Weingarten calculus
and
counting paths on Weingarten graphs

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Random matrices and their applications, Kyoto University
Weingarten calculus

It is a method for computations of mixed moments

\[ E[x_{i_1j_1} x_{i_2j_2} \cdots x_{i_nj_n}] \quad \text{or} \quad E[x_{i_1j_1} x_{i_2j_2} \cdots x_{i_nj_n} x_{k_1l_1} x_{k_2l_2} \cdots x_{k_ml_m}] \]

where \( X = (x_{ij}) \) is a random matrix picked up from a classical compact Lie group.

History:
Don Weingarten (1978), Benoît Collins (2003), B.C. & Piotr Śniady (2006), ...

Today’s topics:
- Weingarten calculus on Lie groups \( U(d), O(d), \text{Sp}(d) \);
- Weingarten calculus on symmetric spaces \( G/K \) (COE, chiral unitary matrix);
- Weingarten graphs (joint work with Benoît Collins).
1 Weingarten Calculus for Unitary Groups

2 Weingarten Calculus for Orthogonal Groups

3 Weingarten Calculus for Symplectic Groups

4 Weingarten Calculus for Symmetric Spaces
   - A I – circular orthogonal ensemble (COE)
   - A III – chiral unitary ensemble (chUE)

5 Weingarten Graphs (joint work with Benoît Collins)
   - Unitary group $U(d)$
   - A III – chiral unitary ensemble (chUE)
Weingarten calculus for unitary groups

\[ G = U(d) = \{ g \in \text{GL}(d, \mathbb{C}) \mid g g^* = I_d \}. \quad (\text{CUE} = \text{circular unitary ensemble}) \]

Any compact Lie group \( G \) has the \textbf{normalized Haar measure} \( \mu = \mu_G \) such that

\[
\int_G f(g_1 g_2) \mu(dg) = \int_G f(g) \mu(dg), \quad \int_G \mu(dg) = 1,
\]

where \( f \) is any continuous function on \( G \), and \( g_1, g_2 \) are any elements in \( G \).
**Weingarten calculus for unitary groups**

\[ G = U(d) = \{ g \in \text{GL}(d, \mathbb{C}) \mid g g^* = I_d \}. \quad (\text{CUE} = \text{circular unitary ensemble}) \]

Any compact Lie group \( G \) has the **normalized Haar measure** \( \mu = \mu_G \) such that

\[
\int_G f(g_1 g g_2) \mu(\text{d}g) = \int_G f(g) \mu(\text{d}g), \quad \int_G \mu(\text{d}g) = 1,
\]

where \( f \) is any continuous function on \( G \), and \( g_1, g_2 \) are any elements in \( G \).

Let \( U = (u_{ij})_{1 \leq i, j \leq d} \) be a random matrix distributed with respect to \( \mu_U(d) \).

Consider

\[
\mathbb{E}[u_{i_1j_1} u_{i_2j_2} \cdots u_{i_nj_n} \overline{u_{i_1'j_1'}} u_{i_2'j_2'} \cdots u_{i_m'j_m'}]
\]

where \( i_p, j_p, i'_p, j'_p \) are entries in \( \{1, 2, \ldots, d\} \). Here \( \mathbb{E} \) stands for the expectation (with respect to \( \mu_U(d) \)). For example, we will compute \( \mathbb{E}[u_{11} u_{22} u_{33} \overline{u_{12}} u_{23} u_{31}] \).

**Fact**

The expectation \( \mathbb{E}[\cdots] \) vanishes unless \( n = m \).
Theorem (Collins, 2003)

Given four sequences

\[ i = (i_1, \ldots, i_n), \quad j = (j_1, \ldots, j_n), \quad i' = (i'_1, \ldots, i'_n), \quad j' = (j'_1, \ldots, j'_n) \]

in \( \{1, 2, \ldots, d\}^n \), we have

\[
\mathbb{E} \left[ u_{i_1j_1} u_{i_2j_2} \cdots u_{i_nj_n} u_{i'_1j'_1} u_{i'_2j'_2} \cdots u_{i'_nj'_n} \right] = \sum_{\sigma \in \mathcal{S}_n} \sum_{\tau \in \mathcal{S}_n} \delta_{\sigma}(i, i') \delta_{\tau}(j, j') W^U_{\mathcal{G}}(\sigma^{-1} \tau, d).
\]

Here \( \mathcal{S}_n \) is the symmetric group on \( \{1, 2, \ldots, n\} \) and

\[
\delta_{\sigma}(i, i') = \begin{cases} 
1 & \text{if } (i_{\sigma(1)}, i_{\sigma(2)}, \ldots, i_{\sigma(n)}) = (i'_1, i'_2, \ldots, i'_n), \\
0 & \text{otherwise}.
\end{cases}
\]

The function \( W^U_{\mathcal{G}}(\cdot, d) \) on \( \mathcal{S}_n \) is given in the next slide.
Unitary Weingarten function

Fourier expansion of $W_g^U$

$$W_g^U(\sigma, d) = \frac{1}{n!} \sum_{\lambda \vdash n, \ell(\lambda) \leq d} \frac{f^\lambda}{\prod_{i=1}^{\ell(\lambda)} \prod_{j=1}^{\lambda_i} (d + j - i)} \chi^\lambda(\sigma) \quad (\sigma \in S_n).$$

- $\lambda \vdash n$: The sum runs over all partitions $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ of $n$ with length $l = \ell(\lambda)$.

