Some annihilating pairs in Harmonic Analysis

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Outline of talk

Definitions

f a function on $G = \mathbb{R}^d$, $\mathbb{T}^d$, $\mathbb{Z}^d$ or $\mathbb{Z}/n\mathbb{Z}$, $\hat{f}$ the Fourier transform of f:

1. $G = \mathbb{R}^d$, $\hat{f}(\xi) = \int_{\mathbb{R}^d} f(t) e^{-2i\pi\langle \xi, t\rangle}\,\mathrm{d}t$, $\xi \in \hat{G} = \mathbb{R}^d \to$ extend to L^2
2. $G = \mathbb{T}^d$, $\hat{f}(k) = \int_{\mathbb{T}^d} f(t) e^{-2i\pi\langle k, t\rangle}\,\mathrm{d}t$, $k \in \hat{G} = \mathbb{Z}^d$ (Fourier coefficient)
3. $G = \mathbb{Z}^d$, $\hat{f}(\xi) = \sum_{k\in\mathbb{Z}^d} f(k) e^{2i\pi\langle k, \xi\rangle}$, $\xi \in \hat{G} = \mathbb{T}^d$ (sum of Fourier series)
4. $G = \mathbb{Z}/n\mathbb{Z}$, $\hat{f}(\ell) = \frac{1}{\sqrt{n}} \sum_{k=0}^{m-1} f(k) e^{2i\pi k\ell/n}$, $\ell \in \hat{G} = \mathbb{Z}/n\mathbb{Z}$ (Discrete Fourier transform).

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- (S, Σ) *is an annihilating pair if*

$$\operatorname{supp} f \subset S \quad \& \quad \operatorname{supp} \widehat{f} \subset \Sigma \quad \Rightarrow \quad f = 0;$$

- (S, Σ) *is a strong annihilating pair if* $\exists C = C(S, \Sigma)$ *s.t.* $\forall f \in L^2(G)$,

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(Ω, s) has Uniform Uncertainty Principle (*Restricted Isometry Property*) if $\exists \delta_s \in (0,1)$ s.t., $\forall S \subset \mathbb{Z}/n\mathbb{Z}$, $|S| = s$ $\forall f \in L^2(\mathbb{Z}/n\mathbb{Z})$, $\operatorname{supp} a \subset S$

$$(1-\delta_s)\frac{|\Omega|}{n}\|f\|_2^2 \leq \|\hat{f}\|^2_{L^2(\Omega)} \leq (1+\delta_s)\frac{|\Omega|}{n}\|f\|_2^2 \tag{1}$$

δ_s is called Restricted Isometry Constant of (T, Ω, s).

$\Rightarrow$ (S, Ω^c) is a strong annihilating pair $\forall S$ s.t. $|S| = s$,

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$\Leftarrow$ If (S, Σ) is a strong annihilating pair $\forall S$ s.t. $|S| = s$, then set $C(\Sigma) = \sup_{|S|=s} C(S, \Sigma)$, (Σ^c, s) satisfies UUP with

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If $S,\Sigma\subset\mathbb{Z}/n\mathbb{Z}$ are s.t. $|S||\Sigma|<n$, then

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Non probabilistic results

$$\|\hat{f}\|_{L^2(\mathbb{Z}/n\mathbb{Z}\setminus\Sigma)} \geq \|\hat{f}\|_{L^2(\mathbb{Z}/n\mathbb{Z})} - \|\hat{f}\|_{L^2(\Sigma)} \geq \left(1-\left(\frac{|S||\Sigma|}{n}\right)^{1/2}\right)\|f\|_{L^2(S)}.$$

