A panorama of Hungarian Mathematics in the Twentieth Century I, ed. J. Horváth, pp. 211–244, Springer, 2005.

A panorama of the Hungarian real and functional analysis in the 20th century

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The theory of real functions is a relatively young chapter of mathematical analysis. In fact, in the 19th century, the expression "function theory" was applied to mean the theory of complex-valued analytic functions of one or more complex variables. The first investigations that initiated the part of analysis called today "theory of real functions" or briefly "real analysis" were constructions of various real-valued functions of a real variable whose characteristic properties are very far from those of analytic functions; as a typical example, we can mention the construction, due to *Karl Weierstrass*, of a real-valued function continuous in an interval but not differentiable at any point of this interval.

The investigations in this direction obtained a very powerful instrument in the theory of sets discovered and developed by *Georg Cantor*. As first important results of the theory, we can mention the investigations of *Camille Jordan* on properties of the functions of bounded variation, or the theory of the area of plane point sets due to the same author. However, the last discovery that finished the acceptance of real analysis as a well-adopted chapter of analysis was the concept and theory of a very general kind of *integral*, due to *Henri Lebesgue*, in the first years of the 20th century. This acceptance was perhaps due to the fact that Lebesgue not only developed the theory of the integral but also reached amazing applications of his theory so that the importance of the new, general theory of the integral convinced everybody interested in mathematical analysis. On the other hand, Lebesgue's integration theory catalyzed the cristallization of the ideas of *functional analysis* in the setting of infinite dimensional function spaces.

There were Hungarian mathematicians who joined the investigations on real analysis in the earliest period of its existence, i.e., in the last decades of the 19th century. We must first mention **Gyula [Julius] Kőnig** (1849–1913) who, in his courses on analysis given at the Technical University Budapest, presented a definition of the *integral* that included not only the classical definition of the Riemannian integral but also the Stieltjes integral. Despite the fact that he did it in about 1890, he formulated these ideas not earlier than 1897 in a paper in Hungarian language {27} so that the priority evidently belongs to *Thomas Jean Stieltjes* who published his discovery in 1894.

However, Kőnig was very probably the first researcher who constructed a real-valued function continuous in an interval and having an extremum in every subinterval {26}.

[1] The authors are grateful to János Horváth, József Szűcs and László Zsidó for many helpful suggestions.

This result certainly influenced the disciple of Kőnig, **Zoárd Geőcze** (1873–1916), who constructed a continuous function that is not rectifiable in any subinterval {10}. Many years later, it turned out that Geőcze's construction yields essentially more; in fact it is one of the simplest constructions furnishing a continuous nowhere differentiable function i.e., it presents the *Weierstrass singularity* (see work {21} of **Sándor Kántor**). On the other hand, the same method of construction, under another choice of the parameters, furnishes a continuous function that is increasing and singular (i.e., its derivative is equal to 0 almost everywhere, see {8}). In the last sentence, the expression "almost everywhere" means, of course, "*with the exception of a set of points of Lebesgue measure zero*". It is an imperishable merit of the history of mathematics in Hungary that we had a researcher who, a few years after its birth, not only made himself a master of Lebesgue's theory of measure and integral, but also added to this theory essential contributions. This researcher is **Frigyes [Frédéric] Riesz** (1880–1956).

Frédéric Riesz was definitely the first mathematician in Hungary who understood the great importance of the new theory of integration. He was born in Győr, a town in the midway between Budapest and Vienna. After a two-year-study at the polytechnic in Zürich, he continued at the science university in Budapest. From 1904 to 1912 he was a high-school teacher and wrote fundamental papers already in this period. Although he published very good works also in Hungarian all his life, he was clever enough to understand that Paris was not only the capital of France but the capital of modern analysis as well. His publications in *Comptes Rendus*, the journal of the French Academy of Sciences, earned a world fame for him very early and he became a professor of the University of Kolozsvár (called Cluj-Napoca now in Roumania) in 1912.

Frédéric Riesz was one of the fathers of functional analysis. Although functional analysis in the sense of nowadays had several roots in the 19th century, such as Fourier expansion of functions and spectral theory of some differential equations, its genesis could be put at the beginning of 20th century. Even at the end of 1800's linear algebra was very finite dimensional and dealt with n-tuples of real numbers, and Fréchet's thesis on metric spaces appeared in 1906. The cristallization of the ideas was catalyzed by Lebesgue's integration theory. Many of the basic concepts of functional analysis were born in the setting of infinite dimensional function spaces. The intimate relation of the Lebesgue integral and infinite dimensional functional analysis is very transparent in the work of Frédéric Riesz.

The space L^2 of square integrable functions on an interval of the real line was the first infinite dimensional space on which functional analysis in the modern sense was studied. The so-called *Riesz-Fischer theorem* (1907) {58} claims that the space L^2 is complete, that is, all Cauchy sequences are convergent in the L^2-sense. That time it was shown that if $f_n \in L^2$ and $\int |f_n(x) - f_m(x)|^2 \, dx$ is arbitrarily small when n and m are large enough, then there exists a function $f \in L^2$ such that

$$\int |f_n(x) - f(x)|^2 \, dx \to 0 \text{ as } n \to \infty \, .$$

This is Fischer's version of the Riesz-Fischer theorem but Riesz was aware of the equivalence. Riesz actually showed that given an orthogonal sequence $f_1, f_2, \dots$ of functions of unit length and a sequence $c_1, c_2, \dots$ of scalars such that $\sum_i |c_i|^2$ is finite, there exists a function f in the L^2 space such that $\langle f_i, f\rangle = c_i$.

Another early result of Riesz, discovered independently by Fréchet relying on an earlier paper of Riesz, tells us that any bounded linear functional A of L^2 is induced by an element g of L^2 in the form of integration of the product {54} :

$$A(f) = \int f(x)g(x)\,dx \qquad (f \in L^2) \tag{1}$$

for some $g \in L^2$ if there exists a constant M_A such that

$$\int |f(x)|^2\,dx \le 1 \text{ implies } |A(f)| \le M_A \qquad (f \in L^2)\,.$$

It is still a pleasure to read the original works of Riesz. In 1910 he published a paper in Hungarian (Integrálható függvények sorozatai {56}), in which he explains his understanding of L^2 and the above mentioned two results. It is remarkable that abstract functional analysis did not exist at that time, nevertheless he understood his own (as well as Lebesgue's, Fréchet's and Fischer's) result in a very modern and intrinsic way. For example, he wrote that *"The extension of the concept of integral due to Lebesgue is an indispensable condition for my theorem, similarly to the fact that validity of certain theorems of algebra or arithmetics require the appropriate extension of the concept of numbers"*. In this paper he defined the *weak topology* on the space L^2 and shows that a bounded sequence contains a weakly convergent subsequence.

In our present language, (1) describes the dual of the L^2 space. The *dual space* was certainly a concept that should be attributed to Riesz. He defined the dual of L^2 in 1907 and in 1909 he dealt with the dual of the space of continuous functions, *Sur les opérations fonctionnelles linéaires.* Let $C[a,b]$ denote the set of all continuous real-valued functions on the interval $[a,b]$. In 1903 Hadamard wanted to describe all linear functionals $U : C[a,b] \to \mathbb{R}$ such that $Uf_n \to Uf$ whenever $f_n \to f$ uniformly. He took a function F such that

$$f(x) = \lim_{n\to\infty} n \int_a^b f(t)F(n(t-x))\,dt$$

uniformly in $x \in [a,b]$, for example $F(x) = \exp(-x^2)$ would do, and he showed that

$$U(f) = \lim_{n\to\infty} \int_a^b f(t)\Phi_n(t)\,dt\,,$$

where $\Phi_n(t)$ is the value of the functional U at the function $x \mapsto nF(n(t-x))$. Riesz described the continuous linear functionals of $C[a,b]$ by means of the Stieljes integral and removed the arbitrariness of the function F in Hadamard's theorem {55}. He proved that there exists a function α of bounded variation such that

$$U(f) = \int_a^b f(x)\,d\alpha(x), \tag{2}$$

moreover α is unique if $\alpha(a) = 0$ and the left continuity of α are required. For any $a < t < b$ he considered the function

$$f_t(x) = \begin{cases} x-a, & \text{if } \ a \le x \le t, \\ t-a, & \text{if } \ t \le x \le b. \end{cases}$$

He showed that the function $A : t \mapsto U(f_t)$ satisfies a Lipschitz condition and took $-\alpha(t)$ as one of the derived numbers of A at the point t. Then it was a standard procedure to modify α to fulfil the additional requirements and to keep (2).