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l > 0, \quad \lambda_i \in \mathbb{Z}_{>0}$$

We identify $\lambda$ with its Young diagram. Example: $(4, 2, 1) = \square \square \square \square \square \square$

- $\chi^\lambda$: the (unnormalized) irreducible character of $S_n$ associated with $\lambda$.

- $f^\lambda$: the degree of $\chi^\lambda$ i.e. $f^\lambda = \chi^\lambda(id_n) \in \mathbb{Z}_{>0}$.

- The product in the denominator runs over all boxes of the Young diagram $\lambda$. The quantity $j - i$ is called the content of the box $(i, j)$. 
Example: unitary Weingarten functions

\[ Wg^U(\sigma, d) = \frac{1}{n!} \sum_{\ell(\lambda) \leq d} \frac{f^\lambda}{\prod_{i=1}^{\ell(\lambda)} \prod_{j=1}^{\lambda_i} (d + j - i)} \chi^\lambda(\sigma) \quad (\sigma \in \mathfrak{S}_n). \]

Example

Consider \( n = 3 \) and \( \sigma = [3, 1, 2] = (1 2 3) = (1 3 2) \). Suppose \( d \geq 3 \).

\[
Wg^U([3, 1, 2], d) = \frac{1}{3!} \left( \frac{1}{d(d + 1)(d + 2)} \cdot 1 + \frac{2}{d(d + 1)(d - 1)} \cdot (-1) + \frac{1}{d(d - 1)(d - 2)} \cdot 1 \right)
= \frac{2}{d(d^2 - 1)(d^2 - 4)}.
\]

Here we use one-row notation for a permutation. We also use cycle expressions.
Example: Weingarten calculus for $U(d)$

Let $U = (u_{ij})$ be a Haar-distributed unitary matrix from $U(d)$. Then

$$
\mathbb{E}[u_{12} u_{23} u_{31} u_{11} u_{22} u_{33}] = \frac{2}{d(d^2 - 1)(d^2 - 4)}.
$$

Input $n = 3, \ i = i' = (1, 2, 3), \ j = (2, 3, 1), \ j' = (1, 2, 3)$.

$$
\mathbb{E}[u_{12} u_{23} u_{31} u_{11} u_{22} u_{33}] = \sum_{\sigma \in S_3} \sum_{\tau \in S_3} \delta_\sigma(i, i') \delta_\tau(j, j') \text{Wg}^U(\sigma^{-1}\tau, d)
$$

Recall

$$
\delta_\sigma(i, i') = \begin{cases} 
1 & \text{if } (i_{\sigma(1)}, i_{\sigma(2)}, \ldots, i_{\sigma(n)}) = (i'_1, i'_2, \ldots, i'_n), \\
0 & \text{otherwise}.
\end{cases}
$$

Only term for $\sigma = \text{id}_3$ and $\tau = [3, 1, 2]$ contributes (i.e. $\delta_\sigma(i, i')\delta_\tau(j, j') = 1$).

$$
= \text{Wg}^U([3, 1, 2], d) = \frac{2}{d(d^2 - 1)(d^2 - 4)}.
$$
An important invariance for $Wg^U(\sigma, d)$

The function $\mathcal{S}_n \ni \sigma \mapsto Wg^U(\sigma, d) \in \mathbb{Q}$ is central (another name is class function). Namely,

$$Wg^U(\tau^{-1}\sigma\tau, d) = Wg^U(\sigma, d) \quad (\forall \sigma, \forall \tau \in \mathcal{S}_n)$$

Equivalently,

- It is constant on each conjugacy class of $\mathcal{S}_n$.
- It depends on only the cycle-type of $\sigma$ (→ a partition of $n$).

We will see that Weingarten functions for other Lie groups $O(d)$ and $Sp(d)$ have different invariances.
1. Weingarten Calculus for Unitary Groups

2. Weingarten Calculus for Orthogonal Groups

3. Weingarten Calculus for Symplectic Groups

4. Weingarten Calculus for Symmetric Spaces
   - A I – circular orthogonal ensemble (COE)
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5. Weingarten Graphs (joint work with Benoît Collins)
   - Unitary group $U(d)$
   - A III – chiral unitary ensemble (chUE)
Preparations: Pair partitions

**Definition**
Denote by $\mathcal{M}_{2n}$ the set of all pair partitions on $\{1, 2, \ldots, 2n\}$.

**Example**
$\mathcal{M}_4$ consists of three elements

$$\{1, 2\}\{3, 4\}, \quad \{1, 3\}\{2, 4\}, \quad \{1, 4\}\{2, 3\}$$

Every element $p$ in $\mathcal{M}_{2n}$ is uniquely expressed as

$$\{p_1, p_2\}\{p_3, p_4\}\cdots\{p_{2n-1}, p_{2n}\}$$

$$p_{2j-1} < p_{2j} \quad (j = 1, \ldots, n), \quad 1 = p_1 < p_3 < \cdots < p_{2n-1}.$$ 