General case:

$$\begin{aligned}\|f\| &\leq \|1_S f\| + \|1_{S^c} f\| \\ &\leq \left(1-\left(\frac{|S||\Sigma|}{n}\right)^{1/2}\right)^{-1}\|\widehat{1_S f}\|_{L^2(\mathbb{Z}/n\mathbb{Z}\setminus\Sigma)} + \|f\|_{L^2(\mathbb{Z}/n\mathbb{Z}\setminus S)}\end{aligned}$$

$$\begin{aligned}\|\widehat{1_S f}\|_{L^2(\mathbb{Z}/n\mathbb{Z}\setminus\Sigma)} &= \|\hat{f} - \widehat{1_{S^c} f}\|_{L^2(\mathbb{Z}/n\mathbb{Z}\setminus\Sigma)} \\ &\leq \|\hat{f}\|_{L^2(\mathbb{Z}/n\mathbb{Z}\setminus\Sigma)} + \|\widehat{1_{S^c} f}\|_{L^2(\mathbb{Z}/n\mathbb{Z}\setminus\Sigma)} \\ &\leq \|\hat{f}\|_{L^2(\mathbb{Z}/n\mathbb{Z}\setminus\Sigma)} + \|\widehat{1_{S^c} f}\|_{L^2(\mathbb{Z}/n\mathbb{Z})} \\ &= \|\hat{f}\|_{L^2(\mathbb{Z}/n\mathbb{Z}\setminus\Sigma)} + \|1_{S^c} f\|_{L^2(\mathbb{Z}/n\mathbb{Z})}\end{aligned}$$

$$\|\hat{f}\|_{L^2(\mathbb{Z}/n\mathbb{Z}\setminus\Sigma)} \geq \|\hat{f}\|_{L^2(\mathbb{Z}/n\mathbb{Z})} - \|\hat{f}\|_{L^2(\Sigma)} \geq \left(1 - \left(\frac{|S||\Sigma|}{n}\right)^{1/2}\right)\|f\|_{L^2(S)}.$$

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Question

Is every set a part of an annihilating pair ?

Yes, the following is a corollary of Bourgain-Tzafriri's restricted invertibility theorem:

Proposition (Ghobber-J.)

$S,\Sigma\subset\mathbb{Z}/n\mathbb{Z}$ s.t. $|S|+|\Sigma|=n$. $\exists\sigma\subset S$ s.t. $|\sigma|\geq\dfrac{(n-|\Sigma|)^2}{240n}$ and, $\forall f\in\ell^2_d$,

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Probability

$k \leq n$, $\delta_0, \ldots, \delta_{n-1}$ n independent random variables $\mathbb{P}(\delta_j = 1) = k/n$ and $\mathbb{P}(\delta_j = 0) = 1 - k/n$.
$\Omega = \{i \in \mathbb{Z}/n\mathbb{Z} \;:\; \delta_i = 1\}$ *random subset of average cardinality* k, $\mathbb{P}\left[|\Omega - k| \geq \frac{k}{2}\right] \leq 2e^{-k/10}$.

Theorem (Rudelson-Vershynin)

$\exists \kappa$ *s.t.* $\forall 0 < \eta < 1$,
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Lemma (Nazarov, $d=1$, Fontes-Merz $d \geq 2$)

- $$p(\theta_1,\ldots,\theta_d)=\sum_{k_1=0}^{m_1}\cdots\sum_{k_d=0}^{m_d} c_{k_1,\ldots,k_d} e^{2i\pi(r_{1,k_1}\theta_1+\cdots r_{d,k_d}\theta_d)}$$
  a trigonometric polynomial in d variables.
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Lemma

With the same notations, $1 \le p < \infty$

$$\|P\|_{L^p(\mathbb{T}^d)} \le \left(\frac{28d}{|E|}\right)^{ord\,P+1/p} \|P\|_{L^p(E)}.$$

Proof: $\forall \varepsilon > 0$,

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Theorem

Let $S, \Sigma \subset \mathbb{R}^d$ have finite measure. Then

- *(Benedicks 1974-1985) (S, Σ) is weakly annihilating.*
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- *(Nazarov $d = 1$ 1993) $C(S, \Sigma) \leq ce^{c|S||\Sigma|}$*
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- $f = e^{-\pi|x|^2} = \widehat{f}$, $S = \Sigma = B(0,R)$

$$\int_{\mathbb{R}^d} |f(x)|^2 \, dx \leq c e^{(2\pi+\varepsilon)R^2} \left(\int_{\mathbb{R}^d \setminus B(0,R)} e^{-2\pi|x|^2} \, dx \right.$$

