

On 1-symmetric logarithmically concave distributions

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Basic Definitions

Let $X = (X_1, \dots, X_n)$ be a random vector in \mathbb{R}^n with full dimensional support. We say that the distribution of X is

- *logarithmically concave*, if X has density of the form $e^{-h(x)}$, $h: \mathbb{R}^n \rightarrow (-\infty, \infty]$ convex,
- *1-symmetric*, if $(\eta_1 X_1, \dots, \eta_n X_n)$ has the same distribution as X for any choice of $\eta_1, \dots, \eta_n \in \{-1, 1\}$,
- *isotropic*, if $\mathbb{E}X_i = 0$ and $\mathbb{E}X_i X_j = \delta_{i,j}$.

If X is logarithmically concave then there exists an affine transformation T such that TX is isotropic.

If X is 1-symmetric and logconcave then there exists a diagonal transformation D such that DX is 1-symmetric, logconcave and isotropic.

Examples of 1-symmetric logconcave isotropic vectors

During this talk we will assume that $X = (X_1, \dots, X_n)$ is logconcave, isotropic and 1-symmetric.

Basic examples include

- Standard normal vector $X = (g_1, \dots, g_n)$, where g_i are i.i.d. $\mathcal{N}(0, 1)$
- X_i independent symmetric with (one dimensional) logarithmically concave distribution normalized in such a way that $\mathbb{E}X_i^2 = 1$
- Uniform distributions on 1-symmetric convex bodies normalized to satisfy $\mathbb{E}X_i^2 = 1$ (for example uniform distributions on $C_{r,n}B_r^n$, $C_{r,n} \sim n^{1/r}$).

Moments and tails

For a random variable S (or more general a random vector with values in a normed space) and $p > 0$, we put $\|S\|_p := (\mathbb{E}|S|^p)^{1/p}$ (resp. $\|S\|_p := (\mathbb{E}\|S\|^p)^{1/p}$).

Moments and tails are strictly related. Namely by Chebyshev's inequality

$$\mathbb{P}(\|S\| \geq e\|S\|_p) \leq e^{-p}.$$

Moreover if $\|S\|_{2p} \leq \alpha\|S\|_p$ then by the Paley-Zygmund inequality

$$\mathbb{P}\left(\|S\| \geq \frac{1}{C(\alpha)}\|S\|_p\right) \geq e^{-\max\{C(\alpha), p\}}.$$

For scalar or vector valued combinations of coordinates of logconcave vectors and $p \geq 2$ we have $\|S\|_{2p} \leq C\|S\|_p$ ($C = 2$ in the scalar case).

- (ε_i) is a Bernoulli sequence (i.e. a sequence of i.i.d. symmetric ± 1 r.v.'s) independent of other random variables.
- (\mathcal{E}_i) is a sequence of i.i.d. symmetric exponential r.v.'s with variance 1 (i.e. the density $\frac{1}{\sqrt{2}} \exp(-\sqrt{2}|x|)$).
- By letter C we denote universal constants (that may take different values at each occurrence).
- For two functions f and g we write $f \sim g$ if $\frac{1}{C}f \leq g \leq Cf$.

Lower estimate of moments - scalar case

For any scalars a_i and $p \geq 1$,

$$\left\| \sum_{i=1}^n a_i X_i \right\|_p = \left\| \sum_{i=1}^n a_i \varepsilon_i |X_i| \right\|_p \geq \left\| \sum_{i=1}^n a_i \varepsilon_i \mathbb{E}|X_i| \right\|_p \geq \frac{1}{C} \left\| \sum_{i=1}^n a_i \varepsilon_i \right\|_p.$$

Montgomery-Smith'90 and Hitczenko'93

$$\left\| \sum_{i=1}^n a_i \varepsilon_i \right\|_p \sim \sum_{i \leq p} a_i^* + \sqrt{p} \left(\sum_{i > p} (a_i^*)^2 \right)^{1/2},$$

where (a_i^*) denotes the nonincreasing rearrangement of (a_i) .

Upper estimate of moments - scalar case

Less trivial is upper bound - it follows by Bobkov-Nazarov'03 result

$$\left\| \sum_{i=1}^n a_i X_i \right\|_p \leq C \left\| \sum_{i=1}^n a_i \mathcal{E}_i \right\|_p \quad p \geq 2.$$

Theorem (Gluskin-Kwapień'95)

Let Y_i be independent symmetric r.v.'s with logconcave tails such that $\mathbb{E} Y_i^2 = 1$. We put $N_i(t) = -\ln \mathbb{P}(|Y_i| \geq t)$ for $t > 1$ and $N_i(t) = t^2$ for $t \in [0, 1]$. Then for any $p \geq 2$,

$$\left\| \sum_{i=1}^n a_i Y_i \right\|_p \sim \sup \left\{ \sum_{i=1}^n a_i b_i : \sum_i N_i(|b_i|) \leq p \right\}.$$

In particular $\left\| \sum_{i=1}^n a_i \mathcal{E}_i \right\|_p \sim p \|a\|_\infty + \sqrt{p} \|a\|_2$.