We then regard $p$ as a permutation in $\mathfrak{S}_{2n}$:

$$\mathcal{M}_{2n} \subset \mathfrak{S}_{2n}, \quad p = [p_1, p_2, \ldots, p_{2n-1}, p_{2n}].$$
Preparations: Hyperoctahedral groups

Definition
Denote by $\mathcal{B}_n \subset \mathfrak{S}_{2n}$ the hyper-octahedral group, which is generated by permutations

$$(2i - 1 \ 2i) \ (i = 1, 2, \ldots, n), \quad (2i - 1 \ 2j - 1)(2i \ 2j) \ (1 \leq i < j \leq n).$$

Example (in cycle notation)

$\mathcal{B}_2 = \{\text{id}_4, \quad (1 \ 2), \quad (3 \ 4), \quad (1 \ 2)(3 \ 4), \quad (1 \ 3)(2 \ 4), \quad (1 \ 4)(2 \ 3), \quad (1 \ 3 \ 2 \ 4), \quad (1 \ 4 \ 2 \ 3)\}$

The set $\mathcal{M}_{2n}$ forms representatives of left cosets of $\mathcal{B}_n$ in $\mathfrak{S}_{2n}$:

$$\mathfrak{S}_{2n} = \bigsqcup_{p \in \mathcal{M}_{2n}} p\mathcal{B}_n,$$

i.e.

$$\mathcal{M}_{2n} \cong \mathfrak{S}_{2n}/\mathcal{B}_n$$

(Recall that $p \in \mathcal{M}_{2n}$ is regarded as a permutation in $\mathfrak{S}_{2n}$.)
Weingarten calculus for $O(d)$

Real orthogonal group $O(d) = \{ g \in GL(d, \mathbb{R}) \mid gg^T = I_d \}$.

**Theorem** ((Collins–Śniady, 2006), (Collins-M, 2009))

Let $R = (r_{ij})_{1 \leq i, j \leq d}$ be a Haar-distributed orthogonal matrix. Given two sequences $i = (i_1, \ldots, i_{2n})$, $j = (j_1, \ldots, j_{2n})$, we have

$$
\mathbb{E}[r_{i_1j_1} r_{i_2j_2} \cdots r_{i_{2n}j_{2n}}] = \sum_{p \in \mathcal{M}_{2n}} \sum_{q \in \mathcal{M}_{2n}} \Delta_p(i) \Delta_q(j) W^O_{\mathbb{R}}(p^{-1}q, d).
$$

Here

$$
\Delta_p(i) = \prod_{\{a, b\} \in p} \delta_{i_a, i_b}.
$$

Moments of odd degree $\mathbb{E}[r_{i_1j_1} \cdots r_{i_{2n+1}j_{2n+1}}]$ always vanish.

Recall $\mathcal{M}_{2n} \subset \mathfrak{S}_{2n}$ (so $p^{-1}q$ does make sense as permutations).

The orthogonal Weingarten function $W^O_{\mathbb{R}}(\cdot, d)$ on $\mathfrak{S}_{2n}$ is described as follows.
Orthogonal Weingarten functions

In order to study $Wg^O(\cdot, d)$, we review (finite) Gelfand pairs.

**Definition**

Let $G$ be a finite group and $H$ its subgroup. Consider the Hecke algebra

\[ \mathcal{H}(G, H) = \{ f : G \to \mathbb{C} \mid f(\zeta_1 \sigma \zeta_2) = f(\sigma) \ (\forall \sigma \in G, \ \forall \zeta_1, \forall \zeta_2 \in H) \} \]

with convolution product $(f_1 * f_2)(\sigma) = \sum_{\tau \in G} f_1(\sigma \tau^{-1}) f_2(\tau)$. The pair $(G, H)$ is called a Gelfand pair if $\mathcal{H}(G, H)$ is commutative: $g * f = f * g$.

**Fact (well known)**

$(\mathcal{S}_{2n}, \mathcal{B}_n)$ is a Gelfand pair.

- The unitary Weingarten function $Wg^U(\cdot, d)$ belongs to the center $Z \mathbb{C}[\mathcal{S}_n] = \bigoplus_{\lambda \vdash n} \mathbb{C} \chi^\lambda$.
- The orthogonal Weingarten function $Wg^O(\cdot, d)$ belongs to the Hecke algebra

\[ \mathcal{H}_n := \mathcal{H}(\mathcal{S}_{2n}, \mathcal{B}_n) \]

\[ = \{ f : \mathcal{S}_{2n} \to \mathbb{C} \mid f(\zeta_1 \sigma \zeta_2) = f(\sigma) \ (\sigma \in \mathcal{S}_{2n}, \ \zeta_1, \zeta_2 \in \mathcal{B}_n) \}. \]
Orthogonal Weingarten functions

- **Zonal spherical functions** $\omega^\lambda (\lambda \vdash n)$ form a linear basis of $\mathcal{H}_n$.

$$\omega^\lambda(\sigma) = \frac{1}{2^n n!} \sum_{\zeta \in \mathcal{B}_n} \chi^{2\lambda}(\sigma \zeta) \quad (\sigma \in \mathfrak{S}_{2n}),$$

where $2\lambda = (2\lambda_1, 2\lambda_2, \ldots)$. $\mathcal{H}_n = \bigoplus_{\lambda \vdash n} \mathbb{C} \omega^\lambda$.

