Statistics of the real roots of real random polynomials

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Optimal and random point configurations
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Real random polynomials

\[ P_n(x) = \sum_{i=0}^{n} a_i x^i \]

\[ a_i \equiv \text{ind. random variables}, \quad \mathbb{E}(a_i) = 0, \quad \mathbb{E}(a_i a_j) = \sigma_i^2 \delta_{ij} \]

\[ P_n(\lambda_i) = 0, \quad \lambda_1, \ldots, \lambda_d \in \mathbb{R} \]
Real random polynomials

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General question: statistics of \( \lambda_i \)'s ?
Bloch, Pólya, *On the roots of certain algebraic equations* (1932),

Littlewood, Offord, *On the number of real roots of a random algebraic equation* (1939),

Kac, *On the average number of real roots of a random algebraic equation*, (1943),

Edelman, Kostlan, *How many zeros of random polynomials are real?* (1999),
Bloch, Pólya, *On the roots of certain algebraic equations* (1932),

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Kac, *On the average number of real roots of a random algebraic equation*, (1943),

Bogomolny, Bohigas, Leboeuf, *Quantum chaotic dynamics and random polynomials*, (1992),

Edelman, Kostlan, *How many zeros of random polynomials are real?* (1999),

Introduction: topics of this talk

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How many zeros of \( P_n(x) \) are real?

Condensation of the roots of \( P_n(x) \) on the real axis

Random polynomials having few or no real roots

Probability of no real root and first-passage problems of stochastic processes

G. Schehr (LPTMS Orsay)
Introduction: topics of this talk

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Outline

1. Condensation of the roots of random polynomials on the real axis
   - Motivations: Kac’s polynomials and beyond
   - Condensation transition
   - Derivation of the results

2. Polynomials having few real roots
   - Motivation: roots of Kac’s random polynomials
   - First passage problems and persistence
   - Derivation: mapping to a Gaussian Stationary Process

3. Conclusion
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Motivations: Kac’s polynomials

Real Kac’s polynomials

\[ K_n(x) = \sum_{i=0}^{n} a_i x^i \]

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Motivations: Kac’s polynomials

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$$K_n(x) = \sum_{i=0}^{n} a_i x^i$$

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Complex roots
Motivations: Kac’s polynomials

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Complex roots

Real roots

\[ \mathbb{E}(N_n) \equiv \text{average number of roots on the real axis} \]
\[ \mathbb{E}(N_n) = \frac{2}{\pi} \log n + \mathcal{O}(1) \]

G. Schehr (LPTMS Orsay)
Motivations: Kac’s polynomials

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\[ \mathbb{E}(N_n) \equiv \text{average number of roots on the real axis} \]

\[ \mathbb{E}(N_n) \sim \frac{2}{\pi} \log n \ll n \]

Real roots

M. Kac ’43
Motivations: Kac’s polynomials

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Complex roots

Real roots

\[ \mathbb{E}(N_n) \equiv \text{average number of roots on the real axis} \]

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Q: how can one increase \( \mathbb{E}(N_n) \) by modifying \( \mathbb{E}(a_i^2) \)?
Beyond Kac’s polynomials

**Weyl polynomials**

\[
W_n(x) = \sum_{i=0}^{n} a_i x^i
\]

\[a_i \equiv \text{independent random variables},\]
\[\mathbb{E}(a_i) = 0, \quad \mathbb{E}(a_i a_j) = \sigma_i^2 \delta_{ij}, \quad \sigma_i = \frac{1}{\sqrt{i!}}\]

**Complex roots**

\[\sim \sqrt{n}\]

**Real roots**

\[\mathbb{E}(N_n) = \frac{2}{\pi} \sqrt{n} + o(\sqrt{n})\]
Beyond Kac’s polynomials

