The Stochastic Semigroup Approach to the Edge of β -ensembles

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Based on work by V. Gorin and M. Shkolnikov, and joint work with M. Shkolnikov.

Problem: Edge Fluctuations of β -ensembles

Given $\beta > 0$, let $\lambda_1^{\beta} \ge \lambda_2^{\beta} \ge \cdots \ge \lambda_N^{\beta}$ be sampled from

$$\frac{1}{\mathcal{Z}_{\beta}} \cdot \prod_{i < j} (x_j - x_i)^{\beta} \cdot \exp\left(-\frac{\beta}{4} \sum_{i=1}^{N} x_i^2\right), \quad x_1 \ge \cdots \ge x_N.$$

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Problem

Given $k \in \mathbb{N}$, understand the fluctuations of $(\lambda_1^{\beta}, \ldots, \lambda_k^{\beta})$ as $N \to \infty$.

Operator Limit

Define the stochastic Airy opterator (SAO) with parameter $\beta > 0$ as

$$[\mathcal{H}^{\beta}f](x) := -f''(x) + xf(x) + \frac{2}{\sqrt{\beta}}W'_{x}f(x), \qquad f: \mathbb{R}_{+} \to \mathbb{R}, f(0) = 0,$$

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Theorem (Dumitriu-Edelman (2002); Edelman-Sutton (2007); Ramírez-Rider-Virág (2011))

Let $\Lambda_1^{\beta} \leq \Lambda_2^{\beta} \leq \cdots$ be the eigenvalues of \mathcal{H}^{β} . For every $k \in \mathbb{N}$ fixed,

$$N^{1/6}(2\sqrt{N} - \lambda_i^\beta)_{1 \le i \le k} \Rightarrow (\Lambda_i^\beta)_{1 \le i \le k}.$$

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Advantages of operator limit approach.

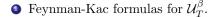
- **(**) Unified method (i.e., for all $\beta > 0$) of studying β -ensembles.
- Study limiting fluctuations through functional analysis, as they arise as the spectrum of a differential operator.

Idea. Study the asymptotic extreme value fluctuations of Gaussian β -ensembles through the semigroups generated by the SAOs:

$$\mathcal{U}_T^{\beta} := \mathrm{e}^{-T \cdot \mathcal{H}^{\beta}/2}, \qquad T \ge 0.$$

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Theorem

Let $(e_t)_{t \in [0,1]}$ be a Brownian excursion, and let $(\ell^a)_{a \ge 0}$ be its local time process on [0,1].

$$\int_0^1 e_t \, \mathrm{d}t - \frac{1}{2} \int_0^\infty (\ell^a)^2 \, \mathrm{d}a \sim N(0, 1/12)$$