- They are constant on each double cosets $\mathcal{B}_n \sigma \mathcal{B}_n$.

**Theorem (Collins-M, 2009)**

$$Wg^O(\sigma, d) = \frac{2^n n!}{(2n)!} \sum_{\lambda \vdash n} \frac{f^{2\lambda}}{\prod_{\ell=1}^{\ell(\lambda)} \prod_{j=1}^{\lambda_j} (d + 2j - i - 1)} \omega^\lambda(\sigma) \quad (\sigma \in \mathfrak{S}_{2n}).$$

$$Wg^U(\sigma, d) = \frac{1}{n!} \sum_{\lambda \vdash n} \frac{f^\lambda}{\prod_{\ell=1}^{\ell(\lambda)} \prod_{j=1}^{\lambda_j} (d + j - i)} \chi^\lambda(\sigma) \quad (\sigma \in \mathfrak{S}_n).$$
Example. Weingarten calculus for $O(d)$

Example

Let $R = (r_{ij})_{1 \leq i,j \leq d}$ be a Haar-distributed orthogonal matrix in $O(d)$. Let us compute

$$\mathbb{E}[r_{11}r_{12}r_{21}r_{22}r_{32}r_{32}].$$

Input $n = 3$, $i = (1, 1, 2, 2, 3, 3)$. $j = (1, 2, 1, 2, 2, 2)$.

Contributions: $p_1 = \{1, 2\}\{3, 4\}\{5, 6\}$ and

$q_1 = \{1, 3\}\{2, 4\}\{5, 6\}$, $q_2 = \{1, 3\}\{2, 5\}\{4, 6\}$, $q_3 = \{1, 3\}\{2, 6\}\{4, 5\}$

$$\mathbb{E}[r_{11}r_{12}r_{21}r_{22}r_{32}r_{32}] = \sum_{p \in \mathcal{M}_6} \sum_{q \in \mathcal{M}_6} \Delta_p(i) \Delta_q(j) Wg^O(p^{-1}q, d)$$

$$= Wg^O(p_1^{-1}q_1, d) + Wg^O(p_1^{-1}q_2, d) + Wg^O(p_1^{-1}q_3, d)$$

$$= \frac{-1}{d(d+4)(d-1)(d-2)} + \frac{2}{d(d+2)(d+4)(d-1)(d-2)} \times 2$$

$$= -\frac{1}{d(d+2)(d+4)(d-1)}.$$
Observation

\( i = (1, 1, 2, 2, 3, 3), \quad j = (1, 2, 1, 2, 2, 2). \)

If \( \Delta_p(i) = \prod_{\{a,b\} \in p} \delta_{i_a, i_b} = 1 \) and \( \Delta_q(j) = \prod_{\{a,b\} \in q} \delta_{j_a, j_b} = 1 \) then we can choose

\[ p_1 = \{1, 2\}\{3, 4\}\{5, 6\}, \quad q_1 = \{1, 3\}\{2, 4\}\{5, 6\}, \quad q_2 = \{1, 3\}\{2, 5\}\{4, 6\}, \quad q_3 = \{1, 3\}\{2, 6\}\{4, 5\} \]
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5 Weingarten Graphs (joint work with Benoît Collins)
   • Unitary group $U(d)$
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Consider a skew-symmetric bi-linear form on $\mathbb{C}^{2d}$ given by

$$\langle v, w \rangle_J = v^T J w,$$

$$J = J_d = \left( \begin{array}{cc} O_d & I_d \\ -I_d & O_d \end{array} \right)$$

**Definition ((unitary) symplectic group)**

$$\text{Sp}(d) = \{ g \in \text{U}(2d) \mid \langle g v, g w \rangle_J = \langle v, w \rangle_J \ (v, w \in \mathbb{C}^{2d}) \}.$$ 

Recall

$$O(d) = \{ g \in \text{GL}(d, \mathbb{R}) \mid \langle g v, g w \rangle = \langle v, w \rangle \ (v, w \in \mathbb{R}^d) \}$$

with the standard inner product $\langle v, w \rangle = v^T w.$
The symplectic Weingarten function $Wg_{Sp}(\cdot,d)$ on $\mathcal{S}_2n$ and $B_n$-twisted:

$$Wg_{Sp}(\zeta_1\sigma\zeta_2,d) = \text{sgn}(\zeta_1)\text{sgn}(\zeta_2)Wg_{Sp}(\sigma,d)$$

$(\sigma \in \mathcal{S}_2n, \zeta_1, \zeta_2 \in B_n)$.