Littlewood & Offord’s random polynomials

\[ L_n(x) = \sum_{i=0}^{n} a_i x^i \]

\[ a_i = \frac{\epsilon_i}{\sqrt{(i!)^i}}, \epsilon_i = \pm 1 \]

\[ \mathbb{E}(a_i) = 0, \mathbb{E}(a_i a_j) = \frac{1}{(i!)^j} \delta_{ij} \]
Beyond Kac's polynomials

Littlewood & Offord’s random polynomials

\[ L_n(x) = \sum_{i=0}^{n} a_i x^i \]

\[ a_i = \frac{\epsilon_i}{\sqrt{(i!)^i}} , \quad \epsilon_i = \pm 1 \]

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* All roots are real with probability one: \( N_n = \mathbb{E}(N_n) = n \)

* (Quasi)-periodic structure:

\[ x_0 = 0 , \quad x_m = m^m m! , \quad m \leq n \]

one root in \([x_{m-1}, x_m]\) or in \([-x_{m-1}, -x_m]\) with probability one
Kac’s polynomials and beyond

1. Kac polynomials

\[ K_n(x) = \sum_{i=0}^{n} a_i x^i, \quad \mathbb{E}(a_i^2) = \sigma^2 \implies \mathbb{E}(N_n) \propto \log n \]

2. Weyl polynomials

\[ W_n(x) = \sum_{i=0}^{n} a_i x^i, \quad \mathbb{E}(a_i^2) = (i!)^{-1} \implies \mathbb{E}(N_n) \propto \sqrt{n} \]

3. Littlewood-Offord polynomials

\[ L_n(x) = \sum_{i=0}^{n} a_i x^i, \quad \mathbb{E}(a_i^2) = (i!)^{-i} \implies \mathbb{E}(N_n) \propto n \]
Kac’s polynomials and beyond

1. Kac polynomials: $K_n(x) = \sum_{i=0}^{n} a_i x^i$, $\mathbb{E}(a_i^2) \sim e^{-i^0}$
   $\mathbb{E}(N_n) \propto \log n$

2. Weyl polynomials: $W_n(x) = \sum_{i=0}^{n} a_i x^i$, $\mathbb{E}(a_i^2) \sim e^{-i \ln i}$
   $\mathbb{E}(N_n) \propto \sqrt{n}$

3. Littlewood-Offord polynomials: $L_n(x) = \sum_{i=0}^{n} a_i x^i$, $\mathbb{E}(a_i^2) \sim e^{-i^2 \ln i}$
   $\mathbb{E}(N_n) \propto n$
Kac’s polynomials and beyond

1. **Kac polynomials**: 
   \[ K_n(x) = \sum_{i=0}^{n} a_i x^i, \quad \mathbb{E}(a_i^2) \sim e^{-i^0} \]
   \[ \mathbb{E}(N_n) \propto \log n \]

2. **Weyl polynomials**: 
   \[ W_n(x) = \sum_{i=0}^{n} a_i x^i, \quad \mathbb{E}(a_i^2) \sim e^{-i \ln i} \]
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3. **Littlewood-Offord polynomials**: 
   \[ L_n(x) = \sum_{i=0}^{n} a_i x^i, \quad \mathbb{E}(a_i^2) \sim e^{-i^2 \ln i} \]
   \[ \mathbb{E}(N_n) \propto n \]

A family of random polynomials indexed by \( \alpha \)

\[ P_n(x) = \sum_{i=0}^{n} a_i x^i, \quad \mathbb{E}(a_i^2) = e^{-i^\alpha} \]
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   - Motivations: Kac’s polynomials and beyond
   - Condensation transition
   - Derivation of the results

2. Polynomials having few real roots
   - Motivation: roots of Kac’s random polynomials
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3. Conclusion
A family of random polynomials indexed by $\alpha$

$$P_n(x) = \sum_{i=0}^{n} a_i x^i, \quad \mathbb{E}(a_i^2) = e^{-i\alpha}$$

$$\langle N_n \rangle \sim \log n \quad \langle N_n \rangle \sim n^{\alpha/2} \quad \langle N_n \rangle \sim n$$

