Support identification of sparse vectors from random and noisy measurements.

Charles Dossal[1] Marie-Line Chabanol[1] Gabriel Peyré[2]
Jalal Fadili[3]

[1]IMB CNRS-Université Bordeaux 1

[2]CEREMADE CNRS-Université Paris Dauphine

[3]GREYC CNRS-ENSICAEN-Université de Caen

Workshop Probability and Geometry in High Dimensions

Outline

- Setting
- ℓ_1 recovery: Overview
  - ℓ_1 minimization and geometry of polytopes.
  - Restricted Isometry Property.
  - Exact support recovery using LASSO.
- Contributions.
- Sketch of proof of the main result.

Setting

Noisy Gaussian measurements of sparse vectors

- Linear random measurements $y = Ax + w \in \mathbb{R}^n$, $x \in \mathbb{R}^N$, $A = (a_{ij})_{i \leqslant n, j \leqslant N} \in \mathbb{R}^{n \times N}$, $a_{ij} \sim \mathcal{N}(0, 1/n)$ and iid, $w \in \mathbb{R}^n$ and $\|w\|_2 \leqslant \varepsilon$.
- x is sparse $\Leftrightarrow$ $\|x\|_0 < N$ is small.
- x is weakly sparse (compressible).

Questions

- Estimate x from y when $n < N$, ill-posed inverse problem.
- Estimate the support I of x from y.
- Stability to noise and robusteness to compressibility.

Sparse Recovery Algorithms

A large choice of methods

- Greedy methods : Matching Pursuit, OMP, Cosamp, MCA ...
- Non convex optimization :

$$\min_{x\in\mathbb{R}^N} \|x\|_p \ s.t. \quad y-Ax \in C,\ p\in(0,1), C=\{0\} \text{ or } C=\mathcal{B}_{\ell_2}(\sigma)$$

- Convex optimization :

$$\min_{x\mathbb{R}^N} \|x\|_1 \quad s.t. \quad y-Ax\in C,$$

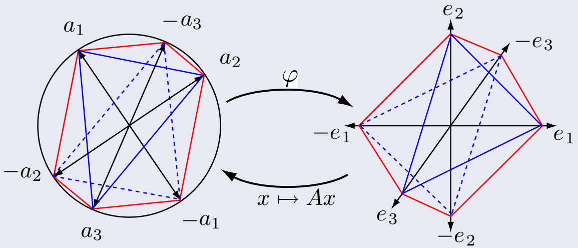
  - $C=\{0\}$: exact ℓ_1 minimization (Basis Pursuit).
  - $C=\{z \quad s.t.\ \|A^t z\|_\infty \leqslant \tau\}$: Dantzig Selector.
  - $C=\mathcal{B}_{\ell_2}(0,r)$: LASSO/BPDN equivalent to

$$\min_{x\in\mathbb{R}^N} \frac{1}{2}\|y-Ax\|_2^2+\gamma\|x\|_1 \qquad \text{(LASSO)}$$

Noiseless observations : Geometry of centrosymmetric polytopes

Donoho [04], Donoho and Tanner [05-07]

- Identifiability is a geometrical property.



- x is identifiable if and only if $\frac{Ax}{\|x\|_1} \in \partial A(\mathcal{B}_{\ell_1})$.
- For $x \in \mathbb{R}^N$, $I = \{i, x_i \neq 0\}$, $f_x = \text{Conv.Hull}(\text{sign}\,(x_i)\, a_i)_{i \in I}$.
- x is identifiable $\Leftrightarrow$ f_x is an exterior facet of $A(\mathcal{B}_{\ell_1})$.