It is described by using the theory of a twisted Gelfand pair.
Comparison of three Weingarten functions

<table>
<thead>
<tr>
<th>Unitary</th>
<th>Orthogonal</th>
<th>Symplectic</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{S}_n$</td>
<td>$\mathcal{M}_{2n}, (\mathcal{S}_2, \mathcal{B}_n)$</td>
<td>$\mathcal{M}_{2n}, (\mathcal{S}_2, \mathcal{B}_n, \text{sgn}</td>
</tr>
<tr>
<td>center $\mathcal{Z}(\mathbb{C}[\mathcal{S}_n])$</td>
<td>Hecke algebra $\mathcal{H}_n$</td>
<td>twisted Hecke algebra $\mathcal{H}_n^{\epsilon_n}$</td>
</tr>
<tr>
<td>irr. char. $\chi^\lambda$</td>
<td>zonal spherical $\omega^\lambda$</td>
<td>twisted spherical $\pi^\lambda$</td>
</tr>
<tr>
<td>central</td>
<td>$\mathcal{B}_n$-invariant</td>
<td>$\mathcal{B}_n$-twisted</td>
</tr>
</tbody>
</table>

$\mathcal{M}_{2n}$: pair partitions, $\mathcal{B}_n$: hyperoctahedral subgroup.

\[
Wg^U(\sigma, d) = \frac{1}{n!} \sum_{\lambda \vdash n} \frac{f^\lambda}{\prod_{i=1}^{\ell(\lambda)} \prod_{j=1}^{\lambda_i} (d+j-i)} \chi^\lambda(\sigma) \quad (\sigma \in \mathcal{S}_n).
\]

\[
Wg^O(\sigma, d) = \frac{2^n n!}{(2n)!} \sum_{\lambda \vdash n} \frac{f^{2\lambda}}{\prod_{i=1}^{\ell(\lambda)} \prod_{j=1}^{\lambda_i} (d+2j-i-1)} \omega^\lambda(\sigma) \quad (\sigma \in \mathcal{S}_2n).
\]

\[
Wg^{Sp}(\sigma, d) = \frac{2^n n!}{(2n)!} \sum_{\lambda \vdash n} \frac{f^{\lambda \cup \lambda}}{\prod_{i=1}^{\ell(\lambda)} \prod_{j=1}^{\lambda_i} (2d-2i+j+1)} \pi^\lambda(\sigma) \quad (\sigma \in \mathcal{S}_2n).
\]
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Let $U$ be a $d \times d$ Haar-distributed unitary matrix picked up from $U(d)$. Then we call

$$V = UU^T$$

a COE matrix. (Note: $U$ itself is also called a CUE matrix (circular unitary ensemble).)

An ensemble of such $V$ is well known as the circular orthogonal ensemble (COE). The random matrix $V$ is symmetric and unitary, and has invariance

$$U_0 V U_0^T \overset{\text{dist}}{=} V \quad \text{for any } d \times d \text{ unitary matrix } U_0.$$

The distribution of $V$ is invariant under the conjugacy action of $O(d)$. 

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**Sho Matsumoto (Kagoshima Univ)**

Weingarten calculus and Weingarten graphs

May 23, 2018 23 / 44
Let $U$ be a $d \times d$ Haar-distributed unitary matrix picked up from $U(d)$. Then we call

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The distribution of $V$ is invariant under the conjugacy action of $O(d)$.

**Aim**

1. We establish Weingarten calculus for a COE matrix.
2. We explain how the COE matrix arises from a framework of the compact symmetric space (CSS) $U(d)/O(d)$.
3. We consider random matrices associated with other CSS.
Weingarten calculus for COE

**Theorem (M, 2012)**

Let $V = (v_{ij})_{1 \leq i, j \leq d}$ be a COE matrix. For two sequences $i = (i_1, i_2, \ldots, i_{2n})$ and $j = (j_1, j_2, \ldots, j_{2n})$, we have

$$
\mathbb{E}[v_{i_1i_2} v_{i_3i_4} \cdots v_{i_{2n-1}i_{2n}} v_{j_1j_2} v_{j_3j_4} \cdots v_{j_{2n-1}j_{2n}}] = \sum_{\sigma \in \mathfrak{S}_{2n}} \delta_\sigma(i, j) Wg^{COE}(\sigma, d).
$$

The Weingarten function $Wg^{COE}(\sigma, d)$ coincides with the orthogonal Weingarten function with a parameter shift:

$$Wg^{COE}(\sigma, d) = Wg^{O}(\sigma, d + 1) \quad (\sigma \in \mathfrak{S}_{2n}).$$

Moments of the form $\mathbb{E}[v_{i_1i_2} v_{i_3i_4} \cdots v_{i_{2n-1}i_{2n}} v_{j_1j_2} v_{j_3j_4} \cdots v_{j_{2m-1}j_{2m}}]$ with $n \neq m$ always vanish.

Note:

- Different from Lie group cases, the formula includes a single summation.
Compact symmetric spaces

$G$: a compact linear Lie group (We deal with only either $U(d)$, $O(d)$, or $Sp(d)$).

$\Omega: G \to G$: an involutive automorphism (called a Cartan involution),

$K = \{ k \in G \mid \Omega(k) = k \}$.

\[
G/K \cong S := \{ g\,\Omega(g)^{-1} \mid g \in G \} \subset G.
\]

We take a Haar-distributed random matrix $Z$ from $G$, and then consider an $S$-valued random matrix

\[
V := Z\,\Omega(Z)^{-1}
\]

associated with the compact symmetric space $G/K$.

