Stolarsky-type identities, energy optimization, uniform tessellations, and one-bit sensing

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Optimal and random point configurations:
From Statistical Physics to Approximation Theory

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Good point distributions

- Lattices
- Energy minimization, polarization
- Monte-Carlo
- Other random point processes (jittered sampling, determinantal)
- Covering/packing problems
- Low-discrepancy sets
- Cubature formulas
- Uniform tessellation, almost isometric embeddings
Discrepancy

- $U$: a set with a natural probability measure $\mu$
  (e.g., $[0, 1]^d$, $S^d$, $\mathbb{R}^d$, etc.)
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• \( \mathcal{A} \) - a collection of subsets of \( U \) (“test sets”, e.g., balls, cubes, convex sets, spherical caps)

• Choose an \( N \)-point set in \( Z \subset U \)

• Discrepancy of \( Z \) with respect to \( \mathcal{A} \):

\[
D_{\mathcal{A}}(Z) = \sup_{A \in \mathcal{A}} \left| \frac{\#(Z \cap A)}{N} - \mu(A) \right|.
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Discrepancy

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- Choose an $N$-point set in $Z \subset U$
- Discrepancy of $Z$ with respect to $A$:

  $$D_A(Z) = \sup_{A \in A} \left| \frac{\#(Z \cap A)}{N} - \mu(A) \right|.$$  

- Optimal discrepancy wrt $A$:

  $$D_N(A) = \inf_{\#Z=N} D_A(Z).$$
Discrepancy

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$$D_\mathcal{A}(Z) = \sup_{A \in \mathcal{A}} \left| \frac{\#(Z \cap A)}{N} - \mu(A) \right|.$$  

- Optimal discrepancy wrt $\mathcal{A}$:

$$D_N(\mathcal{A}) = \inf_{\#Z=N} D_\mathcal{A}(Z).$$

- sup $\rightarrow L^2$-average: $L^2$ discrepancy.
For $x \in \mathbb{S}^d$, $t \in [-1, 1]$ define spherical caps:

$$C(x, t) = \{y \in \mathbb{S}^d : \langle x, y \rangle \geq t \}.$$ 

For a finite set $Z = \{z_1, z_2, ..., z_N\} \subset \mathbb{S}^d$ define

$$D_{cap}(Z) = \sup_{x \in \mathbb{S}^d, t \in [-1, 1]} \left| \frac{\#(Z \cap C(x, t))}{N} - \sigma(C(x, t)) \right|.$$ 

**Theorem (Beck, ’84)**

There exists constants $c_d, C_d > 0$ such that

$$c_d N^{-\frac{1}{2} - \frac{1}{2d}} \leq \inf_{\#Z = N} D_{cap}(Z) \leq C_d N^{-\frac{1}{2} - \frac{1}{2d}} \sqrt{\log N}.$$
Define the spherical cap $L^2$ discrepancy

\[
D_{\text{cap},L^2}(Z) = \left( \int_{S^d} \int_{-1}^{1} \left| \frac{\#(Z \cap C(x, t))}{N} - \sigma(C(x, t)) \right|^2 dt \, d\sigma(x) \right)^{\frac{1}{2}}.
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**Theorem (Stolarsky invariance principle)**

*For any finite set $Z = \{z_1, \ldots, z_N\} \subset S^d$*
Spherical caps: $L^2$ Stolarsky Principle

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Theorem (Stolarsky invariance principle)

For any finite set \( Z = \{ z_1, \ldots, z_N \} \subset S^d \)

\[
\frac{1}{N^2} \sum_{i,j=1}^{N} \| z_i - z_j \| + c_d \left[ D_{L^2,\text{cap}} \right]^2 = \text{const}
\]

\[
= \int_{S^d} \int_{S^d} \| x - y \| \, d\sigma(x) \, d\sigma(y).
\]
Define the spherical cap \( L^2 \) discrepancy

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D_{cap,L^2}(Z) = \left( \int_{S^d} \int_{-1}^{1} \left| \frac{\#(Z \cap C(x,t))}{N} - \sigma(C(x,t)) \right|^2 \, dt \, d\sigma(x) \right)^{1/2}.
\]