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- $f = e^{-\pi|x|^2} = \widehat{f}$, $S = \Sigma = B(0, R)$

$$\int_{\mathbb{R}^d} |f(x)|^2 \, \mathrm{d}x \leq c e^{(2\pi+\varepsilon)R^2} \left(\int_{\mathbb{R}^d \setminus B(0,R)} e^{-2\pi|x|^2} \, \mathrm{d}x + \int_{\mathbb{R}^d \setminus B(0,R)} e^{-2\pi|\xi|^2} \, \mathrm{d}\xi \right).$$

- Optimal: $C(S, \Sigma) \leq c e^{(2\pi+\varepsilon)(|S||\Sigma|)^{1/d}}$.
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Proof 1/2

$|S|, |\Sigma| < +\infty$, $f \in L^2(\mathbb{R})$, $\operatorname{supp} f \subset S$ & $\operatorname{supp} \widehat{f} \subset \Sigma$.

1. WLOG $|S| < 1$
2. $\displaystyle\int_{[0,1]} \sum_k \chi_\Sigma(\xi + k)\,\mathrm{d}\xi = |\Sigma| < +\infty \Rightarrow$ for a.a. $\xi \in \mathbb{R}$, $\mathrm{Card}\,\{k \in \mathbb{Z} : \xi + k \in \Sigma\}$ *finite*
3. $\displaystyle\int_{[0,1]} \underbrace{\sum_k \chi_S(\xi + k)}_{=0 \text{ or } \geq 1}\,\mathrm{d}\xi = |S| < 1 \Rightarrow \exists F \subset [0,1], |F| > 0$ s.t. $\forall x \in F, k \in \mathbb{Z}, f(x + k) = 0$.

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4. by Poisson Summation

$$\sum_{k\in\mathbb{Z}} f(x+k)e^{2i\pi\xi(x+k)} = \sum_{k\in\mathbb{Z}} \widehat{f}(\xi+k)e^{2i\pi kx}.$$

By 2, the RHS is a trigonometric polynomial $Z(f)(x)$ in x (for a.a. ξ)
By 3, the LHS is supported in $[0,1]\setminus F$

5. $Z(f)=0 \Rightarrow \widehat{f}=0 \Rightarrow f=0.$

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Random Periodization

Lemma (Nazarov, $d = 1$)

$\varphi \in L^1(\mathbb{R})$, $\varphi \geq 0$,

$$\int_1^2 \sum_{k\in\mathbb{Z}\quad\setminus\{0\}} \varphi\left(v \quad k\ \right) \mathrm{d}v \qquad\qquad \simeq \int_{\|x\|\geq 1} \varphi(x)\,\mathrm{d}x$$

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Lemma (Nazarov, $d = 1$)

$\varphi \in L^1(\mathbb{R}^d)$, $\varphi \geq 0$,

$$\int_{SO(d)} \int_1^2 \sum_{k \in \mathbb{Z}^d \setminus \{0\}} \varphi\big(v\,\rho(k)\big)\,\mathrm{d}v\,\mathrm{d}\nu_d(\rho) \simeq \int_{\|x\| \geq 1} \varphi(x)\,\mathrm{d}x$$

Random Periodization 2 : Proof

$$\int_{SO(d)}\int_1^2 \sum_{k\in\mathbb{Z}^d\setminus\{0\}} \varphi\big(v\,\rho(k)\big)\,\mathrm{d}v\,\mathrm{d}\nu_d(\rho)$$

$$\simeq \sum_{k\in\mathbb{Z}^d\setminus\{0\}} \int_{1\leq\|x\|\leq 2} \varphi(\|k\|x)\,\mathrm{d}x$$