Example (COE)

$G = U(d), K = O(d), \Omega(g) = \overline{g}.$

\[
U(d)/O(d) \cong S = \{ gg^T \mid g \in U(d) \} = \{ d \times d \text{ symmetric unitary matrices} \}.
\]

The random matrix $V = U\Omega(U)^{-1} = UU^T$ is a COE matrix.
Classification for CSS

Classical CSS are classified by E. Cartan (1927) as follows.

<table>
<thead>
<tr>
<th>Class $C$</th>
<th>CSS</th>
<th>random matrix</th>
</tr>
</thead>
<tbody>
<tr>
<td>A I</td>
<td>$U(d)/O(d)$</td>
<td>circular orthogonal ensemble (COE)</td>
</tr>
<tr>
<td>A II</td>
<td>$U(2d)/Sp(d)$</td>
<td>circular symplectic ensemble (CSE)</td>
</tr>
<tr>
<td>A III</td>
<td>$U(d)/(U(a) \times U(b))$</td>
<td>chiral unitary ensemble (chUE)</td>
</tr>
<tr>
<td>BD I</td>
<td>$O(d)/(O(a) \times O(b))$</td>
<td>chiral orthogonal ensemble (chOE)</td>
</tr>
<tr>
<td>C II</td>
<td>$Sp(d)/(Sp(a) \times Sp(b))$ [d = a + b]</td>
<td>chiral symplectic ensemble (chSE)</td>
</tr>
<tr>
<td>D III</td>
<td>$O(2d)/U(d)$</td>
<td>Bogoliubov-de Gennes (BdG) ensemble</td>
</tr>
<tr>
<td>C I</td>
<td>$Sp(d)/U(d)$</td>
<td></td>
</tr>
</tbody>
</table>

For each CSS, we have a matrix ensemble.

**Theorem (M, 2013)**

*We have established Weingarten calculus for all of them, with an explicit Fourier expansion for each Weingarten function.*
A III case – chiral unitary ensembles (chUE)

\[ G = \text{U}(d), \ K = \text{U}(a) \times \text{U}(b), \ d = a + b. \]

\[ \Omega(g) = l'_{ab}g l'_{ab}, \quad l'_{ab} = \text{diag}(1, \ldots, 1, -1, \ldots, -1) = \begin{pmatrix} l_{a} & 0 \\ O & -l_{b} \end{pmatrix}. \]

For a Haar-distributed unitary matrix \( U \) from \( G = \text{U}(d) \), we consider a Hermitian and unitary random matrix

\[ X = X^{A \text{III}} = U l'_{ab} U^{*} \]

rather than \( V = U \Omega(U)^{-1} = U l'_{ab} U^{*} l'_{ab} \). The matrix \( X \) is called a chiral unitary matrix, or a random matrix of class A III.

Recall the Schur symmetric polynomial

\[ s_{\lambda}(x_{1}, \ldots, x_{d}) = \frac{\det(x_{i}^{d-i} + x_{j}^{d-i})_{1 \leq i, j \leq d}}{\det(x_{j}^{d-i})_{1 \leq i, j \leq d}} \]

for partitions \( \lambda \). This is a character for an irreducible representation of \( \text{U}(d) \).
Theorem (M, 2013)

Let \( X = (x_{ij})_{1 \leq i,j \leq d} \) be a chiral unitary matrix from \( U(a + b)/(U(a) \times U(b)) \). For two sequences \( i = (i_1, i_2, \ldots, i_n) \) and \( j = (j_1, j_2, \ldots, j_n) \), we have

\[
E[x_{i_1j_1} x_{i_2j_2} \cdots x_{i_nj_n}] = \sum_{\sigma \in \mathfrak{S}_n} \delta_{\sigma}(i, j) Wg_{A \text{ III}}^A(\sigma, a, b).
\]

The Weingarten function \( Wg_{A \text{ III}}^A(\sigma, a, b) \) \( (\sigma \in \mathfrak{S}_n) \) has the Fourier expansion

\[
Wg_{A \text{ III}}^A(\sigma, a, b) = \frac{1}{n!} \sum_{\lambda \vdash n} f^{\lambda} \frac{s_{\lambda}(1, \ldots, 1, -1, \ldots, -1)}{s_{\lambda}(1, \ldots, 1)} \chi^{\lambda}(\sigma) \quad (\sigma \in \mathfrak{S}_n).
\]

Different from Weingarten function appeared so far, this Weingarten function \( Wg_{A \text{ III}}^A \) has two parameters \( a, b \).
1. Weingarten Calculus for Unitary Groups

2. Weingarten Calculus for Orthogonal Groups

3. Weingarten Calculus for Symplectic Groups

4. Weingarten Calculus for Symmetric Spaces
   - A I – circular orthogonal ensemble (COE)
   - A III – chiral unitary ensemble (chUE)

5. Weingarten Graphs (joint work with Benoît Collins)
   - Unitary group $U(d)$
   - A III – chiral unitary ensemble (chUE)
Our goal is to reformulate various Weingarten functions via a Weingarten graph.