The geometrical viewpoint

Counting k-faces of centro-symmetric polytopes [Donoho 04]

- If A is gaussian or USE, there is a function $\rho_N(.)$ such that w.o.p. on A, all sparse x with

$$\|x\|_0 \leqslant \rho_N(n/N)n \text{ are } \ell_1 - \text{identifiable.} \quad (1)$$

- If A is gaussian or USE, x with randomly chosen support and sign, there is $\rho_F(.)$ such that w.o.p. on A, most sparse x with

$$\|x\|_0 \leqslant \rho_F(n/N)n \text{ are } \ell_1 - \text{identifiable.} \quad (2)$$

- No stability to noise.
- Sharp phase transition :

$$\rho_N(1/2) \sim 0.089,\ \rho_F(1/2) \sim 0.38$$
$$\rho_N(1/4) \sim 0.065,\ \rho_F(1/4) \sim 0.25.$$

Restricted Isometry Property

Definition of RIP

- For $A \in \mathbb{R}^{n \times N}$, $\delta_S^{\min}$ and $\delta_S^{\max}$ are the smallest real numbers in $(0,1)$ such that for any x, $\|x\|_0 \leqslant S$,

$$(1-\delta_S^{\min})\|x\|_2^2 \leqslant \|Ax\|_2^2 \leqslant (1+\delta_S^{\max})\|x\|_2^2.$$

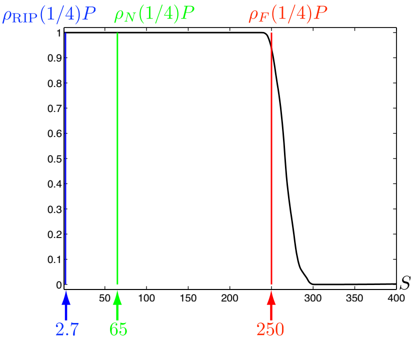
Theorem [Fourcart and Lai 08]

$$\text{If } (4\sqrt{2}-3)\delta_{2S}^{\min} + \delta_{2S}^{\max} < 4(\sqrt{2}-1), \qquad \text{(RIPFL)}$$

- All vectors x such that $\|x\|_0 \leqslant S$ are identifiable.
- Stability to noise, consistency if x is only compressible.
- There exist C_0 and C_1 depending on $\delta_{2S}^{\min}$ and $\delta_{2S}^{\max}$ such that the solution x^* of (LASSO) satisfies

$$\|x^* - x_0\|_2 \leqslant C_0 S^{-1/2} \|x - x_S\|_1 + C_1 \varepsilon.$$

Bounds for gaussian matrices, $N = 4000$, $n = 1000$



- If A is a Gaussian matrix with iid entries, then w.o.p. A satisfies (RIPFL) for $S = O\left(\frac{n}{\log(N/n)}\right)$.
- For $n/N = \frac{1}{4}$, (RIPFL) applies up to $S = 0.0027n$ [Blanchard et al. 09].
- but (RIPFL) doesn't apply if $S \geqslant 0.005n$. [D. et al 09].

Exact Support and sign pattern recovery with LASSO

Theorem [Candes Plan 07]

Let $A \in \mathcal{M}_{n,N}(\mathbb{R})$ which columned are normed and such that $\mu(A) \leqslant \frac{c_1}{\ln N}$. Let $w \in \mathbb{R}^n$ such that $w(i) \sim \mathcal{N}(0, \frac{\varepsilon^2}{n})$. Let $x_0 \in \mathbb{R}^N$ and $T = \min_{i \in I} |x_0(i)|$.

For sufficiently small constant c_0

- if x_0 is randomly chosen among vectors such that $|I| \leqslant \frac{c_0 N}{\|A\|^2 \ln N}$, (support : uniform and sign : Rademacher).
- if $T \geqslant 8\varepsilon\sqrt{\frac{2 \ln N}{n}}$ and $\gamma = 2\varepsilon\sqrt{\frac{2 \ln N}{n}}$

the solution x^* of

$$\min_{x \in \mathbb{R}^N} \frac{1}{2} \|y - Ax\|_2^2 + \gamma \|x\|_1 \qquad \text{(LASSO)}$$

satisfies $Supp(x^*) = Supp(x_0)$ and $\operatorname{sign}(x^*) = \operatorname{sign}(x_0)$ w.o.p.