Extending his work on the space L^2, Riesz devoted a fundamental paper to L^p spaces in 1910, *Untersuchungen über Systeme integrierbarer Funktionen* {57}. L^p is the set of all complex valued measurable functions such that $|f|^p$ is integrable. He restricted himself to the case $p > 1$ and extended the Hölder and Minkowski inequalities

$$\Big|\sum_{k=1}^{n} a_k b_k\Big| \le \Big(\sum_{k=1}^{n} |a_k|^p\Big)^{1/p}\Big(\sum_{k=1}^{n} |b_k|^q\Big)^{1/q}, \text{ where } \frac{1}{p}+\frac{1}{q}=1,$$

$$\Big(\sum_{k=1}^{n} |a_k+b_k|^p\Big)^{1/p} \le \Big(\sum_{k=1}^{n} |a_k|^p\Big)^{1/p} + \Big(\sum_{k=1}^{n} |b_k|^p\Big)^{1/p}$$

to measurable functions. If $f \in L^p$ and $g \in L^q$, then fg is integrable and

$$\Big|\int f(x)g(x)\,dx\Big| \le \Big(\int |f(x)|^p\,dx\Big)^{1/p}\Big(\int |g(x)|^q\,dx\Big)^{1/q}.$$

Moreover, if $f, g \in L^p$, then $f+g \in L^p$ and

$$\Big(\int |f(x)+g(x)|^p\,dx\Big)^{1/p} \le \Big(\int |f(x)|^p\,dx\Big)^{1/p} + \Big(\int |g(x)|^p\,dx\Big)^{1/p}.$$

He extended several definitions and results from the theory of L^2 spaces. He defined *strong convergence* in L^p as $f_n \to f$ if and only if $\int |f_n(x) - f(x)|^p\,dx \to 0$. His first definition of *weak convergence* was different from today's usual one. He said that $f_n \to f$ weakly if

$$\int_a^t f_n(x)\,dx \to \int_a^t f(x)\,dx$$

for all t in the interval on which the functions are defined. He showed that this is equivalent to

$$\int (f(x) - f_n(x))g(x)\,dx \to 0 \text{ for all } g \in L^q.$$

He proved the *weak compactness* of the unit ball of L^p and he was particularly interested in the solution of the infinite system of linear equations

$$\int_a^b f_i(x)\xi(x)\,dx = c_i\,, \tag{3}$$

where $\xi(x)$ is the unknown and the $f_i(x)$'s belong to L^q. (The subscript i can run over an arbitrary set, countable, or not.) One cannot give an easy condition for the existence of the solution. He claimed that the condition is the existence of a constant M such that

$$\Big|\sum_{i\in I} \mu_i c_i\Big|^q \le M^q \int_a^b \Big|\sum_{i\in I} \mu_i f_i(x)\Big|^q\,dx \tag{4}$$

holds for all finite subsets I of the index set and for all complex numbers μ_i. Riesz was aware of the importance of the case where the f_i's are all the functions in L^q. Then (4) is exactly the boundednes of the functional L defined as $L(f_i) = c_i$ and he discovered that the dual of L^q can be identified with L^p. He achieved the first example of what we call today *reflexive Banach space.*

The system of equations (3) is related to the moment problem. This means that given the continuous functions f_i on an interval $[a, b]$ and a sequence c_i of real numbers $(i = 1, 2, \ldots)$, an increasing function α should be found such that

$$c_i = \int_a^b f_i(x)\, d\alpha(x) \qquad (i = 1, 2, \ldots) \tag{5}$$

(cf. (2)). In the original moment problem $f_i(t) = t^i$. A trivial necessary condition for the existence of α is the property that $\sum_{j=1}^n \lambda_j f_j \geq 0$ should imply $\sum_{j=1}^n \lambda_j c_j \geq 0$. If this is fulfilled then

$$U_0\left(\sum_{j=1}^n \lambda_j f_j\right) = \sum_{j=1}^n \lambda_j c_j$$

defines a positive functional on the linear span of the functions f_i and this functional should be extended to all continuous functions. The representation theorem (2) could be used. The moment problem belonged to the circle of ideas Frédéric Riesz worked on. His brother, **Marcell [Marcel] Riesz** (1886–1969), considered the moment problem as the question of extension of a positive functional. His method works in a very general setting, where a linear functional is defined on a subspace and the positivity is determined by a convex cone. His beautiful method is applicable not only to the power moment problem but many related problems in function theory (see Section II.6 in {1}).

In 1920 Frédéric Riesz published a detailed paper dedicated to an elementary presentation of Lebesgue's integral (B4 in [156]). The method is based on a completely elementary particular case of Lebesgue measure, namely on the definition and simplest properties of the sets of measure zero (briefly null sets): A set $A \subset R$ is a null set if it can be covered, for an arbitrary $\varepsilon > 0$, with a sequence of intervals $[a_n, b_n]$ such that $\sum_{n=1}^{\infty}(b_n - a_n) < \varepsilon$. A statement is true a.e. if it is true everywhere with the exception of the points of a null set.

The starting point is the, still quite elementary, definition of the integral of *simple functions*, i.e., functions in $[a, b]$ such that there is a decomposition of $[a, b]$ into finitely many pairwise disjoint subintervals in each of which the function is constant. For simple functions, the (Riemann) integral can be given by a finite sum.

Now a function f, bounded in $[a, b]$, is said to be *integrable* if there exists a bounded sequence of simple functions f_n (i.e., $|f_n| \leq M$ for some M) such that $f_n \to f$ a.e. in $[a, b]$. It can be shown that, under these conditions, the integrals $\int_a^b f_n(x)\, dx$ converge to a limit depending only on f (i.e., independent of the sequence); this limit defines $\int_a^b f(x)\, dx$.

In order to define the integral of an unbounded function, let us first say that a (bounded or unbounded) function is *measurable* in $[a, b]$ if it is the pointwise limit of an a.e. convergent sequence of simple functions. The integral $\int_a^b f(x)\, dx$ of a measurable function f is defined as the limit of the sequence $\int_a^b f_{c_n,d_n}(x)\, dx$, where (c_n) is an arbitrary sequence tending to $-\infty$ and (d_n) is one tending to $+\infty$, while f_{c_n,d_n} is the truncation of f: it is equal to $f(x)$ if $c_n \leq f(x) \leq d_n$, to c_n if $f(x) < c_n$, and to d_n if $f(x) > d_n$; the function f is said to be *integrable* if the above limit exists, is finite and independent of the choice of the sequences (c_n) and (d_n).

Based on these definitions, it is not difficult to deduce the usual properties of the integral (linearity, theorems on the integration of sequences of functions, etc.) It can be easily shown that, for a function integrable in the sense of Riemann, the new integral exists and is equal to the Riemann integral.

Riesz presented his exposition of the theory of the Lebesgue integral in his courses on analysis. A detailed exposition can be found in the monograph [157] which was later translated into several languages.

In two papers (B5 and B6 in [156]) Riesz analyzes the role of *Egoroff's theorem*, which states that a convergent sequence of measurable functions is uniformly convergent eliminating a subset of arbitrarily small measure {9}, in the theory of the Lebesgue integral. In particular, he indicates the modifications necessary for extending the theorem for applications in the theory of the *Lebesgue-Stieltjes integral*. The Stieltjes integral apparently captured Riesz's attention because it played a decisive role in his result concerning the integral representation of bounded linear operations on the function space $C(a, b)$ of continuous functions (see (2)).

In three short papers (B7, B8 and B9 of [156]) and in his letters to *G.H. Hardy*, Riesz gives simple proofs for some integral inequalities in particular, the celebrated maximal inequality of Hardy and Littlewood; in general, arguments are based on the use of the *distribution function* $m(y) = m(\{x \in [a, b] : f(x) < y\})$ associated with a function f measurable in $[a, b]$, where m denotes Lebesgue measure. In B9, he uses the so called *Riesz lemma* to furnish an elementary proof of Lebesgue's theorem: every monotone function is almost everywhere differentiable (see B10, B11 and B12 in [156]). In its simplest form, i.e., for continuous functions, the Riesz lemma is so elementary that its proof can be included here.

Riesz lemma: If f is continuous in the interval $[a, b]$, then the set H of points $x \in [a, b]$ for which there exists some point $x < x' \leq b$ such that $f(x) < f(x')$ is open: $H = \bigcup_k (a_k, b_k)$ and $f(a_k) \leq f(b_k)$ for each k.