Theorem

There exists a constant C such that for any $p \geq 2$,

$$\begin{aligned} & \left\| \sum_{i=1}^n a_i X_i \right\|_p \\ & \sim \inf_{\#I = \min\{\lfloor p \rfloor, n\}} \sup \left\{ \sum_{i \in I} a_i x_i + \sqrt{p} \left(\sum_{i \notin I} a_i^2 \right)^{1/2} : g_I(x) \geq e^{-Cp} \right\} \\ & \sim \sup \left\{ \sum_{i \in I_p} a_i x_i + \sqrt{p} \left(\sum_{i \notin I_p} a_i^2 \right)^{1/2} : g_{I_p}(x) \geq e^{-Cp} \right\}, \end{aligned}$$

where g_I is a density of $(X_i)_{i \in I}$ and $(|a_i|)_{i \in I_p}$ are $\min\{p, n\}$ largest values of $|a_i|$.

Uniform distribution on B_r^n

If X has a uniform distribution on $C_{r,n}B_r^n$ then for $p \geq 2$

$$\left\| \sum_{i=1}^n a_i X_i \right\|_p \sim \min\{p, n\}^{1/r} \left(\sum_{i \leq p} |a_i^*|^{r'} \right)^{1/r'} + \sqrt{p} \left(\sum_{i > p} |a_i^*|^2 \right)^{1/2},$$

where $\frac{1}{r} + \frac{1}{r'} = 1$ (Barthe, Guedon, Mendelson, Naor'05).

In particular

$$\left\| \sum_{i=1}^n a_i X_i \right\|_p \sim \left\| \sum_{i=1}^n a_i X_i^* \right\|_p \quad \text{for } 2 \leq p \leq n, \quad (1)$$

where X_1^*, \dots, X_n^* are independent such that X_i has the same distribution as X_i^* .

One may ask if (1) holds for more general class of logconcave vectors X (1-symmetric and permutation invariant?)

The result of Pilipczuk and Wojtaszczyk'08 implies that

$$\left\| \sum_{i=1}^n a_i X_i \right\|_p \leq C \left\| \sum_{i=1}^n a_i X_i^* \right\|_p \quad \text{for } p \geq 2$$

if X is uniformly distributed on Orlicz ball.

Theorem

For any vectors v_i in a normed space

$$\frac{1}{C} \left\| \sum_{i=1}^n v_i \varepsilon_i \right\|_p \leq \left\| \sum_{i=1}^n v_i X_i \right\|_p \leq C \left\| \sum_{i=1}^n v_i \varepsilon_i \right\|_p.$$

Lower estimate may be shown as in the scalar case. Upper follows from Bobkov-Nazarov's result and Talagrand's estimation of suprema of linear combinations of exponential random variables (generic chaining technique).

Corollary

For any $t > 0$

$$\frac{1}{C} \mathbb{P} \left(\left\| \sum_{i=1}^n v_i \varepsilon_i \right\| \geq Ct \right) \leq \mathbb{P} \left(\left\| \sum_{i=1}^n v_i X_i \right\| \geq t \right) \leq C \mathbb{P} \left(\left\| \sum_{i=1}^n v_i \varepsilon_i \right\| \geq \frac{t}{C} \right)$$

Weak and strong moments

Using Talagrand's two level concentration for the product exponential distribution one can prove that

Theorem

If X_i are independent, symmetric, logconcave then for $p \geq 1$,

$$\left\| \sum_{i=1}^n v_i X_i \right\|_p \sim \left\| \sum_{i=1}^n v_i X_i \right\|_1 + \sup_{\|\varphi\|_* \leq 1} \left\| \sum_{i=1}^n \varphi(v_i) X_i \right\|_p. \quad (2)$$

Conjecture

Estimate (2) holds for any (1-symmetric) logconcave random vector X .

Theorem (L.,Wojtaszczyk'08)

(2) holds for uniform distributions on B_r^n .

Connection with concentration.

Comparison of weak and strong moments is related to the following concentration problem for isotropic (1-symmetric) logconcave measures μ . Is it true that

$$1 - \mu(A + \mathcal{Z}_\mu(p)) \leq e^{-p/C} \text{ if } \mu(A) \geq 1/2,$$

where

$$\mathcal{M}_\mu(p) := \left\{ t \in \mathbb{R}^n : \int |\langle t, x \rangle|^p d\mu(x) \leq 1 \right\}$$

and

$$\begin{aligned} \mathcal{Z}_\mu(p) &:= (\mathcal{M}_\mu(p))^\circ \\ &= \{y \in \mathbb{R}^n : |\langle t, y \rangle|^p \leq \int |\langle t, x \rangle|^p d\mu(x) \text{ for all } t \in \mathbb{R}^n\} \end{aligned}$$