Kac \quad Weyl \quad Littlewood-Offord
Condensation transition: the density

A change of variable:

\[ Y = \left( \frac{2}{\alpha} \ln x \right)^{\frac{1}{\alpha-1}} \]
Condensation transition: the density

- A change of variable:

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\(0\) \quad \text{Kac} \quad \text{Weyl} \quad \text{Littlewood-Offord} \quad \alpha
Condensation transition: the density

- A change of variable:
  \[ Y = \left( \frac{2}{\alpha} \ln x \right) \frac{1}{\alpha - 1} \]

- The density \( \hat{\rho}_n(Y) \) across the transition

\[ \langle N_n \rangle \sim \log n \quad \langle N_n \rangle \sim n^{\alpha/2} \quad \langle N_n \rangle \sim n \]

\[ 0 \quad 1 \quad 2 \quad \alpha \]

Kac \quad Weyl \quad Littlewood-Offord

\[ 1 < \alpha < 2 \]
\[ \alpha = 2 \]
\[ \alpha > 2 \]
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\begin{align*}
P_n(\lambda_i) &= 0, \quad \lambda_1, \cdots, \lambda_d \in \mathbb{R} \\
\rho_n(x) &= \sum_{i=1}^{d} \mathbb{E}[\delta(x - \lambda_i)]
\end{align*}
Average density of real roots

\[ P_n(\lambda_i) = 0, \lambda_1, \cdots, \lambda_d \in \mathbb{R} \]

\[ \rho_n(x) = \sum_{i=1}^{d} \mathbb{E}[\delta(x - \lambda_i)] = \mathbb{E}[|P_n'(x)| \delta(P_n(x))] \]
Average density of real roots

\[ P_n(\lambda_i) = 0 , \lambda_1, \cdots , \lambda_d \in \mathbb{R} \]

\[ \rho_n(x) = \sum_{i=1}^{d} \mathbb{E}[\delta(x - \lambda_i)] = \mathbb{E}[|P'_n(x)| \delta(P_n(x))] \]

\[ = \int_{-\infty}^{\infty} dy |y| \mathbb{E}[\delta(P'_n(x) - y) \delta(P_n(x))] \]
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After some algebra...

\[ \rho_n(x) = \frac{\sqrt{c_n(x)(c_n'(x)/x + c_n''(x))} - [c_n'(x)]^2}{2\pi c_n(x)} , \]

\[ c_n(x) = \mathbb{E}[P_n(x)P_n(x)] = \sum_{k=0}^{n} e^{-k\alpha} x^{2k} \]
Average density of real roots

- Saddle point calculation

\[ \rho_n(x) = \frac{\sqrt{c_n(x)(c_n'(x)/x + c_n''(x)) - [c_n'(x)]^2}}{2\pi c_n(x)}, \]

\[ c_n(x) = \mathbb{E}[P_n(x)P_n(x)] = \sum_{k=0}^{n} e^{-k^\alpha} x^{2k} = \sum_{k=0}^{n} \exp[-\phi(k, x)] \]

\[ \phi(u, x) = u^\alpha - 2u \ln x \]
Average density of real roots

- Saddle point calculation

\[
\rho_n(x) = \frac{\sqrt{c_n(x)(c'_n(x)/x + c''_n(x)) - [c'_n(x)]^2}}{2\pi c_n(x)},
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c_n(x) = \mathbb{E}[P_n(x)P_n(x)] = \sum_{k=0}^{n} e^{-k\alpha} x^{2k} = \sum_{k=0}^{n} \exp[-\phi(k, x)]
\]

\[
\phi(u, x) = u^{\alpha} - 2u \ln x \quad \sim \exp[-\phi(u^*(x), x)]
\]