The set H can be empty; in this case we have nothing to prove. If $H \neq \emptyset$, it is evidently open by the continuity of f so that the representation $H = \bigcup(a_k, b_k)$ is clearly possible. Fix a k and consider $a_k < x < b_k$. Let x_0 be one of the points in the interval $[x, b]$ where the value of f is maximal. Then $x \leq x_0 < b_k$ is impossible since it would imply $x_0 \in H$ and the existence of an $x_0 < x' \leq b$ satisfying $f(x_0) < f(x')$. Thus $b_k \leq x_0 \leq b$ and then $f(b_k) \geq f(x_0)$ as $b_k \notin H$. On the other hand, $f(x) \leq f(x_0)$ by the choice of x_0, hence $f(x) \leq f(b_k)$ and, from the continuity of f, $x \to a_k$ yields $f(a_k) \leq f(b_k)$.

Using the Riesz lemma, the proof of Lebesgue's theorem becomes almost completely elementary. The Riesz lemma quite automatically provides coverings of the exceptional set by systems of intervals of arbitrarily small total length in the proof of Lebesgue's theorem on the a.e. differentiability of a monotone function.

Riesz himself was aware that his lemma can be used for proving further interesting theorems of measure theory (B9 and B13 of [156]) and ergodic theory (G5, G7 and G8 in [156]). Much later, the lemma was generalized to several variables (cf. {5} and [166]).

The fact that Lebesgue's theorem on the differentiability of monotone functions obtained an elementary proof through the Riesz lemma, suggested to Riesz a new approach to Lebesgue's integral theory, based on the differentability of monotone functions. He presents his ideas in two papers (B14 and B15 of [156]) The starting point is the following observation: Let $f \geq 0$ in the interval $[a, b]$ and suppose that there exists a function F, increasing in $[a, b]$ and satisfying $F'(x) = f(x)$ a.e. in (a, b). Then there exists, among these F, one for which the difference $F(b) - F(a)$ is the smallest possible.

After having proved this, we say that $f \geq 0$ is *integrable* in $[a, b]$ if there is an F as above, and define $\int_a^b f$ as the minimum of $F(b) - F(a)$. A function f of arbitrary sign is said to be integrable if $f^+ = \max(f, 0)$ and $f^- = -\min(f, 0)$ are integrable and then we define $\int_a^b f = \int_a^b f^+ - \int_a^b f^-$. From these definitions, one can deduce without any difficulty the usual properties of the integral, e.g., the theorems on the integration of sequences of functions.

In the years after World War II, Riesz wrote some big expository papers on the evolution of the concept of the integral (B16, B17 in [156]) and on the role of null sets in real analysis (B18 in [156]). It is natural that his own ideas played a central role in all these summaries.

The original proof of the Riesz lemma due to Frédéric Riesz, was slightly more complicated; the idea of applying the above point x_0 is due to his brother **Marcell [Marcel] Riesz** (1886–1969). Marcel Riesz was also an outstanding mathematician, he lived most of his life in Sweden and had a wide scientific interest, including functional analysis, partial differential equations and algebra. Assume that a linear operator A is defined on a set of measurable functions and its values are also measurable functions on a different space. Assume that A has a finite norm $C(p, q)$ when it is regarded as a map from L^p to L^q $(1 \leq p \leq \infty, 1 \leq q \leq \infty)$. The *Riesz convexity theorem* of Marcel Riesz tells us that $\log C(p, q)$ is a convex function of the variables $(p^{-1}, q^{-1}) \in [0, 1] \times [0, 1]$. The convexity theorem became a starting point of abstract interpolation theorems. The spaces L^p and $L^{p'}$ are connected by a path of Banach spaces (namely the L^q spaces, when q is between p and p'). Under some conditions a construction works for any two Banach spaces, this is a very concise description of the *interpolation theory* due Calderón, Lions and Peetre which has its roots in the work of Marcel Riesz.

A considerable part of the work of several Hungarian mathematicians in the first part of the 20th century was devoted to an important application of Lebesgue integral, namely to the calculation of the area of surfaces. Surface area is only seemingly an easy two-dimensional analogue of arc length. Since the work of *Hermann Amandus Schwarz*, we know that the theory of surface area is essentially more complicated. Recall that if, say,

$$x = \varphi(t), \quad y = \psi(t), \quad z = \chi(t) \quad (a \leq t \leq b)$$

is the parametric representation of a continuous curve in R^3 (i.e., φ, ψ, χ are continuous in $[a, b]$) then the *length* of the curve can be defined as the least upper bound of the lengths of polygons inscribed in the curve, namely obtained with the help of a subdivision of $[a, b]$ by points $a = t_0 < t_1 < t_2 < \ldots < t_n = b$ and taking as vertices of the polygon the points of the curve with parameters t_i; the curve is *rectifiable* iff this least upper bound is finite.

Schwarz discovered that the area of a surface *cannot* be defined in a similar way. Even for very simple surfaces, e.g., for a circular cylinder, it can happen that the areas of all inscribed polyhedra are unbounded from above, and the surface can be uniformly approximated by inscribed polyhedra so that their area tends to an arbitrary limit not less than the usual (elementary) area of the surface.

Consider, for the sake of simplicity, a surface having an equation $z = f(x, y)$ where f is continuous in a rectangle $R = [a, b] \times [c, d]$. In this case, an idea due to Lebesgue again produces a suitable definition of the area of the surface. Consider a subdivision of R into pairwise non-overlapping triangles $T_1, ..., T_n$ and a function g continuous in R and linear in each of the triangles T_i. The equation $z = g(x, y)$ corresponding to the *piecewise linear* function g can be considered as representing a polyhedron P having an elementary area $a(P)$. Let us consider a sequence of subdivisions having the property that the functions g_n converge uniformly to f and the areas $a(P_n)$ have a limit l; this limit may depend on the sequence (g_n) and then the smallest possible limit can be considered as the area of the surface; we shall call it the *Lebesgue area* $L(f)$ of $z = f(x, y)$.

In the case of a good function f (e.g. if f has continuous partial derivatives f_x and f_y in R) it is not difficult to show that the Lebesgue area can be computed with the help of the classical formula

$$L(f) = \iint \sqrt{f_x^2 + f_y^2 + 1}\, dxdy\,; \tag{6}$$

however, in the general case of a continuous f, the definition of $L(f)$ does not directly involve any method for calculating it.

This was the motivation for **Zoárd Geőcze**, in one of his first papers on the theory of surfaces, presented as a Thesis to the Sorbonne in Paris in 1908 (Quadrature des surfaces courbes, Ungar. Ber., 26 (1910), 1–88), to introduce the following expressions:

$$G_1(f, I) = \int_\alpha^\beta |f(x, \delta) - f(x, \gamma)|\, dx,$$

$$G_2(f, I) = \int_\gamma^\delta |f(\beta, y) - f(\alpha, y)|\, dy,$$

$$G(f, I) = (G_1(f, I)^2 + G_2(f, I)^2 + |I|)^{1/2}\,,$$

where $I = [\alpha, \beta] \times [\gamma, \delta]$ is a subinterval of R and $|I|$ denotes the area $(\beta - \alpha)(\delta - \gamma)$ of I. He considered further the limit that we call nowadays the *Burkill integral* of the interval function G; in order to define it, let us consider a subdivision $\mathcal{I} = \{I_1, ..., I_n\}$ of R into pairwise non-overlapping subintervals I_i, then take the sum

$$s(\mathcal{I}) = \sum_1^n G(f, I_i)$$

and the (always existing) limit $H(f, R)$ of $s(\mathcal{I})$, i.e., a (finite or infinite) number to which $(s(\mathcal{I}_n))$ converges whenever the subdivision $\mathcal{I}_n$ is *infinitely refining* (i.e., varies such a manner that the maximum of the diameters of the intervals belonging to $\mathcal{I}_n$ tends to 0).

Now Geőcze proposes to consider the value $H(f, R)$ as the area of the surface $z = f(x, y)$. This is motivated by the result that $H(f, R) = L(f)$ whenever the function f satisfies a *Lipschitz condition* (i.e., there is a constant M such that $|f(x', y') - f(x, y)| < M(|x' - x| + |y' - y|)$ whenever $(x, y), (x', y') \in R$). This proposal is well-motivated because **Tibor Radó** (1895–1967) proved later that the equality $H(f, R) = L(f)$ is valid for any continuous function f {47}. Thus Geőcze found in fact a method for calculating the Lebesgue area of an arbitrary continuous surface defined by an equation $z = f(x, y)$ $((x, y) \in R)$.