**Theorem (Stolarsky invariance principle)**

*For any finite set \( Z = \{z_1, \ldots, z_N\} \subset S^d* *

\[
c_d \left[ D_{cap,L^2}(Z) \right]^2 = \int_{S^d} \int_{S^d} \|x - y\| \, d\sigma(x) \, d\sigma(y) - \frac{1}{N^2} \sum_{i,j=1}^{N} \|z_i - z_j\|.
\]
Spherical caps: $L^2$ Stolarsky Principle

Define the spherical cap $L^2$ discrepancy

$$D_{cap,L^2}(Z) = \left( \int_{S^d} \int_{-1}^1 \left| \frac{\#(Z \cap C(x,t))}{N} - \sigma(C(x,t)) \right|^2 dt \, d\sigma(x) \right)^{\frac{1}{2}}.$$ 

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- Proofs: Stolarsky (’73), Brauchart, Dick (’12), DB (’16).
Define the spherical cap discrepancy of fixed height $t$:

$$D_{L^2, \text{cap}}(Z) := \left( \int_{S^d} \left| \frac{1}{N} \sum_{j=1}^{N} 1_{C(x,t)}(z_j) - \sigma(C(x,t)) \right|^2 \, d\sigma(x) \right)^{1/2}$$
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- 

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- Using this definition, we have:

$$\left[ D_{L^2, \text{cap}}^{(t)}(Z) \right]^2 = \frac{1}{N^2} \sum_{i,j=1}^{N} \sigma(C(z_i,t) \cap C(z_j,t)) - (\sigma(C(p,t)))^2.$$

- Averaging over $t \in [-1, 1]$

$$\int_{-1}^{1} \sigma(C(x,t) \cap C(y,t)) \, dt = 1 - C_d \|x - y\|$$

$$\int_{-1}^{1} (\sigma(C(p,t)))^2 \, dt = 1 - C_d \int_{\mathbb{S}^d} \int_{\mathbb{S}^d} \|x - y\| \, d\sigma(x) d\sigma(y).$$
$L^2$ discrepancy for spherical cap discrepancy of fixed height $t$ satisfies:

$$\left[ D_{L^2, \text{cap}}^{(t)}(Z) \right]^2 = \frac{1}{N^2} \sum_{i,j=1}^{N} \sigma(C(z_i, t) \cap C(z_j, t)) - \left( \sigma(C(p, t)) \right)^2.$$
Hemisphere discrepancy

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- Taking $t = 0$ (i.e. hemispheres)
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Theorem (Stolarsky for hemispheres, DB '16, Skriganov '16)

$$\left[D_{L^2,\text{hem}}(Z)\right]^2 = \left[D_{L^2,\text{cap}}^{(0)}(Z)\right]^2$$

$$= \frac{1}{2} \left( \int \int_{S^d \times S^d} d(x, y) \, d\sigma(x) \, d\sigma(y) - \frac{1}{N^2} \sum_{i,j=1}^{N} d(z_i, z_j) \right).$$
\[ [D_{L^2, \text{hem}}(Z)]^2 = \frac{1}{2} \left( \frac{1}{2} - \frac{1}{N^2} \sum_{i,j=1}^{N} d(z_i, z_j) \right). \]
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- For any \( Z = \{z_1, \ldots, z_N\} \subset S^d \)
  \[
  \frac{1}{N^2} \sum_{i,j=1}^{N} d(z_i, z_j) \leq \frac{1}{2}
  \]
Hemisphere Stolarsky: simple corollaries

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- For even \( N \):

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- For odd \( N \) the maximal value is
  \[
  \frac{1}{N^2} \sum_{i,j=1}^{N} d(z_i, z_j) = \frac{1}{2} - \frac{1}{2N^2}.
  \]
Fejes-Toth ’59: $d = 1$ and conjectured for $d \geq 2$.
Sperling, ’60 (even $N$)
Larcher, ’61 (odd $N$)
Hemisphere Stolarsky for general measures

Let $\mu$ be a probability measure on $\mathbb{S}^d$. Define the geodesic distance energy integral

$$I_g(\mu) = \int_{\mathbb{S}^d} \int_{\mathbb{S}^d} d(x, y) \, d\mu(x) d\mu(y).$$

Then the following version of the Stolarsky principle holds:

$$\int_{\mathbb{S}^d} \left( \mu(H(x)) - \frac{1}{2} \right)^2 \, d\sigma(x) = \frac{1}{2} \cdot \left( I_g(\mu) \right).$$

