < \ldots < l_p \le n} \varphi(\lambda_{l_1}^{(n)}, \ldots, \lambda_{l_p}^{(n)}),$$

$$\mathcal{N}_{p,n}[\varphi] = \sum_{l_1 = \ldots = l_p = 1}^{n} \varphi(\lambda_{l_1}^{(n)}, \ldots, \lambda_{l_p}^{(n)}).$$

We have:

1. with probability 1 (LLN):

$$\lim_{n\to\infty} n^{-p}\mathcal{U}_{p,n}[\varphi] = \lim_{n\to\infty} n^{-p}\mathcal{N}_{p,n}[\varphi] = \int p \text{ times} \int \varphi(\lambda_1, \ldots, \lambda_p) N_{scl}(d\lambda_1) \ldots N_{scl}(d\lambda_p);$$

2. in distribution

$$\lim_{n\to\infty} n^{-p+1}\mathcal{U}_{p,n}[\varphi] = \lim_{n\to\infty} n^{-p+1}\mathcal{N}_{p,n}[\varphi] = \lim_{n\to\infty} \mathcal{N}_{1,n}[\varphi^*],$$

where

$$\varphi^*(\lambda) = \int (p-1) \text{ times} \int \varphi(\lambda, \lambda_2 ..., \lambda_p) \times N_{scl}(d\lambda_2)...N_{scl}(d\lambda_p),$$

i.e., the CLT.

Both assertions are valid in the cases, where there are corresponding results for $p = 1$.

Borel type theorem

Theorem

Let U_n be a $n \times n$ unitary random matrix, whose probability law is the normalized Haar measure on $U(n)$, and A_n be a $n \times n$ matrix such that

$$\lim_{n\to\infty} n^{-1}\mathrm{Tr}A_n^*A_n = 1.$$

Then $\mathrm{Tr}U_nA_n$ converges in distribution to the standard complex Gaussian variable: $\gamma = \gamma_1 + i\gamma_2$, $\mathbf{E}\{\gamma_1\} = \mathbf{E}\{\gamma_1\} = 0$, $\mathbf{E}\{\gamma_1^2\} = \mathbf{E}\{\gamma_2^2\} = 1/2$.

E. Borel 05 $(A_n = \{\delta_{j1}\delta_{k1}\}_{j,k=1}^n)$, *Diaconis et al 03; Snyady-Stolz 06.*
On the other hand, by using analogs of the differentiation formula and the Poincaré type inequality for $U(n)$ and $O(n)$ (*P.-Vasilchuk 06*) and the above scheme, a short and simple proof of the assertion can be obtained. Analogous assertions are valid for $O(n)$ and $Sp(n)$.