Geőcze found also a necessary and sufficient condition for the value $H(f, R)$ to be finite, i.e., by Radó's theorem, for the continuous surface $z = f(x, y)$ to have a finite Lebesgue area. This is the following: let the function f be of bounded variation as a function of x in the interval $[a, b]$ for almost every fixed $y \in [c, d]$ and as a function of y in the interval $[c, d]$ for almost every fixed $x \in [a, b]$; let us denote by $V_1(y)$ the total variation of $f(x, y)$ as a function of x over the interval $[a, b]$ and by $V_2(x)$ the total variation of $f(x, y)$ as a function of y over the interval $[c, d]$; the condition postulates that V_1 should be (Lebesgue) integrable in $[c, d]$ and V_2 be integrable in $[a, b]$.

This condition due to Geőcze was rediscovered by *Leonida Tonelli* {64}; a function satisfying this condition is said to be of *bounded variation in the Tonelli sense.* Tonelli found also a necessary and sufficient condition for the classical formula (6) to give the Lebesgue area of the continuous surface $z = f(x, y)$. A function f satisfying this condition is said to be *absolutely continuous in the Tonelli sense.* This theory fills Chapter V of the brilliant monograph {61} of Stanislaw Saks, where works due to Geőcze and Radó are often quoted.

The problem of calculation of the area of surfaces is essentially more complicated if we consider continuous surfaces having a parametric representation; suppose the surface S is given in the form

$$x = f(u, v), \quad y = g(u, v), \quad z = h(u, v)\,,$$

where f, g, h are continuous in a rectangle $R = [a, b] \times [c, d]$ of the uv-plane. Geőcze made a few first steps in this direction in {12} and in the works "A rectifiabilis felületről" {14} "A felület területének Peano-féle definitiójáról" {15} written in Hungarian. However, the thorough discussion of this problem was mainly accomplished by Radó who not only published a long series of papers on this subject but is also the author of a great monograph [144] containing a deep analysis of the serious difficulties of the problem.

Besides the Lebesgue area $L(S)$ of the surface S, defined with the help of sequences of polyhedra quite similarly as in the case of surfaces represented in the form $z = f(x, y)$, it is convenient to introduce another kind of area $a(S)$ playing a role similar to the expression $H(f, R)$ in the theory of surfaces $z = f(x, y)$. This is done, based on ideas of Geőcze {12} by Radó in {48, 49, 50} and in [144]. The concept of $a(S)$ can be used in examining the properties of the surfaces of zero area {13, 52}.

The role of the surface area $a(S)$ in calculating the Lebesgue area $L(S)$ is emphasized by the fact that, in many cases, we have $a(S) = L(S)$ and, at the same time, $a(S)$ is often equal to the value of the classical integral formula

$$\iint_R W(u,v)\, dudv\,, \tag{7}$$

where

$$W(u,v) = (J_1(u,v)^2 + J_2(u,v)^2 + J_3(u,v)^2)^{1/2}$$

and

$$J_1(u,v) = \frac{\partial(g,h)}{\partial(u,v)}, \quad J_2(u,v) = \frac{\partial(h,f)}{\partial(u,v)}, \quad J_3(u,v) = \frac{\partial(f,g)}{\partial(u,v)}$$

are Jacobians. Radó $\{51\}$ has shown that the value of (7) is always $\leq L(S)$, whenever the partial derivatives $f_x, f_y, g_x, g_y, h_x, h_y$ exist a.e. in R.

In the general case, Radó has shown in $\{53\}$ that, instead of the concept of functions of bounded variation and absolutely continuous in the Tonelli sense, it is possible to introduce the concept of a surface S of *essential bounded variation* and *essentially absolutely continuous*, respectively, further instead of the ordinary Jacobians J_i, *essential generalized Jacobians* and, with the help of them, a generalized function $W_e(u,v)$. Now if $L(S)$ is finite, then S is of essential bounded variation, $W_e(u,v)$ exists a. e. on R and we have the inequality

$$\iint_R W_e(u,v)\, du\, dv \leq L(S).$$

The sign of equality holds if S is essentially absolutely continuous. Moreover, if the partial derivatives $f_x, ..., h_y$ exist a.e. in R, then $W_e(u,v)$ can be replaced here by $W(u,v)$. As to the equality $a(S) = L(S)$, it holds whenever $L(S)$ is finite and also if $a(S) = 0$.

Radó's results in the theory of surface area play, of course, an important role in his monograph on a famous question in differential geometry [142]. He also published a monograph together with Reichelderfer [145], where the methods developed in the theory of surface area play an essential role. In his last papers, he combines the methods of this theory with methods of general measure theory $\{51\}$ and of three papers written in collaboration with *E.J. Mickle* $\{31, 32, 33\}$.

Geőcze and Radó were determining personalities in the theory of surface area and their works are quoted everywhere in the literature of this important chapter of Analysis.

György [George] Alexits (1899–1978) became later a famous researcher in the theory of orthogonal series; however, one of his early papers $\{2\}$ is an essential contribution to an important chapter of real analysis, namely to the theory of Baire functions. A paper of **Pál [Paul] Veress** (1893–1945) $\{65\}$ is concerned with the same theory. Both papers are quoted in the monograph of *Hans Hahn* (Reelle Funktionen, Leipzig, 1932). Veress was the author of the first textbook on real analysis in Hungarian .

At the beginnings of functional analysis *integral equations* enjoyed a lot of attention. They are of the form

$$\varphi(s) = f(s) + \lambda \int_a^b K(s,t) f(t)\, dt\,,$$

where φ is a continuous function on $[a,b]$, λ is some complex parameter, $K(s,t)$ is continuous on $[a,b] \times [a,b]$ and $f(t)$ is the unknown function. For example, the Dirichlet problem could be reduced to such an integral equation. *David Hilbert*, in his very famous and fundamental paper of 1906, replaced the integral equation by an older concept of an infinite system of linear equations. Let u_n be a complete orthonormal sequence of continuous functions on $[a, b]$. If f is a solution of the equation, then we can consider the generalized Fourier coefficients

$$k_{ij} = \int_a^b \int_a^b k(s,t)u_i(s)u_j(t)\,dsdt,$$

$$x_i = \int_a^b \varphi(s)u_i(s)\,ds \quad \text{and} \quad y_i = \int_a^b f(s)u_i(s)\,ds.$$

In this way we arrive at the infinite system of linear equations

$$y_i + \lambda \sum_{j=1}^{\infty} k_{ij}y_j = x_i\,,$$

where the sequences x_i and y_i are square summable and k_{ij} is an infinite matrix (of a certain bilinear form). Hilbert himself worked with square integrable sequences and introduced the important concepts of continuity and complete continuity, mostly for symmetric bilinear forms. It is not our aim to give more details about Hilbert's work on integral operators, we want to turn to the work of Riesz on *completely continuous operators*.

His lecture delivered in a session of the Hungarian Academy of Sciences in 1916 appeared in the journal Mathematikai és Természettudományi Értesítő with the title *Lineáris függvényegyenletekről* in Hungarian {59} in 1917, and the German translation *Über lineare Funktionalgleichungen* {60} was published in 1918. The subject of this paper is the invertibility of certain transformations and Riesz gave the definition and spectral theory of completely continuous transformations. He works on the space of all continuous functions on an interval, but he notes that similar methods work on other function spaces, i.e. on L^2, where they are even simpler . He uses the *norm* of a function f: $\|f\|$, which is the maximal value of the function $|f(x)|$. The same concept and the same notation is standard today. He carried over Hilbert's definition of a completely continuous bilinear form (based on the weak topology) to the new situation. He defined a linear mapping as completely continuous if the image of a bounded sequence is compact. (In today's language one would replace "compact" by "precompact" or by "relative compact".) The novelty of his paper is that he realized that Fréchet's concept of compactness is the proper tool to deal with completely continuous operators and he uses only the axiomatic definition of norm years before it was introduced under the name of "Banach space". He gives in an entirely geometric language what is known nowadays as the *Riesz-Fredholm theory of compact operators.*

In 1918 Riesz left Kolozsvár, after the World War I the town became part of Roumania. For two years Riesz lived in Budapest and in 1922 he became professor of the newly founded university at Szeged.

