# On the existence of subgaussian directions for log-concave measures

joint work with A. Giannopoulos and P. Valettas.

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# Notation

$\mathcal{P}_{[n]}$ : the class of all probability measures in $\mathbb{R}^n$ which are absolutely continuous with respect to the Lebesgue measure.

$\mu$ on $\mathbb{R}^n$ is called log-concave if for any Borel sets $A, B$ and any $\lambda \in (0,1)$, $\mu(\lambda A + (1-\lambda)B) \geq \mu(A)^{\lambda}\mu(B)^{1-\lambda}$.

$\mu \in \mathcal{P}_{[n]}$ is called centered if for all $\theta \in S^{n-1}$, $\int_{\mathbb{R}^n} \langle x, \theta \rangle d\mu(x) = 0$.

Let $K$ a convex body in $\mathbb{R}^n$. Then $d\mu := \mathbf{1}_K dx$ is log-concave.

# Notation

A direction $\theta \in S^{n-1}$ is subgaussian for $\mu$ with constant $r > 0$ if

$$\|\langle \cdot, \theta \rangle\|_{\psi_2} \leq r m_\theta,$$

where $m_\theta$ is the median of $|\langle \cdot, \theta \rangle|$ with respect to $\mu$, and

$$\|f\|_{\psi_2} = \inf \left\{ t > 0 : \int_{\mathbb{R}^n} \exp\left( (|f(x)|/t)^2 \right) \, d\mu(x) \leq 2 \right\}.$$

Let $K$ centered convex body in $\mathbb{R}^n$, $|K| = 1$,

$$h_{Z_2(K)}(\theta) = \left( \int_K |\langle x, \theta \rangle|^2 dx \right)^{\frac{1}{2}}, \ \theta \in S^{n-1}$$

,

We say that $K$ is isotropic if $Z_2(K) = L_K B_2^n$.

# The Question

[Bourgain] If $r_K(\theta) = O(1)$ for all $\theta$ , then $L_K = O(1)$.

**Question:** [V. Milman] is it true that every convex body $K$ has at least one "subgaussian" direction (with constant $r = O(1)$)?
[Bobkov- Nazarov] Yes, if $K$ is 1-uncoditional.
[Klartag] Yes, with $r = \log^2 n$.
[G-P-P] Yes, with $r = \log n$.
[G-P-V] Yes, with $r = \sqrt{\log n}$.

# The result

**Theorem** (i) If $K$ is a centered convex body of volume 1 in $\mathbb{R}^n$, then there exists $\theta \in S^{n-1}$ such that

$$|\{x \in K : |\langle x, \theta \rangle| \geq cth_{Z_2(K)}(\theta)\}| \leq e^{-\frac{t^2}{\log(t+1)}}$$

for all $t \geq 1$, where $c > 0$ is an absolute constant.

(ii) If $\mu$ is a centered log-concave probability measure on $\mathbb{R}^n$, then there exists $\theta \in S^{n-1}$ such that

$$\mu\left(\{x \in \mathbb{R}^n : |\langle x, \theta \rangle| \geq ct\mathbb{E}|\langle \cdot, \theta \rangle|\}\right) \leq e^{-\frac{t^2}{\log(t+1)}}$$

for all $1 \leq t \leq \sqrt{n \log n}$, where $c > 0$ is an absolute constant.

# The $\Psi_2(K)$ body

Let $K$ be a centered convex body of volume 1 in $\mathbb{R}^n$. Definition of $\Psi_2(K)$: it is the symmetric convex body with support function

$$h_{\Psi_2(K)}(\theta) = \sup_{1\leq p\leq n} \frac{\left(\int_K |\langle x,\theta\rangle|^p dx\right)^{\frac{1}{p}}}{\sqrt{p}} = \sup_{1\leq p\leq n} \frac{h_{Z_p(K)}(\theta)}{\sqrt{p}}.$$