Contibutions

Results for Gaussian matrices

- Refinement of Theorem [Candes Plan 07] for Gaussian matrices
  - without any prior on the distribution of x_0 and w.
  - with explicit and optimal constants.
  - robustness to compressibility.
- Without any hypothesis on $\frac{T}{\varepsilon}$
  - $Supp(x^*)$ is controlled.
  - ℓ_2 consistency results.
- Explicit bounds may be better that the ones derived from RIP.
- Justify debiasing.

Support and sign pattern identification

Theorem 1

Let $(a, b) \in (0, 1)^2$, $N > n$, $y = Ax_0 + w$ *where* A *is a Gaussian matrix and* $\|w\|_2 \leqslant \varepsilon$.

- *If* $\|x_0\|_0 = S \leqslant \frac{ab}{2} \frac{n}{\ln N}$,
- *if* $\gamma = \frac{\varepsilon}{\sqrt{1-a}} \sqrt{\frac{2 \ln N}{n}}$ *and if* $T \geqslant \frac{6\varepsilon}{\sqrt{1-a}} \sqrt{\frac{2 \ln N}{n}}$

then w.o.p. $Supp(x^*) = Supp(x_0)$ *and* $\mathrm{sign}\,(x^*) = \mathrm{sign}\,(x_0)$ *and*

$$\|x^* - x_0\|_2 \leqslant \varepsilon\left(\sqrt{\frac{a}{1-a}} + 1\right)$$

Support inclusion

Theorem 2

Let $(a, b) \in (0, 1)^2$, $N > n$, $y = Ax_0 + w$ *where* A *is a Gaussian matrix and* $\|w\|_2 \leqslant \varepsilon$.

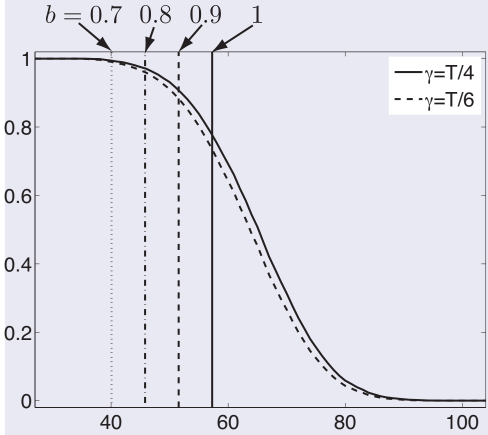
- *If* $\|x_0\|_0 = S \leqslant \frac{ab}{2} \frac{n}{\ln N}$,
- *if* $\gamma = \frac{\varepsilon}{\sqrt{1-a}} \sqrt{\frac{2 \ln N}{n}}$,

then w.o.p $Supp(x^*) \subset Supp(x_0)$ *and*

$$\|x^* - x_0\|_2 \leqslant \varepsilon\left(\sqrt{\frac{a}{1-a}} + 1\right)$$

Numerical experiments

Example with $a = 0.95$, $n = 1000$ and $N = 4000$



- Rates of exact support recovery versus sparsity level.
- $\varepsilon = \frac{T}{6}\sqrt{\frac{(1-a)n}{2\ln N}}$.

Sketch of proof of Theorem 1

Notations

- For a vector x, let's denote I its support,
- A_I the associated active matrix and $\overline{x}$ the restriction of x to I.
- We have $Ax = A_I\overline{x}$.
- Let's denote $P_{A_I^\perp}$ the orthogonal projection on $V^\perp$ with $V = \mathrm{Span}\{(a_i)_{i\in I}\}$.

Remarks

- If A is gaussian, $\forall(y,\gamma) \in \mathbb{R}^n \times \mathbb{R}^{+*}$, the solution x^* of

$$\min \frac{1}{2} \|y - Ax\|_2^2 + \gamma \|x\|_1 \qquad \text{(LASSO)}$$

  - is unique with probability 1 and
  - $(A_I^t A_I)$ associated to x^* is inversible with probability 1.