A partially ordered real linear space L has an order structure which is compatible with the linear structure. This means, that for any pair of elements f and g in L satisfying $f \leq g$ it follows that $f + h \leq g + h$ holds for all $h \in L$ and $af \leq ag$ holds for all real numbers $a \geq 0$. If, in addition, the order structure in L is a lattice structure, then L is called a *Riesz space*. In the present language Frédéric Riesz was interested in the ordered dual of an ordered vector space and the basic example was the space of continuous functions. His lecture at the International Mathematical Congress at Bologna in 1928 was devoted to this subject and he returned to it in an 1940 Annals of Mathematics paper (which was the translation of his 1937 inaugural lecture at the Hungarian Academy of Science). Riesz put emphasis on the following decomposition property: If $f_1 + f_2 = g_1 + g_2$, then there are elements f_{11}, f_{12}, f_{21} and f_{22} such that $f_1 = f_{11} + f_{12}$, $f_2 = f_{21} + f_{22}$, $g_1 = f_{11} + f_{21}$ and $g_2 = f_{12} + f_{22}$. In the space of continuous functions one can choose $f_{11} := \min(f_1, g_1)$ and $f_{22} = f_1 + f_2 - \max(f_1, g_1)$.

Between 1938 and 1948 Riesz dealt in eight papers with *ergodic theorems*. The ergodic and quasi-ergodic hypothesis were born in statistical mechanics and von Neumann gave the following mathematical formulation. Let $\mathcal{H}$ be a Hilbert space and T be a bounded linear transformation on $\mathcal{H}$. According to *von Neumann's mean ergodic theorem* the averages

$$s_n(f) := \frac{1}{n}(f + Tf + \ldots + T^{n-1}f)$$

converge to a T-invariant vector for every vector $f \in \mathcal{H}$, when T is a unitary operator. Riesz gave a very elegant proof for von Neumann's result. Riesz showed that the orthogonal complement of the set

$$\{f - s_n(f) : n \in \mathbb{N}, f \in \mathcal{H}\}$$

is the fixed point set of T. From this fact one can prove the convergence of the averages and the proof requires only the hypothesis $\|Tf\| \le \|f\|$ for every $f \in \mathcal{H}$, that is T is a contraction. The Hilbert space version of the mean ergodic theorem corresponds to L^2 spaces and Riesz considered other L^p spaces as well. Much later ergodic theory appeared again in Hungarian functional analysis in the context of operator algebras: **István Kovács** and **József Szűcs** obtained the first mean ergodic theorem in von Neumann algebras {25}.

Their result implies that if α is an automorphism of a von Neumann algebra admitting a faithful normal invariant state, then the averages

$$s_n := \frac{1}{n}(I + \alpha + \ldots + \alpha^{n-1})$$

converge to a conditional expectation onto the fixed point algebra, pointwise in the strong operator topology.

John von Neumann was born Neumann János in 1903 in Budapest. He was a child prodigy, a prodigious student and he left his mark not only on pure mathematics but on theoretical physics, on meteorology, on economics, on digital computers and on more. He was the mathematician admired by most scholars outside of his own discipline. In its December 24 issue in 1999, The Financial Times has declared John von Neumann to be *"Man of the Century"*.

In the years 1914-21 von Neumann studied in Budapest's Lutheran Gymnasium. In 1921 he went to become a chemical engineer first to Berlin University and then in 1923 he took the entrance examination for the prestigious course in the chemical engineering department of the famous Eidgenössische Technische Hochschule in Zürich. When Hermann Weyl was absent from Zürich, the undergraduate chemist von Neumann took over the teaching of some of his classes. During his university years at the ETH, von Neumann was passing courses in Budapest University (which he never attended) from where received his Ph.D. with highest honors in 1925.

In early autumn of 1926 von Neumann arrived in Göttingen. He immediately learnt quantum theory from Heisenberg's lectures. Von Neumann became an axiomatizer of quantum mechanics on behalf of the so-called Copenhagen school (which did not include Schrödinger.) To Hilbert's delight, von Neumann's mathematical exposition made much use of Hilbert's own concept of Hilbert space. However, it is not sure that axiomatization of the Hilbert space and its linear operators (as a substitute for infinite matrices) by the twenty-three-year-old von Neumann was to Hilbert's delight. Our present concept of Hilbert space, infinite dimensional complex vector space endowed with an inner product whose metric is complete and separable, was formulated by von Neumann. The rigorous quantum mechanics required the use of unbounded operators defined only on a subspace of a Hilbert space. Von Neumann developped several technicalities concerning such operators. The role of the graph, the difference between symmetric and selfadjoint operators, the spectral decomposition of unbounded selfadjoint operators were discovered by him. In his excellent textbook {29} **Peter Lax** makes the following historical comment: *In the 1960s Friedrichs met Heisenberg, and used the occasion to express to him the deep gratitude of the community of mathematicians for having created quantum mechanics, which gave birth to the beautiful theory of operators in a Hilbert space. Heisenberg allowed that this was so; Friedrichs then added that the mathematicians have, in some measure, returned the favor. Heisenberg looked noncommittal, so Friedrichs pointed out that it was a mathematician, von Neumann, who clarified the difference between a selfadjoint operator and one is merely symmetric. "What's the difference," said Heisenberg.*

After some earlier work on single operators, von Neumann turned to families of operators. He initiated the study of rings of operators, which are commonly called *von Neumann algebras* today. The papers which constitute the series *"Rings of operators"* opened a new field in mathematics and influenced research for half a century (or even longer). In the standard theory of modern operator algebras, many concepts and ideas have their origin in von Neumann's work.

A von Neumann algebra consists of bounded linear Hilbert space operators. The characteristic feature of the concept of von Neumann algebra is its very rich structure. A von Neumann algebra contains the spectral projections of all selfadjoint operators belonging to the algebra. In particular, there are many orthogonal projections in the algebra itself. Roughly speaking, the point in the concept of von Neumann algebra is that formation of product and spectral diagonalization of selfadjoint elements are posssible within the algebra. It turns out that the projections of a von Neumann algebra form a lattice in the sense that any two of them have a least upper bound and a greatest lower bound with respect to an appropriate and natural ordering. The lattice of projections is the starting point in the classification of von Neumann algebras and a ground for quantum logic. Von Neumann algebras are classified in terms of the range of a dimension function defined on the lattice of projections. The dimension function is the extension of the simple concept of rank (for matrices) and the peculiarity of the subject begins with the observation that in nontrivial cases this "rank" can be noninteger. Below the classification of von Neumann algebras is described. Also, the influence of measure theory on early operator algebra theory is demonstrated by a comparison of a measure-theoretic construction of **Alfréd Haar** with the dimension function of *Murray* and von Neumann. This example shows that the connection with measure theory and ergodic theory has been very important for operator algebras since the very beginning.

We denote by $B(\mathcal{H})$ the set of all bounded operators acting on the Hilbert space $\mathcal{H}$. For a subset $\mathcal{S} \subseteq B(\mathcal{H})$, its commutant $\mathcal{S}'$ is defined as the set of operators commuting with $\mathcal{S}$:

$$\mathcal{S}' = \{K \in B(\mathcal{H}) : KS = SK \quad \text{for all} \quad S \in \mathcal{S}\}\,.$$

Note that $\mathcal{S} \subseteq (S')'$ holds obviously for any $\mathcal{S} \subseteq B(\mathcal{H})$. A family of operators acting on a Hilbert space and containing the identity operator is called a von Neumann algebra if it contains the adjoint, the linear combinations and the products of its elements and forms a closed subspace of the space of all bounded operators with respect to the topology of pointwise convergence. A von Neumann algebra is linearly spanned by its selfadjoint elements and the spectral resolution of the latter ones lies conveniently in the algebra. One of the first results of von Neumann, the *von Neumann's double commutant theorem*, was an equivalent algebraic definition of von Neumman algebras. Von Neumann's double commutant theorem asserts that a family of operators is a von Neumann algebra if and only if it contains the adjoint of its elements and coincides with its second commutant (that is, the commutant of its commutant). The remarkable point in the double commutant theorem is the lack of any topological requirement. In the concept of von Neumann algebra, topology and pure algebra are in great harmony.

The selfadjoint idempotents, called (orthogonal) projections, of a von Neumann algebra form an orthomodular, complete lattice with respect to the lattice operations $\wedge, \vee, \perp$ and the partial ordering $\leq$. Below we describe how these operations are defined in terms of the algebaric operations. The projections are in natural correspondence with the closed subspaces of the underlying Hilbert space and the set theoretical inclusion of subspaces induces a partial ordering on the projections. This ordering is equivalently defined as

$$p \leq q \quad \text{if} \quad pq = p\,. \tag{8}$$

The smallest projection with respect to this ordering is 0 and the largest one is the identity. For projections p and q, their meet (that is, greatest lower bound) $p \wedge q$ is the orthogonal projection onto the intersection of the range spaces of p and q. The projection $p \wedge q$ may be obtained as the strong limit of $(pq)^n$ as $n \to \infty$. The projection $p \vee q$ is defined as the smallest upper bound in the lattice of projections. ($p \vee q$ projects onto the closed subspace spanned by the range spaces of p and q.)