where \( \partial_u \phi(u^*(x), x) = 0 \)
Average density of real roots

- **Saddle point calculation**

\[
\rho_n(x) = \frac{\sqrt{c_n(x)(c'_n(x)/x + c''_n(x)) - [c'_n(x)]^2}}{2\pi c_n(x)},
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c_n(x) = \mathbb{E}[P_n(x)P_n(x)] = \sum_{k=0}^n e^{-k\alpha} x^{2k} = \sum_{k=0}^n \exp[-\phi(k, x)]
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\[
\phi(u, x) = u^\alpha - 2u \ln x \quad \simeq \:\exp[-\phi(u^*(x), x)]
\]

where \(\partial_u \phi(u^*(x), x) = 0\)

- **3 different cases depending on \(\alpha\)**

1. \(\alpha < 1\): \(u^*(x) = n\)
2. \(1 < \alpha < 2\): \(u^*(x) < n\) \& \(\partial_u^2 \phi(u^*(x), x) \to 0, \: x \to \infty\)
3. \(\alpha > 2\): \(u^*(x) < n\) \& \(\partial_u^2 \phi(u^*(x), x) \to \infty, \: x \to \infty\)
Condensation transition: to summarize

- A change of variable:
  \[ Y = \left( \frac{2}{\alpha} \ln x \right) \left( \frac{1}{\alpha-1} \right) = u^*(x) \]

- The density \( \hat{\rho}_n(Y) \) across the transition

\[ \langle N_n \rangle \sim \log n \quad \langle N_n \rangle \sim n^{\alpha/2} \quad \langle N_n \rangle \sim n \]

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Motivations: Real random polynomials

Real Kac’s polynomials

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Complex roots

![Diagram of complex roots](image.png)
**Motivations : Real random polynomials**

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**Real roots**

\[ \mathbb{E}(N_n) \equiv \text{mean number of roots on the real axis} \]  

M. Kac ’43

\[ \mathbb{E}(N_n) \sim \frac{2}{\pi} \log n \]
Motivations: **Real roots** of Kac's polynomials

\[
q_0(n) \equiv \text{Probability that } K_n(x) \text{ has no real root in } [0, 1]
\]

\[
q_0(n) \propto n^{-\gamma}
\]

with \( \gamma = 0.19(1) \) \hspace{1cm} \text{(Numerics)}
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Persistence probability $p_0(t)$

- $X(t) \equiv$ stochastic random variable evolving in time $t$, $\mathbb{E}[X(t)] = 0$
- Persistence probability
  
  $p_0(t) \equiv$ Proba. that $X$ has not changed sign up to time $t$
Persistence probability $p_0(t)$

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Persistence in *spatially extended systems*

- phase ordering kinetics
Introduction: Phase ordering kinetics

- Glauber dynamics of 2d Ising model at $T = 0$, $H_{\text{Ising}} = -J \sum_{\langle i,j \rangle} \sigma_i \sigma_j$ with $\sigma_j = \pm 1$

$$t_1 = 0$$
Introduction: Phase ordering kinetics

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$t_1 = 0$

$t_2 = 10^2$
Introduction: Phase ordering kinetics

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$t_3 = 10^4$
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  H_{\text{Ising}} = -J \sum_{\langle i,j \rangle} \sigma_i \sigma_j \\
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  \]

\( t_1 = 0 \)

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\( t_4 = 10^6 \)
Introduction

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Persistence in spatially extended systems

- phase ordering kinetics ('94-)

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Persistence in spatially extended systems

- phase ordering kinetics ('94-)
- diffusion field ('96-)
- height of a fluctuating interface ('97-)
- ...
**Introduction**

**Persistence probability $p_0(t)$**
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**Persistence in spatially extended systems**
- phase ordering kinetics ('94-)
- diffusion field ('96-)
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- ... 