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Average order

Lemma

- Σ *be a relatively compact open set with* $0 \in \Sigma$
- $\Lambda = \Lambda(\rho, v) := \{v\,{}^t\rho(j) \ : \ j \in \mathbb{Z}^d\}$ *a random lattice*
- $\mathcal{M}_{\rho,v} = \{k \in \mathbb{Z}^d \ : \ v\,{}^t\rho(k) \in \Sigma\} = \Lambda \cap \Sigma$
- *then*

$$\mathbb{E}_{\rho,v}\left(\operatorname{ord} \mathcal{M}_{\rho,v} - d\right) \leq C\omega(\Sigma).$$

remark

If order $\to$ size of support, $\mathbb{E}_{\rho,v}\left(\operatorname{Card} \mathcal{M}_{\rho,v} - d\right) \leq C|\Sigma|$.

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Scale to have $|S| = 2^{-d-1}$ and take $f \in L^2$ with $\operatorname{supp} f \subset S$

Set $\Gamma_{\rho,v}(t) = \dfrac{1}{v^{d/2}} \displaystyle\sum_{k\in\mathbb{Z}^d} f\left(\frac{\rho(k+t)}{v}\right)$

Set $E_{\rho,v} = \{t \in [0,1] : \Gamma_{\rho,v}(t) = 0\}$

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End of Proof 2/4

From the Lattice averaging lemma, one can choose ρ, v s.t.

— $\|R_{\rho,v}\|_2^2 \leq C \displaystyle\int_{\mathbb{R}^d \setminus \Sigma} |\widehat{f}(\xi)|^2 \, \mathrm{d}\xi$ (w.h.p)

— $\operatorname{ord} P_{\rho,v} \leq C(\omega(\Sigma) + d)$ (w.h.p)

— $|E_{\rho,v}| \geq 1/2$ (certain)

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ρ, v s.t. all 4 properties hold.

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End of Proof 3/4

On $E_{\rho,v}$, we have $\Gamma_{\rho,v} = 0$, thus $P_{\rho,v} = -R_{\rho,v}$ so

$$\int_{E_{\rho,v}} |P_{\rho,v}(t)|^2 \,\mathrm{d}t = \int_{E_{\rho,v}} |R_{\rho,v}(t)|^2 \,\mathrm{d}t \leq C \int_{\mathbb{R}^d \setminus \Sigma} |\widehat{f}(\xi)|^2 \,\mathrm{d}\xi$$

So $E := \{t \in E_{\rho,v} \ : |P_{\rho,v}(t)|^2 \leq 16C^2 \int_{\mathbb{R}^d \setminus \Sigma} |\widehat{f}(\xi)|^2 \,\mathrm{d}\xi\}$ has $|E| \geq 1/4$.

End of Proof 4/4

$$
\begin{aligned}
|\widehat{f}(0)|^2 &\leq |\widehat{P_{\rho,v}}(0)|^2 \leq \left(\sum_{k\in\mathbb{Z}^d} |\widehat{P_{\rho,v}}(k)|\right)^2 \leq \left(\sup_{x\in\mathbb{T}^d} |P_{\rho,v}(x)|\right)^2 \\
&\leq \left[\left(\frac{14d}{|E|}\right)^{\mathrm{ord}P_{\rho,v}-1} \sup_{x\in E} |P_{\rho,v}(x)|\right]^2 \\
&\leq \left[\left(\frac{14d}{1/4}\right)^{\mathrm{ord}P_{\rho,v}-1} 4\left(C\int_{\mathbb{R}^d\setminus\Sigma} |\widehat{f}(\xi)|^2\,\mathrm{d}\xi\right)^{1/2}\right]^2 \\
&\leq Ce^{C\omega(\Sigma)}\int_{\mathbb{R}^d\setminus\Sigma} |\widehat{f}(\xi)|^2\,\mathrm{d}\xi.
\end{aligned}
$$

Apply to $f \to f_y(x) = f(x)e^{-2i\pi xy}$, $\Sigma \to \Sigma_y = \Sigma - y$ and integrate over $y \in \Sigma$ QED