The orthocomplementation $\perp$ is defined as $p^\perp = I - p$. The orthomodularity of the lattice of projections means that the following so-called orthomodularity condition is fulfilled in the lattice:

$$q = p \vee (p^\perp \wedge q) \quad \text{for} \quad p \leq q. \tag{9}$$

This relation is a weakening of the distributivity condition and is an essential property of the lattice of projections.

Let p and q be two projections in a von Neumann algebra $\mathcal{M}$. The projections p and q are called equivalent (with respect to $\mathcal{M}$), $p \sim q$ in notation, if there is an operator x in $\mathcal{M}$ such that $p = x^*x$ and $q = xx^*$. In terms of the underlying Hilbert space, the equivalence of p and q means that there exists a partial isometry x in the given von Neumann algebra which sends the range space of p isometrically onto the range of q. An extended positive-valued function $D : \mathcal{P}(\mathcal{M}) \to [0, \infty]$ on the set $\mathcal{P}(\mathcal{M})$ of all projections of $\mathcal{M}$ is called a dimension function if it satisfies the following requirements:

(a) $D(p) > 0$ if $p \neq 0$ and $D(0) = 0$.

(b) $D(p) = D(q)$ if p and q are equivalent projections.

(c) $D(\sum_i p_i) = \sum_i D(p_i)$ if $p_i p_j = 0$ whenever $i \neq j$.

It is fundamental in the theory of von Neumann algebras that the dimension function is determined up to a positive multiple if the center of the algebra is trivial, that is, the algebra is a *factor*.

We sketch how the dimension function was obtained in {34} . A nonzero projection is called finite if it is not equivalent to a smaller projection. "Smaller" is understood here in the sense of the partial ordering (8). Murray and von Neumann proved in {34} that if f is a finite and e is an arbitrary projection in a factor then there exists a unique integer k such that

$$f = q_1 + q_2 + \ldots + q_k + p\,,$$

where $q_1,\ q_2, \ldots, q_k$ are pairwise orthogonal projections equivalent to e, p is a projection orthogonal to all q_i and equivalent to a subprojection of f. This integer k is denoted by

$$\left[\frac{f}{e}\right] \tag{10}$$

and this is the number of projections equivalent to e which may be packed into f in a pairwise orthogonal way. (10) is an integer and is only an approximate measure of the ratio of the subspaces corresponding to f and e. Now we fix a finite non-zero projection e_0 and a sequence e_n of non-zero finite projections converging to 0. The limit

$$\lim_{n\to\infty} \frac{\left[\frac{f}{e_n}\right]}{\left[\frac{e_0}{e_n}\right]} = \left(\frac{f}{e_0}\right) \tag{11}$$

forms a quantitative ratio of relative dimensionality, when the sequence e_n converges to 0 strongly. (Heuristically, the projection e_0 will have dimension 1, first we estimate the dimension e_n by comparison with e_0 and then the dimension of f is estimated by comparison with e_n.)

The relative dimension was defined in {34} as

$$D(e) = \begin{cases} 0 & \text{if } e = 0, \\ \left(\dfrac{e}{e_0}\right) & \text{if } e \text{ is finite,} \\ +\infty & \text{if } e \text{ is not finite.} \end{cases}$$

The use of the relative dimension in the classification of factors will be discussed below. Now we make a detour and compare the construction of the dimension function with that of the Haar measure on a locally compact topological group. The existence of a measure on an abstract locally compact group which is invariant under right translations was proven in 1932 by the Hungarian mathematician **Alfréd Haar** {17}. Von Neumann was in contact with Haar and knew his celebrated result before it was published. It is instructive to trace back the dimension function of a ring of operators to Haar's beautiful idea for the construction of the invariant measure.

Let G be a locally compact topological group and for a relatively compact $B \subset G$ and an open $U \subset G$ denote by $h(B;U)$ the number which gives at least how many right-translates of the set U are needed to cover the set B. This is an integer showing the size of B compared to U. $h(B;U)$ is translation invariant by construction. Of course, the smaller the U, the larger the $h(B;U)$. The latter one may increase to infinity when U runs over the neighbourhoods of a point. We need a normalization of $h(B;U)$. A compact set S of nonempty interior is chosen to normalize the measure. (S will be a set of unit measure.)

$$\lim_n \frac{h(B;R_n)}{h(S;R_n)} = \mu(B) \tag{12}$$

gives the measure of a compact set B if (R_n) is the filter of neighbourhoods of a point. The set function μ is right-translation invariant and additive on disjoint compact sets. After the measure μ of compact sets is obtained, measure-theoretic arguments are used to extend μ to a larger class of sets.

It is difficult to refrain from comparing Haar's idea with the construction of dimension function of projections in a von Neumann algebra: the similarity between the formulas (12) and (11) is striking. (12) yields the right-translation invariant size of subsets of a group G and (11) defines an invariant under partial isometries for projections in a von Neumann algebra. This example demonstrates how measure-theoretic arguments can survive in the apparently different discipline of operator algebras. Von Neumann devoted two papers to Haar measure. In {39}, he gave another proof for the existence and uniqueness in the compact case and in {40} he obtained uniqueness in the general locally compact case. Von Neumann presented several courses on measure theory and invariant measures at the *Institute for Advanced Study*. For him operator algebra theory was a *noncommutative outgrowth* of measure theory.

Now we continue the comparison of the relative dimension and Haar measure. The objective of integration theory is to construct a linear functional, called integral, from a certain measure. Murray and von Neumann extended the relative dimension functional to arbitrary selfadjoint elements of the given von Neumann algebra $\mathcal{M}$. Let $A = A^* \in \mathcal{M}$

and let $\int \lambda\, dE(\lambda)$ be its spectral resolution with a projection-valued measure E on the real line. Then by property (c) of the relative dimension, $D(E)$ is an ordinary measure and

$$\mathrm{Tr}_{\mathcal{M}}(A) = \int \lambda\, dD(E)(\lambda) \tag{13}$$

determines a real number when the integral on the right-hand side exists. The inconveniency of definition (13) is in the fact that for noncommuting self-adjoint operators A and B one cannot say much about the spectral resolution of $A+B$ in terms of the spectral resolutions of A and B. Murray and von Neumann expected that

$$\mathrm{Tr}_{\mathcal{M}}(A+B) = \mathrm{Tr}_{\mathcal{M}}(A) + \mathrm{Tr}_{\mathcal{M}}(B)$$

but this was proven in {34} only for commuting A and B. The general case was postponed to the subsequent paper {35}. It was established there that the abstract trace functional $\mathrm{Tr}_{\mathcal{M}}$ is linear. $\mathrm{Tr}_{\mathcal{M}}$ yields an analogue of an integral. (This analogy has developed into an operator-algebraic integration theory, including L^p spaces, measurable operators and so on. For this Segal proposed the term *"noncommutative integration"* in {62} since a commutative von Neumann algebra admits representations by functions.)

In {37} von Neumann established the structure of commutative von Neumann algebras: The selfadjoint part of a commutative von Neumann algebra consists of all bounded measurable functions of a certain selfadjoint operator. The classification of nonabelian algebras was carried out in {34}. Murray and von Neumann recognized that the center of the algebra plays an important role in the structure problem. The center of a von Neumann algebra $\mathcal{M}$ is a von Neumann algebra again and if it contains a projection z, then $\mathcal{M}$ becomes the direct sum of $z\mathcal{M}$ and $(I-z)\mathcal{M}$. Hence to decrease the complexity of an algebra, one may assume that its center does not contain a nontrivial projection. A von Neumann algebra is called a *factor* if its center is trivial, that is, if it contains the multiples of the identity operator only. On a von Neumann factor, the dimension function is unique up to a scalar multiple. Murray and von Neumann proved that there are the following possibilities for the range of the dimension function of projections:

(I_n) $\{0, 1, \dots, n\}$, where n is a natural number.

(I_∞) $\{0, 1, \dots, n, \dots, \infty\}$.

(II_1) The interval $[0, 1]$.