$p_0(t) \propto t^{-\theta_p}$


“Persistence and First-Passage Properties in Non-equilibrium Systems”
Motivations: persistence for the diffusion equation

Diffusion equation with random initial conditions

\[ \partial_t \phi(x, t) = \nabla^2 \phi(x, t) \]
\[ \mathbb{E}(\phi(x, 0)\phi(x', 0)) = \delta^d(x - x') \]
Motivations: persistence for the diffusion equation

Diffusion equation with random initial conditions

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- Diffusion equation (or heat equation) is universal and ubiquitous in nature
- Ordering dynamics for $O(N)$-symmetric spin models in the limit $N \to \infty$
- see A. Dembo, S. Mukherjee, Ann. Probab. 15
Motivations: persistence for the diffusion equation

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Single length scale
\[ \ell(t) \propto t^{1/2} \]

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Single length scale

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Persistence \( p_0(t, L) \) for a \( d \)-dim. system of linear size \( L \)

\[ p_0(t, L) \equiv \text{Proba. that } \phi(x, t) \text{ has not changed sign up to } t \]

S. N. Majumdar, C. Sire, A. J. Bray and S. J. Cornell, PRL 96

B. Derrida, V. Hakim and R. Zeitak, PRL 96
Motivations: persistence for the diffusion equation

Diffusion equation with random initial conditions

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Single length scale \( \ell(t) \propto t^{1/2} \)

Persistence \( p_0(t, L) \) for a \( d \)-dim. system of linear size \( L \)

\[ p_0(t, L) \equiv \text{Proba. that } \phi(x, t) \text{ has not changed sign up to } t \]

\[ p_0(t, L) \sim t^{-\theta(d)} L^{-2\theta(d)} \]

G. Schehr (LPTMS Orsay) Real roots of real random polynomials IHP, June 27, 2016 25 / 39
Diffusion equation with random initial conditions

\[ \partial_t \phi(x, t) = \nabla^2 \phi(x, t) \]
\[ \mathbb{E}(\phi(x, 0) \phi(x', 0)) = \delta^d(x - x') \]

Single length scale
\[ \ell(t) \propto t^{1/2} \]

Persistence \( p_0(t, L) \) for a \( d \)-dim. system of linear size \( L \)

\[ p_0(t, L) \equiv \text{Proba. that } \phi(x, t) \text{ has not changed sign up to } t \]

\[ p_0(t, L) \propto L^{-2\theta(d)} h(t/L^2) \]

\[ \theta(1) = 0.1207 \]
\[ \theta(2) = 0.1875, \quad \text{Numerics} \]
Generalized Kac’s polynomials

\[ K_n(x) = a_0 + \sum_{i=1}^{n-1} a_i i^{(d-2)/4} x^i \]

\[ a_i \equiv \text{Gaussian random variables, } \mathbb{E}(a_i) = 0, \mathbb{E}(a_i a_j) = \delta_{ij} \]
Generalized Kac’s polynomials

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Proba. of no real root in $[0, 1]$

$$q_0(n) \propto n^{-b(d)}$$

Persistence of diffusion

$$\rho_0(t, L) \propto L^{-2\theta(d)}$$
Purpose: a link between random polynomials & diffusion equation

Generalized Kac’s polynomials

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G. S., S. N. Majumdar 07

G. Schehr (LPTMS Orsay)
Purpose: a link between random polynomials & diffusion equation

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G. S., S. N. Majumdar 07
A. Dembo, S. Mukherjee 15
1. Condensation of the roots of random polynomials on the real axis
   - Motivations: Kac’s polynomials and beyond
   - Condensation transition
   - Derivation of the results

2. Polynomials having few real roots
   - Motivation: roots of Kac’s random polynomials
   - First passage problems and persistence
   - Derivation: mapping to a Gaussian Stationary Process

