Random embedding of ℓ_p^n into ℓ_r^N

$$0 < r < p < 2 \qquad \frac{2p}{p+2} \leq r \leq 1$$

Omer FRIEDLAND Olivier GUÉDON

Université Pierre et Marie CURIE
Université Paris-Est Marne La Vallée

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History

- **Johnson – Schechtman '82** proved the existence of a random embedding for non-Euclidean spaces :
- Let $1 < p < 2$. Then for any $\varepsilon > 0$

$$\ell_p^n \overset{1+\varepsilon}{\hookrightarrow} \ell_1^N \quad , \qquad N = C(p, \varepsilon) n.$$

- More precisely, they gave an explicit definition of a random operator, $T : \ell_p^n \to \ell_1^N$, and proved that :

$$1 - \varepsilon \le |T\alpha|_1 \le 1 + \varepsilon \quad , \qquad \forall \alpha \in S_p^{n-1}.$$

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- **Figiel – Lindenstrauss – Milman ’77** proved, following Milman’s approach to Dvoretsky theorem :

$$\ell_2^n \overset{1+\varepsilon}{\hookrightarrow} \ell_1^N \quad , \qquad N = C(\varepsilon)n.$$

- **Kashin ’77**, with a different approach, proved :

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	$1 \leq p < 2$	$p = 2$
almost-isometric $\varepsilon-$embed	**JS '82**	**FLM '77**
isomorphic with $N = (1+\eta)n$	?	**K '77**

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Questions

- Whether there is an embedding that satisfies these conditions ?
- Is there a random embedding ?

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- **Naor – Zvavitch '01** provided an explicit definition of a random operator which satisfies the desired property :

$$\ell_p^n \overset{C}{\hookrightarrow} \ell_1^N \quad , \qquad N = (1+\eta)n,$$

where $C = (c \log n)^{(1-1/p)(1+1/\eta)}$.

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- **Pisier '83** extended this result to the case of *a general finite normed space* E *of dimension* N :

$$\ell_p^n \overset{1+\varepsilon}{\hookrightarrow} E,$$

where n depends only on ε and on the stable-type p constant of E.

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- **Johnson – Schechtman '82** used a discretization method to approximate $p-$stable random variables.
- **Naor – Zvavitch '01** used truncated $p-$stable random variables.
- **Pisier '83** used a completely different approach.

Definitions

- Let $(e_i)_{1\le i\le N}$ be the canonical basis of $\mathbb{R}^N$.
- Let Y be a random vector taking the values $\{\pm e_1, \ldots, \pm e_N\}$, with probability $\frac{1}{2N}$.
- We define the following operator :

$$T : \ell_p^n \to \ell_r^N$$

$$\alpha = (\alpha_1, \ldots, \alpha_n) \mapsto \frac{\sigma_{p,r}}{N^{1/q}} \sum_{i=1}^{n} \alpha_i \sum_{j\ge 1} \frac{1}{j^{1/p}} Y_{i,j},$$

where $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, and $(Y_{i,j})$ are independent copies of Y.

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Theorem [Random embedding of ℓ_p^n into ℓ_r^N]

Let $0 < r < p < 2$ and $\frac{2p}{p+2} \le r \le 1$.

For any $\eta > 0$, and any integers n, $N = (1+\eta)n$ we have

$$\mathbb{P}\left\{\forall \alpha \in S_p^{n-1},\ \ c(p,r)^{1/\eta} \le |T\alpha|_r \le C(p,r)\right\} \ge 1 - c\ \exp(-c_{p,r}n),$$

where $c(p,r), C(p,r), c_{p,r}$ depend only on p and r,
and c is an absolute constant.

Remark

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Stable random variables

- A real-valued symmetric r.v. θ is called standard $p-$stable :

$$\mathbb{E}\exp(it\theta) = \exp(-|t|^p) \quad , \qquad \forall t \in \mathbb{R}^n.$$

- Why "stable" ?

$$\sum_i \alpha_i \theta_i \stackrel{D}{=} \left(\sum_i |\alpha_i|^p\right)^{1/p} \cdot \theta_1,$$

  where $\alpha_i \in \mathbb{R}$, θ_i is standard $p-$ stable r.v., and for any finite sequence.