**Definition (Weingarten graph for unitary groups)**

We define an infinite directed graph $G^U = (V, E^{\text{red}} \sqcup E^{\text{blue}})$ as follows.

- **Vertex set $V$**. $V = \bigsqcup_{n=0}^{\infty} \mathcal{S}_n$ with $\mathcal{S}_0 = \{\emptyset\}$. We call the vertex $\emptyset$ the root.
- **Red edges**. (keep level)

  $$\mathcal{S}_n \ni \sigma \iff \exists \tau \in \mathcal{S}_n : \exists \text{ tranposition } (i \ n) \text{ such that } \tau = (i \ n)\sigma.$$

- **Blue edges**. (lower level)

  $$\mathcal{S}_n \ni \sigma \longrightarrow \sigma' \in \mathcal{S}_{n-1}$$

  if

  - the letter $n$ is fixed by $\sigma$;
  - $\sigma' \in \mathcal{S}_{n-1}$ is obtained from $\sigma$ by removing the trivial cycle $(n)$. 

Joint work with Benoît Collins (2017)
Weingarten graph for $U(d)$

A part of Weingarten graph.

- $[3, 2, 1] \leftrightarrow [2, 3, 1]$: they are switched by the transposition $(2 \ 3)$.

Sho Matsumoto (Kagoshima Univ)
Asymptotics for $U(d)$

**Theorem (Collins-M, 2017)**

Let $\sigma \in S_n$ and let $p(\sigma, k)$ be the number of paths from $\sigma$ to $\emptyset$ going through exactly $k$ red edges on the Weingarten graph $G^U$. Assume $d \geq n$. Then

$$Wg^U(\sigma, d) = (-1)^{|\sigma|} \sum_{j \geq 0} p(\sigma, |\sigma| + 2j) d^{-(n+|\sigma|+2j)}.$$

Here $|\sigma| = n - \ell(\mu)$, where $\ell(\mu)$ is the length of the cycle-type $\mu$ of $\sigma$.

**Example**

Consider $\sigma = [2, 1, 3] \in S_3$. Then $|\sigma| = 1$.

$$Wg^U([2, 1, 3], d) = -(p([2, 1, 3], 1)d^{-4} + p([2, 1, 3], 3)d^{-6} + \cdots).$$

Let us compute the coefficients $p([2, 1, 3], 1)$ and $p([2, 1, 3], 3)$. 
\( p([2, 1, 3], 1) = 1 \)

\( p([2, 1, 3], 1) \) is the number of path(s) from \([2, 1, 3]\) to \(\emptyset\) going through red edge(s) exactly 1 time on the Weingarten graph.
\[ p([2, 1, 3], 3) = 5 \]

\( p([2, 1, 3], 3) \) is the number of paths from \([2, 1, 3]\) to \(\emptyset\) going through red edges exactly 3 times on the Weingarten graph.

\[ Wg^U([2, 1, 3], d) = -(1d^{-4} + 5d^{-6} + \cdots). \]
Uniform bound for $Wg^U$

It is well known that

$$p(\sigma, |\sigma|) = \prod_{i=1}^{\ell(\mu)} \text{Cat}(\mu_j - 1)$$

with Catalan numbers $\text{Cat}(n) = \frac{1}{n+1} \binom{2n}{n}$.

Theorem (Collins-M, 2017)

For any $\sigma \in S_n$ and nonnegative integer $j$, we have

$$(n - 1)^j p(\sigma, |\sigma|) \leq p(\sigma, |\sigma| + 2j) \leq (6n^{7/2})^j p(\sigma, |\sigma|).$$

Corollary

For any $\sigma \in S_n$ and $d > \sqrt{6n^{7/4}}$,

$$\frac{1}{1 - \frac{n-1}{d^2}} \leq \frac{(-1)^{|\sigma|}d^{n+|\sigma|}Wg^U(\sigma, d)}{p(\sigma, |\sigma|)} \leq \frac{1}{1 - \frac{6n^{7/2}}{d^2}}.$$
Connection to monotone factorizations

The expansion

\[ W^U_g(\sigma, d) = (-1)^{|\sigma|} \sum_{j \geq 0} p(\sigma, |\sigma| + 2j) d^{-(n+|\sigma|+2j)}. \]

is equivalent to the result in [M-Novak (2013)].

**Definition (Monotone factorizations)**

Let \( \sigma \) be a permutation in \( \mathfrak{S}_n \). A sequence \( f = (\tau_1, \ldots, \tau_k) \) of \( k \) transpositions is called a **monotone factorization** of length \( k \) for \( \sigma \) if:

- \( \tau_i = (s_i, t_i) \) with \( 1 \leq s_i < t_i \leq n \);
- \( \sigma = \tau_1 \tau_2 \cdots \tau_k \);
- \( n \geq t_1 \geq t_2 \geq \cdots \geq t_k \geq 1 \) (monotonicity).