For any probability measure $\mu$:

$$I_g(\mu) \leq \frac{1}{2}.$$

$I_g(\mu) = \frac{1}{2}$ (i.e. $\mu$ is a maximizer) iff $\mu(H(x)) = \frac{1}{2}$ for $\sigma$-a.e. $x \in \mathbb{S}^d$ iff $\mu$ is symmetric, i.e. $\mu(E) = \mu(-E)$. 
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Let $H(x) = C(x, 0)$ denote the hemisphere with center at $x$. Then the following version of the Stolarsky principle holds:

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Points on a sphere
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- $I_g(\mu) = \frac{1}{2}$ (i.e. $\mu$ is a maximizer) iff $\mu(H(x)) = \frac{1}{2}$ for $\sigma$-a.e. $x \in \mathbb{S}^d$ iff $\mu$ is symmetric, i.e. $\mu(E) = \mu(-E)$. 
Let $\mu$ be a Borel probability measure on $\mathbb{S}^d$. Then

$$I_E(\mu) = \int_{\mathbb{S}^d} \int_{\mathbb{S}^d} \|x - y\| \, d\mu(x)d\mu(y)$$

has a unique maximizer $\mu = \sigma$ (Bjorck, ’56)
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However,

$$I_g(\mu) = \int_{\mathbb{S}^d} \int_{\mathbb{S}^d} d(x, y) \, d\mu(x) \, d\mu(y)$$

is maximized by any symmetric measure $\mu$. 
Let $\mu$ be a Borel probability measure on the sphere $S^d$. For $\lambda > 0$ define the energy integral

$$I_\lambda = \int_{S^d} \int_{S^d} |x - y|^\lambda d\mu(x) d\mu(y)$$

Maximizers (Bjorck ’56):

- $0 < \lambda < 2$: unique maximizer is surface measure,
- $\lambda = 2$: any measure with center of mass at 0,
- $\lambda > 2$: mass $\frac{1}{2}$ at two opposite poles.
Let $\mu$ be a Borel probability measure on the sphere $\mathbb{S}^d$. For $\lambda > 0$ define the energy integral

$$I_\lambda = \int_{\mathbb{S}^d} \int_{\mathbb{S}^d} (d(x, y))^\lambda d\mu(x) d\mu(y)$$

Maximizers (DB, F. Dai ’16):

- $0 < \lambda < 1$: unique maximizer is $\sigma$,
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$d = 1$: Brauchart, Hardin, Saff, ’12
Let $x, y \in \mathbb{S}^d$.
choose a random hyperplane $z^\perp$, $z \in \mathbb{S}^d$. 

\[ P(z^\perp \text{separates } x \text{ and } y) = d(x, y), \]
where $d$ is the normalized geodesic distance on the sphere, i.e.
\[ d(x, y) = \cos^{-1} \left( \frac{\langle x, y \rangle}{\|x\| \|y\|} \right). \]
Let \( x, y \in \mathbb{S}^d \)
choose a random hyperplane \( z^\perp, z \in \mathbb{S}^d \).

Then

\[
\mathbb{P}(z^\perp \text{ separates } x \text{ and } y) = \mathbb{P}(\text{sign}(z, x) \neq \text{sign}(z, y)) = d(x, y),
\]

where \( d \) is the normalized geodesic distance on the sphere, i.e.

\[
d(x, y) = \frac{\cos^{-1}\langle x, y \rangle}{\pi}.
\]
Hamming distance

Consider a set of vectors $Z = \{z_1, z_2, \ldots, z_N\}$ on the sphere $S^d$. Define the Hamming distance as

$$d_H(x, y) := \frac{\#\{z_k \in Z : \text{sign}(x \cdot z_k) \neq \text{sign}(y \cdot z_k)\}}{N},$$

i.e. the proportion of hyperplanes $z_k^\perp$ that separate $x$ and $y$.