3. Conclusion
Persistence of diffusion equation

\[ \partial_t \phi(x, t) = \nabla^2 \phi(x, t) \]

\[ \mathbb{E}(\phi(x, 0)\phi(x', 0)) = \delta^d(x - x') \]

\[ G(x, t) = (4\pi t)^{-\frac{d}{2}} \exp\left(-\frac{x^2}{4t}\right) \]

\[ \phi(x, t) = \int_{|y|<L} d^d y G(x - y, t) \phi(y, 0) \]
Persistence of diffusion equation

\[ \partial_t \phi(x, t) = \nabla^2 \phi(x, t) \]
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\[ G(x, t) = (4\pi t)^{-\frac{d}{2}} \exp(-x^2/4t) \]

Mapping of \( \phi(x, t) \) to a Gaussian stationary process

1. Normalized process \( X(t) = \frac{\phi(x,t)}{[\mathbb{E}[\phi(x,t)^2]]^{1/2}} \)
\[ \mathbb{E}(X(t)X(t')) \sim \begin{cases} 
    \left(4\frac{tt'}{(t+t')^2}\right)^\frac{d}{4}, & t, t' \ll L^2 \\
    1, & t, t' \gg L^2 
\end{cases} \]

2. New time variable \( T = \log t, \) for \( t \ll L^2 \)
\[ \mathbb{E}(X(T)X(T')) = \left[\cosh((T - T')/2)\right]^{-d/2} \]
Persistence for a Gaussian stationary process (GSP)

- \( X(T) \) is a GSP with correlations
  
  \[
  \mathbb{E}(X(T)X(T')) = a(T - T') \\
  a(T) = (\cosh(T/2))^{-d/2}
  \]

- Persistence probability \( \mathcal{P}_0(T) \) (by Slepian's lemma)

  For \( T \gg 1 \) \( a(T) \propto \exp(-\frac{d}{2}T) \Rightarrow \mathcal{P}_0(T) \propto \exp(-\theta(d)T) \)

- Reverting back to \( t = \exp(T) \)
  
  \[
  p_0(t, L) \sim t^{-\theta(d)} \quad 1 \ll t \ll L^2
  \]
Persistence of diffusion equation

- Normalized process $X(t) = \frac{\phi(x,t)}{[\mathbb{E}[\phi(x,t)^2]]^{1/2}}$

\[
\mathbb{E}(X(t)X(t')) \sim \begin{cases} 
\left(4\frac{tt'}{(t+t')^2}\right)^\frac{d}{4} & , \quad t, t' \ll L^2 \\
1 & , \quad t, t' \gg L^2
\end{cases}
\]
Persistence of diffusion equation

- Normalized process \( X(t) = \frac{\phi(x,t)}{[\mathbb{E}[\phi(x,t)^2]]^{1/2}} \)

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(4 \frac{tt'}{(t+t')^2})^{\frac{d}{4}}, & t, t' \ll L^2 \\
1, & t, t' \gg L^2 
\end{cases}
\]

\[ p_0(t, L) \propto L^{-2\theta(d)} h(t/L^2) \]

\[ h(u) \sim \begin{cases} 
u \sim \nu^{-\theta(d)}, & \nu \ll 1 \\
u \sim c^{\text{st}}, & \nu \gg 1 \end{cases} \]
Real roots of generalized Kac’s polynomials

\[ K_n(x) = a_0 + \sum_{i=1}^{n-1} a_i i^{(d-2)/4} x^i \]

Averaged density of real roots

\[ \rho_n(x) = \mathbb{E}[|K_n'(x)| \delta(K_n(x))] \]

Real roots concentrate around \( x = \pm 1 \)

\[ \rho_n(\pm 1) \sim A_d n \]

\[ A_d = \frac{2 \sqrt{d/(d + 4)}}{\pi (d + 2)} \]
Real roots of generalized Kac’s polynomials

\[ K_n(x) = a_0 + \sum_{i=1}^{n-1} a_i i^{(d-2)/4} x^i \]