Sketch of proof of Theorem 1

A necessary condition

- If $Supp(x^*) = I = Supp(x_0)$ and $\mathrm{sign}\left(\overline{x^*}\right) = \mathrm{sign}\left(\overline{x_0}\right)$ then the solution of (LASSO) is defined by

$$\overline{x^*} = \overline{x_0} - \gamma(A_I^t A_I)^{-1}\mathrm{sign}\left(\overline{x_0}\right) + (A_I^t A_I)^{-1} A_I^t w. \quad (3)$$

A sufficient condition

- Let's denote $T = \min_{i\in I} |x_0(i)|$, if

$$\left\|\gamma(A_I^t A_I)^{-1}\mathrm{sign}\left(\overline{x_0}\right) - (A_I^t A_I)^{-1} A_I^t w\right\|_\infty < T \quad \text{(SC1)}$$

and

$$|\langle a_j, \gamma A_I (A_I^t A_I)^{-1}\mathrm{sign}\left(\overline{x_0}\right) - P_{A_I^\perp}(w)\rangle| \leqslant \gamma, \forall j \notin I \quad \text{(SC2)}$$

the vector x^* defined by (3) is the solution of (LASSO).

Sketch of proof of Theorem 1

SC1 : $\left\|\gamma(A_I^tA_I)^{-1}\mathrm{sign}\,(\overline{x_0}) - (A_I^tA_I)^{-1}A_I^tw\right\|_\infty < T$

- If $|I| \leqslant \frac{abn}{2\ln N}$
  - $\left\|(A_I^tA_I)^{-1}\mathrm{sign}\,(x_0)\right\|_\infty \leqslant 1 + 3\sqrt{ab} \leqslant 4$ with w.o.p
    - Properties of Wishart matrices (signs of coefficients and spectral norm).
  - $\left\|(A_I^tA_I)^{-1}A_I^tw\right\|_\infty \leqslant 2\varepsilon\sqrt{\frac{\ln N}{n}}$ with w.o.p.
    - Rotation Invariance of $(A_I^tA_I)^{-1}A_I^t$, χ^2 concentration lemmas, and spectral norm of Wishart matrices.
- If $\gamma \leqslant \frac{T}{6}$ and $\varepsilon \leqslant \frac{T}{6}\sqrt{\frac{n}{2\ln N}}$ then condition (SC1) applies :

$$\left\|\gamma(A_I^tA_I)^{-1}\mathrm{sign}\,(\overline{x_0}) - (A_I^tA_I)^{-1}A_I^tw\right\|_\infty < T$$

Sketch of proof of Theorem 1

SC2 : $|\langle a_j, \gamma A_I (A_I^t A_I)^{-1} \mathrm{sign}(\overline{x_0}) - P_{A_I^\perp}(w)\rangle| \leqslant \gamma, \forall j \notin I$

- If u and a_j are independent, then $\langle a_j, u\rangle \sim \mathcal{N}(0, \frac{\|u\|_2}{n})$.
- If $j \notin I$, $u = \gamma A_I (A_I^t A_I)^{-1} \mathrm{sign}(\overline{x_0}) - P_{A_I^\perp}(w)$ and a_j are independent.
- It follows that w.o.p.

$$\max_{j \notin I} |\langle a_j, u\rangle| \leqslant \sqrt{\frac{2 \ln N}{n}} \|u\|_2 \tag{4}$$

- $\|u\|_2^2 \leqslant \gamma^2 \left\|A_I (A_I^t A_I)^{-1} \mathrm{sign}(\overline{x_0})\right\|_2^2 + \varepsilon^2$ using Pythagore !!!
- $\left\|A_I (A_I^t A_I)^{-1} \mathrm{sign}(\overline{x_0})\right\|_2^2$ is bounded using a classical Wishart concentration lemma.
- It follows that if $|I| \leqslant \frac{abn}{2 \ln N}$ and $\gamma \geqslant \varepsilon \sqrt{\frac{(1-a)n}{2 \ln N}}$, condition SC2 applies.

Take Away Messages

Conclusions

- Optimal bounds for exact support recovery with Gaussian measurements.
- Partial support recovery if $\frac{T}{\varepsilon}$ is too small.
- New bounds for ℓ_2 recovery different from RIP.
- Robustness to noise and compressibility without RIP.

Going Further

- Extension to subgaussain matrices (USE and Bernoulli)

- Paper available online very soon.