- In particular, it suggests that ℓ_p^n is isometric to a subspace of L_1 :

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- Let $(\lambda_i)_i$ be independent random variables with common exponential distribution $\mathbb{P}\{\lambda_i > t\} = \exp(-t)$, $t \geq 0$.
- Set $\Gamma_j = \sum_{i=1}^{j} \lambda_i$, for $j \geq 1$.
- We recall that Y is the random vector taking the values $\{\pm e_1, \ldots, \pm e_N\}$, with probability $\frac{1}{2N}$.
- By a result of **LePage – Woodroofe – Zinn '81** :

$$\Theta = \sum_{j \geq 1} \Gamma_j^{-1/p} Y_j,$$

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- Let us define the following operator :

$$\tilde{T} : \ell_p^n \to \ell_1^N \quad , \qquad \tilde{T}\alpha = \frac{\sigma_p}{N^{1/q}} \sum_{i=1}^{n} \alpha_i \Theta_i.$$

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Ideas behind this result

- Fix $\alpha \in S_p^{n-1}$. We have

$$\mathbb{P}\{\big||T\alpha|_1 - |\alpha|_p\big| \geq t\} \leq 2\exp(-c_p N t^q),$$

  note that $|\alpha|_p = 1$.

- It means

$$1 - t \leq |T\alpha|_1 \leq 1 + t$$

$$t = \varepsilon > 0 \quad , \qquad N = Cn.$$

- But

$$|T\alpha|_1 \leq 1 + t \quad , \qquad \forall t > 0.$$

- Large deviation is useful for almost-isometric results and for obtaining upper bounds in general.

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- In our situation : $N = (1+\eta)n$ and $t \in (0,1)$.
- We may assume in addition that $\alpha \in S_p^{n-1}$ has a small support : $|\operatorname{supp}(\alpha)| \leq \delta n$.

$$\begin{array}{ccc} Cn & X & n \\ (1+\eta)n & X & \delta n \end{array}$$

- It means that for such vectors with $\delta \simeq \frac{1}{C}$, we may use this large deviation inequality again, and have a lower bound.

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Division of S_p^{n-1} (following Rudelson-Vershynin)

- Let $\delta, \rho \in (0,1)$.
- We define
$$\mathrm{Sparse}(\delta) = \{\alpha \in \ell_p^n : |\operatorname{supp}(\alpha)| \leq \delta n\}.$$
- We partition S_p^{n-1} into two sets with respect to $\mathrm{Sparse}(\delta)$ and ρ.
- We define the following sets :
$$AS(\delta,\rho) = \{\alpha \in S_p^{n-1} : \mathrm{dist}_p(\alpha, \mathrm{Sparse}(\delta)) \leq \rho\},$$
$$NAS(\delta,\rho) = S_p^{n-1} \setminus AS(\delta,\rho),$$

where $AS(\delta,\rho)$ is the ρ-enlargement (for the ℓ_p^n metric) of the set of sparse vectors intersected with S_p^{n-1}.

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Small ball estimate

- For $\alpha \in NAS(\delta, \rho)$

$$\mathbb{P}\{|T\alpha|_1 \leq t\} \leq (c_p t)^N \quad , \qquad t > 0.$$

- It means $\qquad t \leq |T\alpha|_1 \quad$ w.h.p

- Basic properties of NAS vector :

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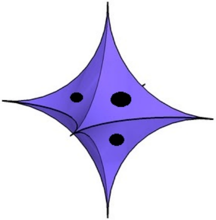

Theorem [Multi-dimensional Esseen type inequality]

Let X be a random vector in $\mathbb{R}^N$, such that the function

$$\xi \mapsto \mathbb{E}\exp(i\langle \xi, X\rangle)$$

belongs to $L_1(\mathbb{R}^N)$.

Then for any compact star-shape $K \subset \mathbb{R}^N$, for any $t > 0$

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Application

- For $\alpha \in NAS(\delta, \rho)$ $\qquad \mathbb{P}\{\big||T\alpha|_1\big| \le t\} \le (c_p t)^N$.
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$$|T\alpha|_1 = \|X\|_K.$$

- Lemma : For any vector $\alpha \in NAS(\delta, \rho)$, the function $\xi \mapsto \mathbb{E}\exp(iN\langle \xi, T\alpha\rangle)$, belongs to $L_1(\mathbb{R}^N)$. Moreover,

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