Averaged density of real roots for \( n \to \infty \)

\[ \rho_\infty(x) = \frac{(\text{Li}_{-1-d/2}(x^2)(1 + \text{Li}_{1-d/2}(x^2)) - \text{Li}_{-d/2}^2(x^2))^{1/2}}{\pi |x|(1 + \text{Li}_{1-d/2}(x^2))} \]
Real roots of generalized Kac’s polynomials

\[ K_n(x) = a_0 + \sum_{i=1}^{n-1} a_i x^{(d-2)/4} x^i \]

Mean number of real roots in \([0, 1]\): Kac-Rice formula

\[ \mathbb{E}(N_n[0, 1]) = \int_0^1 \rho_n(x) \, dx \sim \frac{1}{2\pi} \sqrt{\frac{d}{2}} \log n \]
Probability of no real root for $K_n(x)$

Dembo et al. ’02

Statistical independence of $K_n(x)$ in the 4 sub-intervals

⇒ Focus on the interval $[0, 1]$

$$P_0(x, n) \equiv \text{Proba. that } K_n(x) \text{ has no real root in } [0, x]$$
Probability of no real root for $K_n(x)$

- **Two-point correlator**

$$C_n(x, y) = \mathbb{E}(K_n(x)K_n(y)) = \sum_{i=0}^{n-1} i^{(d-2)/2} (xy)^i$$

- **Normalization**

$$C_n(x, y) = \frac{C_n(x, y)}{(C_n(x, x))^{1/2} (C_n(y, y))^{1/2}}$$

- **Change of variable**

$$x = 1 - \frac{1}{t}, \quad t \gg 1$$
Probability of no real root for $K_n(x)$

Normalized correlator in the scaling limit

Scaling limit

\[ t \gg 1 \quad , \quad n \gg 1 \quad \text{keeping} \quad \tilde{t} = \frac{t}{n} \quad \text{fixed} \]

\[ C_n(t, t') \rightarrow C(\tilde{t}, \tilde{t}') \quad \text{with the asymptotic behaviors} \]

\[
C(\tilde{t}, \tilde{t}') \sim \begin{cases} 
\left( \frac{4 \tilde{t} \tilde{t}'}{(\tilde{t} + \tilde{t}')^2} \right)^{\frac{d}{4}} , & \tilde{t}, \tilde{t}' \ll 1 \\
1 , & \tilde{t}, \tilde{t}' \gg 1 
\end{cases}
\]
Persistence of diffusion equation (reminder)

\[ \partial_t \phi(x, t) = \nabla^2 \phi(x, t) \]
\[ \mathbb{E}(\phi(x, 0)\phi(x', 0)) = \delta^d(x - x') \]
\[ \phi(x, t) = \int d^d y G(x - y) \phi(y, 0) \]
\[ G(x, t) = (4\pi t)^{-\frac{d}{2}} \exp(-x^2/4t) \]

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1. Normalized process \( X(t) = \frac{\phi(x,t)}{\langle \phi(x,t)^2 \rangle^{1/2}} \)

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\left( \frac{tt'}{(t+t')^2} \right)^{\frac{d}{4}}, & t, t' \ll L^2 \\
1, & t, t' \gg L^2
\end{cases} \]

2. Persistence probability \( p_0(t, L) \)

\[ p_0(t, L) \propto L^{-2\theta(d)} h(t/L^2) \]
Probability of no real root for $K_n(x)$

$$C(\tilde{t}, \tilde{t}') \sim \begin{cases} \left(4\frac{\tilde{t}\tilde{t}'}{(\tilde{t}+\tilde{t}')^2}\right)^\frac{d}{4} , & \tilde{t}, \tilde{t}' \ll 1 \\ 1 , & \tilde{t}, \tilde{t}' \gg 1 \end{cases}$$

$$P_0(x, n) \equiv \text{Proba. that } K_n(x) \text{ has no real root in } [0, x]$$