From the definition, one has $Z_p(K) \subseteq \sqrt{p}\Psi_2(K)$ for all $1 \leq p \leq n$. In particular, $Z_2(K) \subseteq \sqrt{2}\Psi_2(K)$, which implies that

$$|\Psi_2(K)|^{1/n} \geq c\frac{L_K}{\sqrt{n}}.$$

**Conjecture:** $|\Psi_2(K)|^{1/n} \leq c' \frac{L_K}{\sqrt{n}}$.

# The $\Psi_2(K)$ body

**Theorem** Let $K$ be a centered convex body of volume 1 in $\mathbb{R}^n$. Then,

$$|\Psi_2(K)|^{1/n} \leq c\frac{\sqrt{\log n}}{\sqrt{n}} L_K.$$

Note that $\Psi_2(K) = \mathrm{conv}\left\{\frac{Z_p(K)}{\sqrt{p}}, p \in [1, n]\right\}$,
and using the fact that $Z_{2p}(K) \simeq Z_p(K)$, we may write

$$\Psi_2(K) \simeq \mathrm{conv}\left\{\frac{Z_p(K)}{\sqrt{p}}, p = 2^k,\ k = 1, \ldots, \log_2 n\right\}.$$

Known (P., L-Z) : $|Z_p(K)|^{\frac{1}{n}} \leq c\sqrt{\frac{p}{n}} L_K$, $|Z_p(K)|^{\frac{1}{n}} \geq c\sqrt{\frac{p}{n}}$.

# The $\Psi_2(K)$ body

For any $A, B$, $|A| \leq N(A,B)|B|$.

**Lemma**

*Let $A_1, \ldots, A_s$ be subsets of $RB_2^k$. For every $t > 0$,*

$$N(\mathrm{conv}(A_1 \cup \cdots \cup A_s), 2tB_2^k) \leq \left(\frac{cR}{t}\right)^s \prod_{i=1}^{s} N(A_i, tB_2^k).$$

Regularity of the covering numbers of $Z_q(K)$: (G-P-P)

$$\log N(Z_q(K), ct\sqrt{q}L_K B_2^n) \leq \frac{n}{t}, \;\; t \geq 1.$$

## Regularity of the covering numbers of $Z_q(K)$

**Proposition** Let $K$ be an isotropic convex body in $\mathbb{R}^n$, let $1 \leq q \leq n$ and $t \geq 1$. Then,

$$\log N\left(Z_q(K), c_1 t\sqrt{q} L_K B_2^n\right) \leq c_2 \frac{n}{t^2} + c_3 \frac{\sqrt{n}\sqrt{q}}{t},$$

where $c_1, c_2, c_3 > 0$ are absolute constants.

**Corollary** Let $K$ be an isotropic convex body in $\mathbb{R}^n$ and let $1 \leq q \leq n$. Define $\beta \geq 1$ by the equation $q = n^{1/\beta}$. Let $\alpha := \min\{\beta, 2\}$. Then,

$$N\left(Z_q(K), c_1 t\sqrt{q} L_K B_2^n\right) \leq \exp\left(c_2 \frac{n}{t^\alpha}\right),$$

where $c_1, c_2 > 0$ are absolute constants.

# Sudakov-type estimates

[Talagrand] Let $\gamma_n$ be the n-dimensional Gaussian measure and let $C \subseteq \mathbb{R}^n$ be a symmetric convex body. Then, for any $s, t > 0$ we have:

$$N(B_2^n, tC^\circ) \leq e^{(2s/t)^2} [\gamma_n(sC^\circ)]^{-1}.$$

Let $m_1$ such that $\gamma_n(m_1 C^\circ) = \frac{1}{2}$. Also

$$m_1 \simeq I_1(\gamma_n, C^\circ) = \int_{\mathbb{R}^n} \|x\|_{C^\circ} d\gamma_n(x) \simeq \sqrt{n} W(C).$$

# Sudakov-type estimates

Let $0 < p$ and let $m_p$ such that $\gamma_n(m_p C^\circ) = \frac{1}{2^p}$. Also
If we write $I_{-p}$ for

$$I_{-p} \equiv I_{-p}(\gamma_n, C) := \left( \int_{\mathbb{R}^n} \|x\|_K^{-p} \, d\gamma_n(x) \right)^{-1/p}.$$