**Example**

\[ f = ((3, 5), (2, 5), (2, 4), (1, 2)) \]

is a monotone factorization of length 4 of \( \sigma = [4, 1, 5, 3, 2] \).
Connection to monotone factorizations

Proposition

The number of monotone factorizations of length $k$ for $\sigma$ is equal to $p(\sigma, k)$. Specifically, we have one-to-one correspondence between the following two objects:

- monotone factorizations of length $k$ for $\sigma$;
- paths from $\sigma$ to $\emptyset$ going through $k$ red edges on the Weingarten graph $G^U$.

Example

\[
[4, 1, 5, 3, 2] \xrightarrow{(3,5)} [4, 1, 3, 5, 2] \xrightarrow{(2,5)} [4, 1, 3, 2, 5] \rightarrow [4, 1, 3, 2] \\
\xrightarrow{(2,4)} [2, 1, 3, 4] \rightarrow [2, 1, 3] \rightarrow [2, 1] \xrightarrow{(1,2)} [1, 2] \rightarrow [1] \rightarrow \emptyset.
\]

\[
f = ((3, 5), (2, 5), (2, 4), (1, 2))
\]
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Chiral unitary matrix – class A III

Let us consider the compact symmetric space $U(d)/(U(a) \times U(b))$, with $d = a + b$, of class A III again. The corresponding random matrix is, the chiral unitary matrix, which is a unitary and Hermitian matrix given by

$$X = (x_{ij})_{1 \leq i,j \leq d} := U \cdot \begin{pmatrix} I_a & 0 \\ 0 & -I_b \end{pmatrix} \cdot U^*,$$

where $U$ is a $d \times d$ Haar-distributed unitary matrix.

(Recall the Fourier expansion for $Wg_{A III}$.)

We now change the parameters

$$d = a + b, \quad e = a - b \quad \in \quad \mathbb{Z}.$$ 

The corresponding Weingarten function

$$Wg_{A III}(\sigma, d, e) = \frac{1}{n!} \sum_{\lambda \vdash n} f^{\lambda} \frac{s^{(d+e)/2}_\lambda(1, \ldots, 1, -1, \ldots, -1)}{s^{(d-e)/2}_\lambda(1, \ldots, 1)} \chi^{\lambda}(\sigma) \quad (\sigma \in \mathbb{S}_n).$$

is a class function on a symmetric space $\mathbb{S}_n$. Suppose $d \geq n$. 

Sho Matsumoto (Kagoshima Univ)
May 23, 2018 39 / 44
We define an infinite directed graph \( G^{A_{III}} = (V, E^{\text{red}} \sqcup E^{\text{blue}} \sqcup E^{\text{green}}) \) as follows.

- **Vertex set** \( V \). \( V = \bigsqcup_{n=0}^{\infty} S_n \).
- **Red edges** and **Blue edges** are the same with the unitary case.
- **Green edges**. (lower the level by two)

\[
S_n \ni \sigma \quad \longrightarrow \quad \sigma^\flat \in S_{n-2}
\]

if

- the letter \( n \) belongs to a 2-cycle \((j, n)\) of \( \sigma \);
- \( \sigma^\flat \in S_{n-2} \) is obtained by removing the 2-cycle \((j, n)\) from \( \sigma \) and by shifting letters \( 1, 2, \ldots, \hat{j}, \ldots, n - 1 \) to \( 1, 2, \ldots, n - 2 \) while keeping order.
Weingarten graph for $A_{III}$
Asymptotics for $Wg^{A_{III}}$

**Theorem (Collins-M, 2017)**

Let $\sigma \in S_n$. Assume $d \geq n$. Then

$$Wg^{A_{III}}(\sigma, d, e) = \sum_{p: \sigma \to \emptyset} (-1)^{\text{red}(p)} e^{\text{blue}(p)} d^{-(\text{red}(p)+\text{blue}(p)+\text{green}(p))}$$

summed over all paths from $\sigma$ to $\emptyset$ on $G^{A_{III}}$. Here red/blue/green$(p)$ stands for the number of the edges in $p$ with the color.

**Example**

Let $X = (x_{ij})_{1 \leq i,j \leq d} = UU^*_{ab}$ be a chiral unitary matrix. Put $e = a - b$. Take $\sigma = [2, 1] \in S_2$.

$$Wg^{A_{III}}([2, 1], d, e) = \mathbb{E}[x_{12}x_{21}] = \mathbb{E}[|x_{12}|^2]$$

$$= \sum_{p: [2,1] \to \emptyset} (-1)^{\text{red}(p)} e^{\text{blue}(p)} d^{-(\text{red}(p)+\text{blue}(p)+\text{green}(p))}$$
Paths from $\sigma = [2, 1]$

Example

$$Wg^\text{III}([2, 1], d, e) = \sum_{p: \sigma \rightarrow \emptyset} (-1)^{\text{red}(p)} e^{\text{blue}(p)} d^{- (\text{red}(p) + \text{blue}(p) + \text{green}(p))}$$

$$= \sum_{j \geq 0} (-1)^{2j} e^0 d^{-(2j+0+1)} + \sum_{j \geq 0} (-1)^{2j+1} e^2 d^{-(2j+1+2+0)}$$

$$= \frac{d^2 - e^2}{d(d^2 - 1)}.$$
Thank you!
Let’s enjoy Weingarten calculus together.

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