(II_∞) The interval $[0, +\infty]$.

(III) The two-element set $\{0, +\infty\}$.

In this classification all von Neumann factors were found to belong to the classes type I, type II or type III. (However, it is worth mentioning that at the time of the discovery of the classification it was not known whether type III factors exist.)

Factors are the building blocks of von Neumann algebras, hence the understanding of their structure has primary interest. According to the range of the dimension function of projections, a factor might be "trivial", "regular" or "singular". The trivial or type

I is characterized by integer dimension, in the regular or type II case the dimension function has a continuous range and the singular or type III case is free of finite nonzero projections. To investigate the type I and type II cases Murray and von Neumann could utilize the dimension function; however, that tool was insufficient for type III factors. To have a feeling about the "singularity" of type III factors, one can think of a measure space in which all nonempty measurable sets have infinite measure. The full understanding of the type III case needed half a century. Ergodic theory was the first source of factors. Classification of von Neumann algebras is strongly related to conjugacy classes of transformations of measure spaces. The *Tomita-Takesaki theory* provided the new tools and revolutionized operator algebras in the 1970's. (The book {63} by Serban Strătilă and **László Zsidó** is a suggested introductory reading about von Neumann algebras.)

Factors of type I are characterized by the existence of minimal projections. If a maximal pairwise orthogonal family of minimal projections has cardinality n, then the factor is isomorphic to $B(\mathcal{H})$, where $\mathcal{H}$ is a Hilbert space of dimension n. In particular, for every $s \in \mathbb{N} \cup \{+\infty\}$, there exists only one factor of type I_s up to isomorphism. The existence of factors of type II and type III is not at all apparent, however. Murray and von Neumann constructed factors of type II_1 and type II_∞ by means of ergodic theory in {34}. Below we describe a method called *"group measure space construction"*. This construction yields factors of different type.

Let $(X, \mathcal{B}, \mu)$ be a measure space and let G be a countable group of measure-preserving transformations of X. The group measure space construction yields a von Neumann algebra acting on the Hilbert space $L^2(\mu) \otimes l^2(G)$, which is regarded as a set of functions ξ defined on G and with values in $L^2(\mu)$. (In this identification $\delta_g \otimes f$ corresponds to $\delta_g \times f$ for $g \in G$ and $f \in L^2(\mu)$.) For every $f \in L^\infty(\mu)$ define a bounded operator M_f acting on $L^2(\mu) \otimes l^2(G)$ as

$$((M_f\xi)(g))(x) = f(g^{-1}x)(\xi(g)(x)) \qquad (\xi \in L^2(\mu) \otimes l^2(G),\ g \in G\,) \tag{14}$$

and for every $g \in G$ we define a unitary V_g by the formula

$$V_g(\xi)(g')(x)) = \xi(g^{-1}g')(x) \qquad (\xi \in L^2(\mu) \otimes l^2(G),\, g' \in G\,). \tag{15}$$

Let $\mathcal{M}(\mu, G)$ be the von Neumann algebra generated by the operators

$$\{M_f : f \in L^\infty(\mu)\} \cup \{V_g : g \in G\}\,.$$

Then the choice of the unit circle with Lebesgue measure and (the powers of) an irrational rotation yields a factor of type II_1. The real line with Lebesgue measure and the rational translations give a factor of type II_∞. A factor of type III was constructed only in the third paper of the "Rings of Operators" series {42}. Von Neumann modified the above measure theoretic procedure by allowing measurable transformations preserving measure 0, nowadays they are called nonsingular transformations. In this way he produced a factor of type III from the Lebesgue measure of the real line and the group of all rational linear transformations. (Although Murray and von Neumann used the group measure space construction for the production of factors, which are called *Krieger* *factors* nowadays, the difficult question of isomorphism of factors that arose from different actions was clarified only 40 years later {28}. Krieger proved that two ergodic nonsingular transformations of a Lebesgue space give rise to isomorphic factors if and only if the two transformations are orbit equivalent.)

Von Neumann believed that among all factors the case II_1 has the strongest interest and expected that not all factors of type II_1 are isomorphic to each other. Von Neumann preferred the type II_1 case for two main reasons. One of these is the nice behavior of the unbounded operators affiliated with a type II_1 factor. It is well-known that addition and multiplication of such operators are particularly troublesome. The crux of the difficulty lies in the unrelatedness of the domain and range of such an operator with the domain of another one. Much of the difficulties evaporates, however, if one considers selfadjoint operators with spectral resolution in a factor of type II_1. The other reason why von Neumann attributed great importance to continuous finite factors is that he interpreted this lattice as the proper logic of a quantum system. The lattice of projections of such a factor is modular, that is, in addition to the orthomodularity property (9), the stronger condition

$$p \vee (p' \wedge q) = (p \vee p') \wedge q \quad \text{for} \quad p \le q$$

holds for every p' (and not only $p' = p^{\perp}$). (Non-modularity of the projection lattice of an infinite dimensional factor of type I was considered by von Neumann as a pathology of the usual Hilbert space quantum mechanics as a non commutative probability theory.)

The paper “Rings of Operators IV” {36} has two important achievements concerning type II_1 factors. It is proved that there exist nonisomorphic type II_1 factors, and that there is only one hyperfinite type II_1 factor. A von Neumann factor is called *hyperfinite* if it is generated by an increasing sequence of finite dimensional subalgebras. (Nowadays such algebras are preferably called approximately finite dimensional, or AFD for short.) The hyperfinite type II_1 factor $\mathcal{R}$ may be produced in many different ways; for example, the above group measure space construction yields $\mathcal{R}$. The uniqueness of $\mathcal{R}$ reminds us of the uniqueness of a finite, atomless separable measure space. Factors of type II_1 did not play much role in the theory of von Neumann algebras until recent years. After *Jones* founded his index theory {19}, the study of subfactors of type II_1 factors has received much interest. Even a concise review of index would require a lot of space (cf. {23}) but its flavour is given below.

Let $\mathcal{N}$ be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$ and having commutant $\mathcal{N}'$. Assume that both $\mathcal{N}$ and $\mathcal{N}'$ are type II_1 factors and let $\mathrm{Tr}_{\mathcal{N}}$ and $\mathrm{Tr}_{\mathcal{N}'}$ be the canonical normalized traces. For any vector $\xi \in \mathcal{H}$ the projection $[\mathcal{N}\xi]$ onto the closure of $\mathcal{N}\xi$ belongs to $\mathcal{N}'$ and similarly $[\mathcal{N}'\xi] \in \mathcal{N}$. The quotient

$$\dim_{\mathcal{N}}(\mathcal{H}) \equiv \frac{\mathrm{Tr}_{\mathcal{N}'}([\mathcal{N}\xi])}{\mathrm{Tr}_{\mathcal{N}}([\mathcal{N}'\xi])} \tag{16}$$

is known to be independent of the vector ξ and is called the coupling constant since the work of Murray and von Neumann. In a certain sense the coupling constant is the dimension of the Hilbert space $\mathcal{H}$ with respect to the von Neumann algebra $\mathcal{N}$. (When $\mathcal{N} \equiv \mathbb{C}I$, the coupling constant is the usual dimension of $\mathcal{H}$, hence the notation $\dim_{\mathcal{N}}(\mathcal{H})$.) V. Jones used the coupling constant to define the size of a subfactor of a finite factor. He was inspired by the notion of the index of a subgroup of a group, he therefore called this the relative size index.

Let $\mathcal{N}$ be a subfactor of a type II_1 von Neumann factor $\mathcal{M}$ possessing a unique canonical normalized trace $\mathrm{Tr}_{\mathcal{M}}$. The index is obtained as the quotient

$$[\mathcal{M}:\mathcal{N}] = \frac{\dim_{\mathcal{N}}(\mathcal{H})}{\dim_{\mathcal{M}}(\mathcal{H})}\,. \tag{17}$$

The number $[\mathcal{M}:\mathcal{N}]$ is not always an integer, and the possible values of the index form the following set:

$$\{t \in \mathbb{R} : t \geq 4\} \cup \{4\cos^2(\pi/p) : p \in \mathbb{N},\, p \geq 3\}. \tag{18}$$

This is the fundamental result of Jones which influenced a huge amount of subsequent research and renewed the almost forgotten coupling constant. *Vaughan F.R. Jones* was awarded the Fields Medal in 1992 for discovering a surprising relationship between von Neumann algebras and geometric topology (see {4} for a review). The index theorem was the first step towards his discovery.