Then (Markov's inequality) $m_p \geq \frac{1}{2} I_{-p}$.
Assume that for some $p$ and some $\alpha > 1$ we have the following "regularity" condition:

$$I_{-p} \leq \alpha I_{-2p}.$$

Then, by applying the Paley-Zygmund inequality we get

$$\gamma_n(x : \|x\|_C^\circ \leq 2I_{-p}) \geq 2^{-2p \log(2\alpha)}.$$

It follows that

$$m_{2p \log(2\alpha)} \leq 2I_{-p}.$$

Also

$$I_{-p}(\gamma_n, C^\circ) \simeq \sqrt{n} W_{-p}(C).$$

# Sudakov-type estimates

**Corollary** Let $C$ be a symmetric convex body in $\mathbb{R}^n$ and let $1 \le p \le n/2$ be such that $W_{-2p}(C) \simeq W_{-p}(C)$. Then,

$$\log N\left(C, c_1\sqrt{n/p}W_{-p}(C)B_2^n\right) \le c_2 p,$$

where $c_1, c_2 > 0$ are absolute constants.
*Proof:* Choose $s := m_p \simeq \sqrt{n}W_{-p}(C)$. Then

$$N(B_2^n, tK^\circ) \le e^{\frac{nW_{-p}(C)^2}{t^2}}[\gamma_n(sK^\circ)]^{-1} \le e^{\frac{nW_{-p}(C)^2}{t^2}}e^p.$$

Choose $t = \sqrt{n/p}W_{-p}(C)$, then $\log N(B_2^n, tK^\circ) \le cp$. Then "duality of entropy" (A-M-S). $\square$

# Back to $Z_q(K)$

Let $-n < p \leq n$, $I_q(K) = \left(\int_K \|x\|_2^p dx\right)^{\frac{1}{p}}$. Let $q > 0$, then (P.),

$$I_q(K) \simeq \sqrt{\frac{n}{q}} W_q(Z_q(K)),$$

$$I_{-q}(K) \simeq \sqrt{\frac{n}{q}} W_{-q}(Z_q(K)).$$

Then

$$c_1\sqrt{q} \leq W_{-n}(Z_q(K)) \leq W_{-q}(Z_q(K)) \leq c_2\sqrt{q} L_K.$$

# Convolutions

**Proposition** Let $K$ be an isotropic convex body in $\mathbb{R}^n$. There exists an isotropic convex body $K_1$ in $\mathbb{R}^n$ with the following properties:

1. $L_{K_1} \leq c_1$.
2. $c_2 Z_p(K_1) \subseteq \frac{Z_p(K)}{L_K} + \sqrt{p} B_2^n \subseteq c_3 Z_p(K_1)$ for all $1 \leq p \leq n$.
3. $c_4 \Psi_2(K_1) \subseteq \frac{\Psi_2(K)}{L_K} \subseteq c_5 \Psi_2(K_1)$.

The constants $c_i$, $i = 1, \ldots, 5$ are absolute positive constants.

# Back to $Z_q(K)$

**Proposition** Let $K_1$ be an isotropic convex body as before , let $1 \leq q \leq n/2$ and $1 \leq t \leq \sqrt{n/q}$. Then,

$$\log N\left(Z_q(K_1), c_1 t\sqrt{q} B_2^n\right) \leq c_2 \frac{n}{t^2}.$$

*Proof:* For $q \leq r \leq n$, $W_{-r}(Z_q(K)) \simeq \sqrt{q}$. Then

$$\log N\left(Z_q(K_1), \sqrt{\frac{n}{r}} W_{-r}(Z_q(K)) B_2^n\right) \leq r.$$

Set $t = \sqrt{\frac{n}{r}}$. $\square$

Let $2^{k_1} = \frac{n}{\log n}$. Let $V_1 := \mathrm{conv}\left\{\frac{Z_p(K_1)}{\sqrt{p}}, p = 2^k, k = 1, \ldots, k_1\right\}$
and $V_2 := \mathrm{conv}\left\{\frac{Z_p(K_1)}{\sqrt{p}}, p = 2^k, k = k_1, \ldots, \log_2 n\right\}$.
Note that $\Psi_2(K_1) \simeq \mathrm{conv}\{V_1, V_2\}$.
Then $\log N\left(V_1, \sqrt{\log n} B_2^n\right) \leq n$ and $\log N\left(V_2, \log\log n B_2^n\right) \leq n$.
So, $\log N\left(\Psi_2(K_1, c\sqrt{\log n} B_2^n\right) \leq n$ and

$$|\Psi_2(K_1)|^{\frac{1}{n}} \leq c\frac{\sqrt{\log n}}{\sqrt{n}}$$

So,

$$|\Psi_2(K)|^{\frac{1}{n}} \leq c\frac{\sqrt{\log n}}{\sqrt{n}} L_K$$