**Scaling form for $P_0(x, n)$**

$$P_0(x, n) \propto n^{-\theta(d)}\tilde{h}(n(1 - x))$$

$$\tilde{h}(u) \sim \begin{cases} c^{st} , & u \ll 1 \\ u^{\theta(d)} , & u \gg 1 \end{cases}$$
Probability of no real root for $K_n(x)$

Scaling form for $P_0(x, n)$

$$P_0(x, n) \propto n^{-\theta(d)} \tilde{h}(n(1 - x))$$

$$\tilde{h}(u) \sim \begin{cases} 
\text{cst}, & u \ll 1 \\
 u^{\theta(d)}, & u \gg 1
\end{cases}$$

$q_0(n) \equiv$ Probability that $K_n(x)$ has no real root in $[0, 1]$

$$q_0(n) = P(1, n) \sim n^{-\theta(d)}$$
Numerical check of the scaling form

Numerical computation of $P_0(x, n)$ for $d = 2$

![Graph showing $P_0(x, n)$ for different values of $n$. The graph plots $1-x$ on the x-axis and $P_0(x, n)$ on the y-axis.]
Numerical check of the scaling form

Numerical computation of $P_0(x, n)$ for $d = 2$

$$P_0(x, n) \propto n^{-\theta(d)} \tilde{h}(n(1 - x))$$
Conclusion

- A condensation phenomenon of the roots on the real axis
- A link between diffusion equation and random polynomials

**Proba. of no real root**

\[ q_0(n) \propto n^{-b(d)} \]

**Persistence of diffusion**

\[ p_0(t, L) \propto L^{-2\theta(d)} \]

\[ b(d) = \theta(d) \]

1. Universality see A. Dembo, S. Mukherjee 15

2. Towards exact results for \( \theta(d) \), \( 1/(4\sqrt{3}) \leq \theta(2) \leq 1/4 \)
   - G. Molchan 12
   - W. Li, Q. M. Shao 02

see also D. Zaporozhets 06
A heuristic argument

Diffusion equation

\[ \phi(x = 0, t) = (4\pi t)^{-d/2} \int_{0 < |x| < L} d^d x \exp\left(-\frac{x^2}{4t}\right) \phi(x, 0) \]

\[ = \frac{S_d^{1/2}}{(4\pi t)^{d/2}} \int_0^L dr \ r^{\frac{1}{2}(d-1)} e^{-\frac{r^2}{4t}} \psi(r) \]

\[ \psi(r) = S_d^{-1/2} r^{-\frac{1}{2}(d-1)} \lim_{\Delta r \to 0} \frac{1}{\Delta r} \int_{r < |x| < r + \Delta r} d^d x \ \phi(x, 0) \]

\[ E(\psi(r)\psi(r')) = \delta(r - r') \]
A heuristic argument

- Diffusion equation

\[ \phi(x = 0, t) \propto \int_0^{L^2} du \, u \frac{d-2}{4} e^{-\frac{u}{t}} \tilde{\Psi}(u) \]

\[ \mathbb{E}(\tilde{\Psi}(u)\tilde{\Psi}(u')) = \delta(u - u') \]
A heuristic argument

- Diffusion equation

\[ \phi(x = 0, t) \propto \int_{0}^{L^2} du \ u^{d-2} e^{-\frac{u}{t}} \tilde{\Psi}(u) \]

\[ \mathbb{E}(\tilde{\Psi}(u)\tilde{\Psi}(u')) = \delta(u - u') \]

- Random polynomials: \( K_n(x) = a_0 + \sum_{i=1}^{n} a_i x^i \)

\[ K_n(1 - 1/t) \sim a_0 + \sum_{i=1}^{n} i^{d-2} e^{-\frac{i}{4 t}} a_i \]

\[ \sim \int_{0}^{n} du \ u^{d-2} e^{-\frac{u}{t}} a(u) \]

\[ \mathbb{E}(a(u)a(u')) = \delta(u - u') \]