Construction of factors was the main activity in the field of operator algebras after the papers "Rings of operators" for many years. It is out of the scope of this survey to summarize the constructions that were used to get more and more factors. Instead, we turn to the very end of the story. By the time the paper "Rings of Operators IV" was published (year 1943) it was known that the classes of type I_n, II_1 contain a unique (up to algebraic isomorphism) hyperfinite von Neumann factor. However, the types II_∞ and III remained unclear for many years. In 1956 **Lajos Pukánszky** constructed two different factors of type III {46, 24}. After his breakthrough infinitely many factors were constructed but the final classification was not achieved until the discovery of new invariants. Operator algebras achieved a revolutionary development in the late 60's after a relative isolation of 30 years.

Type III factors may be produced by means of infinite tensor product. Let $M_2(\mathbb{C})$ be the algebra of 2-by-2 matrices. Fixing $0 < \lambda < 1$ we can define a state φ on this algebra as follows.

$$\varphi(A) = \mathrm{Tr}(AD), \quad \text{where} \quad D = \begin{pmatrix} \frac{1}{\lambda+1} & 0 \\ 0 & \frac{\lambda}{\lambda+1} \end{pmatrix}.$$

(The matrix D is called the density matrix inducing φ.) A representation of the inductive limit of the n-fold tensor product of copies of $M_2(\mathbb{C})$ can be constructed by means of tensor product states of copies of φ. (The so-called *Gelfand-Naimark-Segal* construction is involved here, but we do not give more details.) The generated von Neumann algebra is a hyperfinite factor. For $\lambda = 1$, the type II_1 factor shows up, for $\lambda = 0$ we obtain a type I_∞ factor and for $0 < \lambda < 1$ a type III_λ factor $\mathcal{R}_\lambda$ appears. In fact, $\mathcal{R}_\lambda$ is the only hyperfinite type III_λ factor. Confined to hyperfinite type III_λ factors with $0 < \lambda < 1$ the Connes spectrum is a complete invariant due to the results of *Alain Connes*. He received the Fields Medal in 1983 for his work on von Neumann algebras including the classification of type III factors, approximately finite dimensional factors and automorphisms of the hyperfinite type II_1 factor {3}. After the work of Connes, the uniqueness of the hyperfinite type III_1 factor remained undecided. This question was answered positively somewhat later by *Uffe Haagerup* {16}. (In the case of type III_0, there are infinitely many nonisomorphic hyperfinite factors.)

Quantum mechanics influenced von Neumann to develop several ideas. He was the first person who summarized quantum theory in a comprehensive and mathematical form, his monograph [119] has been a standard reference in mathematical physics. Operator algebras consist of bounded operators but quantum mechanics needs unbounded ones. Von Neumann understood the importance of maximal symmetric operators on Hilbert spaces and introduced the entropy of statistical operators ([119] and {45}). The von Neumann entropy got a new information theoretic interpretation recently.

Béla Szőkefalvi-Nagy was born in Kolozsvár in Transylvania on July 29, 1913. His father was also a mathematician and his mother was a high school teacher. In his scientific papers he did not use his full name but the abbreviation B. Sz.-Nagy. Native Hungarians have always been surprised about the strange pronunciation of his name by foreigners. After World War I, the family moved to Szeged (Hungary).

During his university studies Sz.-Nagy was deeply influenced by *Frigyes Riesz, Béla Kerékjártó and Alfréd Haar*. Von Neumann's book on the foundations of quantum mechanics [119] and van der Waerden's book on group theory and quantum mechanics were his favorite readings. This was the time when quantum theory revolutionized both physics and mathematics. Between 1937 and 1939 Sz.-Nagy spent some time in Leipzig, Grenoble and Paris. From 1939 he worked for the University of Szeged; he became full professor in 1948.

Sz.-Nagy had rather wide mathematical interests. In one of his first papers he gave a new proof for Stone's theorem about the spectral representation of a strongly continuous one-parameter semigroup of unitary Hilbert space operators. Such a semigroup $U(t)$ is obtained in the form

$$U(t) = \int_{-\infty}^{\infty} e^{i\lambda t}\, dE(\lambda)\,,$$

by means of a projection-valued measure E on the real line. Later he extended this result to semigroups of normal operators. He also wrote a very concise book on spectral theory. Generations learnt the spectral theorem from [177] published in 1942 by Springer Verlag.

Although Sz.-Nagy contributed to the theory of Fourier series and to approximation theory, the center of his interest was the Hilbert space and its linear operators. The basic example of a Hilbert space is the L^2 space, the space of square integrable functions over a measure space. On top of the standard Hilbertian structure L^2 has an order structure which is determined by the cone of positive functions. In an early paper Sz.-Nagy gave an abstract characterization of the positive cone. In other words, he listed the requirements of a cone of an abstract Hilbert space under which an isomorphism of the space with an L^2 space exists, such that the positive cones correspond to each other. He also proved that an invertible Hilbert space operator whose positive and negative powers are uniformly bounded is similar to a unitary operator.

The analysis of Hilbert space operators mostly concerns some particular classes of operators such as self-adjoint, unitary etc. The highlight of the scientific activity of Sz.-Nagy was the theory of contractions. It started with the unitary dilation theorem obtained in 1953. Let $\mathcal{H}$ be a Hilbert space and let T be a general bounded linear operator. Hence $\|T\|$ is finite and multiplying T by some constant we can achieve that $\|T\| \le 1$. (Such a T is called a contraction.) The dilation theorem says that there exist a Hilbert space $\mathcal{K} \supset \mathcal{H}$ and a unitary operator U on $\mathcal{K}$ such that

$$T^n f = PU^n f \quad \text{and} \quad (T^*)^n f = PU^{-n} f \qquad (f \in \mathcal{H})$$

for any $n \in \mathbb{N}$, where P denotes the orthogonal projection from $\mathcal{K}$ onto $\mathcal{H}$. The space $\mathcal{K}$ could be the direct sum of infinitely many copies of $\mathcal{H}$ and U can be written in the form of an infinite matrix (with operator entries) as follows:

$$U = \begin{bmatrix} & \cdot & \cdot & \cdot & \cdot & \\ & \cdot & \cdot & \cdot & \cdot & \\ & \cdot & \cdot & \cdot & \cdot & \\ \cdots & I & 0 & 0 & 0 & \cdots \\ \cdots & 0 & D_* & T & 0 & \cdots \\ \cdots & 0 & -T^* & D & 0 & \cdots \\ \cdots & 0 & 0 & 0 & I & \cdots \\ & \cdot & \cdot & \cdot & \cdot & \\ & \cdot & \cdot & \cdot & \cdot & \\ & \cdot & \cdot & \cdot & \cdot & \end{bmatrix},$$

where $D = (I - T^*T)^{1/2}$ and $D_* = (I - TT^*)^{1/2}$. Since the structure of a unitary operator is rather well-understood, the contractions could be investigated through the dilation.

In the study of contractions Sz.-Nagy had a longstanding cooperation with *Ciprian Foiaş* from 1956 to the end of his life. They wrote together 50 papers and the monograph [44]. An interesting class of operators is formed by the completely non-unitary contractions, they do not act unitarily on any subspace. The class C_0 is formed by the completely non-unitary contractions T for which there exists a function $0 \ne w \in H^\infty$ such that $w(T) = 0$. An operator $T \in C_0$ has the following remarkable properties:

(1) For every vector f, $T^n f \to 0$ and $(T^*)^n \to 0$ as $n \to \infty$.

(2) T has a nontrivial invariant subspace.

Recall that a linear operator T of a finite dimensional space always admits a polynomial p such that $p(T) = 0$. The definition and several properties of the class C_0 resemble the finite dimensional scenario. Sz.-Nagy and Foiaş found a quasisimilarity model for the C_0-contractions and a unitary equivalence model for arbitrary completely non-unitary contractions. Their lifting theorem is connected with the minimal isometric dilation of a contraction T. Let T_i be a contraction acting on a Hilbert space $\mathcal{H}_i$ and let V_i be the minimal isometric dilation of T_i acting on the space $\mathcal{K}_i$, $i = 1, 2$. If a bounded linear operator X from the space $\mathcal{H}_1$ to $\mathcal{H}_2$ has the property $T_2 X = X T_1$, then there exist an operator Y from the space $\mathcal{K}_1$ to $\mathcal{K}_2$ such that $V_2 Y = Y V_1$. The lifting theorem of Sz.-Nagy and Foiaş extends earlier results of *T. Ando* and *D. Sarason.* Many applications are known, in particular to interpolation problems.

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