Weaker question

Is it true that

$$\left\| \sum_{i=1}^n v_i X_i \right\|_p \leq C \left(\left\| \sum_{i=1}^n v_i X_i \right\|_1 + \sup_{\|\varphi\|_* \leq 1} \left\| \sum_{i=1}^n \varphi(v_i) \mathcal{E}_i \right\|_p \right).$$

or equivalently that

$$\left\| \sum_{i=1}^n v_i X_i \right\|_p \leq C \left(\left\| \sum_{i=1}^n v_i X_i \right\|_1 + \sup_{\|\varphi\|_* \leq 1} p \|\varphi(v_i)\|_\infty + \sqrt{p} \|\varphi(v_i)\|_2 \right).$$

This is related to the following question for isotropic 1-symmetric logconcave measures: Is it true that

$$\mu(A + \sqrt{t}B_2^n + tB_1^n) \geq \min\left\{\frac{1}{2}, e^{t/C} \mu(A)\right\}?$$

Recent results of Klartag and E. Milman implies that

$$\mu(A + t \log n B_2^n) \leq \min\left\{\frac{1}{2}, e^{t/C} \mu(A)\right\}.$$

Proposition (L., Wojtaszczyk)

Let μ be an isotropic 1-symmetric, permutation invariant logconcave measures and $t \geq 1$. Then either

$$\mu((A + tB_1^n) \cap C\sqrt{n}B_2^n) \geq \frac{1}{2}\mu(A)$$

or

$$\mu(A + tB_1^n) \geq e^{t/C}\mu(A).$$

Sudakov minoration

How to estimate $\mathbb{E} \sup_{t \in T} \langle t, X \rangle$ for $T \subset \mathbb{R}^n$?

Suppose that $\#T \leq e^p$ and t_0 is any vector then

$$\begin{aligned} \mathbb{E} \sup_{t \in T} \langle t, X \rangle &= \mathbb{E} \sup_{t \in T} \langle t - t_0, X \rangle \leq \mathbb{E} \sup_{t \in T} |\langle t - t_0, X \rangle| \\ &\leq (\mathbb{E} \sup_{t \in T} |\langle t - t_0, X \rangle|^p)^{1/p} \leq (\mathbb{E} \sum_{t \in T} |\langle t - t_0, X \rangle|^p)^{1/p} \\ &\leq e \sup_{t \in T} \|\langle t - t_0, X \rangle\|_p. \end{aligned}$$

May one in some way reverse this statement?

Conjecture (Sudakov-type minoration)

Suppose that $T \subset \mathbb{R}^n$, $\#T \geq e^p$, $p \geq 2$ and for any $s, t \in A$, $s \neq t$ one has $\|\langle t - s, X \rangle\|_p \geq A$. Then $\mathbb{E} \sup_{t \in T} \langle t, X \rangle \geq \frac{1}{C} A$.

Examples

If $G = (g_1, \dots, g_n)$ is standard normal vector then $\langle t, G \rangle \sim \mathcal{N}(0, \|t\|_2^2)$ and $\|\langle t - s, G \rangle\|_p \sim \sqrt{p} \|t - s\|_2$. Hence the conjectured estimate in this situation is equivalent to $\mathbb{E} \sup_{t \in T} \langle t, G \rangle \geq \frac{1}{C} a \sqrt{\log N(T, aB_2^n)}$, that is Sudakov minoration for Gaussian processes.

More general if X_i are symmetric independent logconcave then Sudakov minoration holds (Talagrand'94, L'97).

Minoration conjecture holds if X has uniform distribution on B_r^n ball.

Two observations

Suppose that $T \subset \mathbb{R}^n$, $\#T \geq e^p$, $p \geq 2$ and for any $s, t \in A$, $s \neq t$ one has $\|\langle t - s, X \rangle\|_p \geq A$.

Let $\mathcal{E} = (\mathcal{E}_1, \dots, \mathcal{E}_n)$, then $\|\langle t - s, \mathcal{E} \rangle\|_p \geq \frac{1}{C}A$ so by Talagrand's result $\mathbb{E} \sup_{t \in T} \langle t, \mathcal{E} \rangle \geq \frac{1}{C}A$. But

$$\mathbb{E} \sup_{t \in T} \langle t, X \rangle \geq \frac{1}{C} \mathbb{E} \sup_{t \in T} \sum_{i=1}^n t_i \varepsilon_i \geq \frac{1}{C \log n} \mathbb{E} \sup_{t \in T} \langle t, \mathcal{E} \rangle \geq \frac{1}{C \log n} A,$$

thus minoration conjecture holds up to logarithmic factor.

Slightly more complicated argument shows that

$$\mathbb{E} \sup_{t \in T} \langle t, X \rangle \geq \frac{1}{C \max\{1, \log \frac{n}{p}\}} A$$

in particular minoration conjecture holds for $p \geq n$.