In other words,

$$d_H(x, y) = \frac{1}{2N} \cdot \|\phi_Z(x) - \phi_Z(y)\|_1,$$

where $\phi_Z : S^d \to \mathcal{H}^N = \{-1, +1\}^N \subset \mathbb{R}^N$ is given by

$$\phi_Z(x) = \{\text{sign}(x \cdot z_k)\}_{k=1}^N = \text{sign}(Zx).$$
Define

$$\Delta_Z(x, y) := d_H(x, y) - d(x, y).$$
The main question

Define
\[ \Delta_Z(x, y) := d_H(x, y) - d(x, y). \]

Let \( K \subset S^d \). We say that \( Z \) induces a \( \delta \)-uniform tessellation of \( K \) if
\[ \sup_{x, y \in K} |\Delta_Z(x, y)| \leq \delta. \]
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Examples of \( K \):

- \( K = S^d \)
- \( K \) finite
- sparse vectors
Definition

Let $X$, $Y$ be metric spaces. A $\delta$-isometric embedding of $X$ into $Y$ (a $\delta$-RIP map) is a map $f : X \to Y$ such that for each $x, y \in X$

$$|d_X(x, y) - d_Y(f(x), f(y))| \leq \delta.$$
Motivation: almost isometric embeddings

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is a $\delta$-RIP map from $K$ into the Hamming cube $\mathcal{H}^{N} = \{-1, 1\}^{N}$. 

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**Question:** Given $K \subset S^d$ and $\delta > 0$, what is the smallest value of $N$ so that $K$ can be $\delta$-isometrically embedded into $H_N$?

**Prior results:**

Plan, Vershynin, '13: $N = C\delta^{-6} \omega(K)^2$ random points yield a $\delta$-uniform tessellation of $K$ with high probability.
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Plan, Vershynin, ’13: $N = C\delta^{-6} \omega(K)^2$ random points yield a $\delta$-uniform tessellation of $K$ with high probability.
Motivation: cells with small diameter

Lemma

Every cell of a $\delta$-uniform tessellation of $K$ by hyperplanes has diameter at most $\delta$.

Picture from Baraniuk, Foucart, Needell, Plan, Wooters

Points on a sphere
Motivation: cells with small diameter

Lemma

Every cell of a δ-uniform tessellation of $K$ by hyperplanes has diameter at most $\delta$.

Proof:
if $x$ and $y$ are in the same cell then

$$d(x, y) = |d(x, y) - d_H(x, y)| \leq \delta.$$
Motivation: one-bit compressed sensing

- Let $x \in K \subset S^{n-1} \subset \mathbb{R}^n$ represent a signal.
- $\langle x, z_k \rangle$ are linear measurements, $k = 1, \ldots, m$, $m \ll n$. 

Jaques, Laska, Boufounos, Baraniuk: embeddings to Hamming cube through $\phi_Z(x) = \text{sign}(Zx)$.
Motivation: one-bit compressed sensing

- Let $x \in K \subset S^{n-1} \subset \mathbb{R}^n$ represent a signal.
- $\langle x, z_k \rangle$ are linear measurements, $k = 1, \ldots, m$, $m \ll n$.
- $\text{sign} \langle x, z_k \rangle$ are quantized linear measurements.
Motivation: one-bit compressed sensing

- Let $x \in K \subset \mathbb{S}^{n-1} \subset \mathbb{R}^n$ represent a signal.
- $\langle x, z_k \rangle$ are linear measurements, $k = 1, \ldots, m$, $m \ll n$.
- $\text{sign}\langle x, z_k \rangle$ are quantized linear measurements.
- Can one reconstruct/approximate $x$ from these measurements?
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Points on a sphere
Let $\gamma$ be the standard Gaussian vector in $\mathbb{R}^{d+1}$. The Gaussian mean width of $K$ is defined as

$$\omega(K) = \mathbb{E} \sup_{x,y \in K} |\langle \gamma, x-y \rangle|.$$
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“Hemisphere” process: mean zero Gaussian process with $\mathbb{E}G_x^2 = \frac{1}{4}$ with increments

$$(\mathbb{E}|G_x - G_y|^2)^{1/2} = \|1_{H(x)} - 1_{H(y)}\|_2 = \sqrt{d(x, y)},$$

where $H(x)$ is the hemisphere $H(x) = \{z \in \mathbb{S}^d : z \cdot x > 0\}$. Hemisphere mean width

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Mean Gaussian width and “hemisphere” width

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- Sudakov’s inequality:
  $$\sqrt{\log N(K, \delta)} \lesssim \begin{cases} \delta^{-1} \omega(K) \\ \delta^{-1/2} H(K) \end{cases}$$
Main results (DB, Lacey, ’15-16)

- **Small cells:** If $m \gtrsim \delta^{-1} \log N(K, c\delta)$, then w.h.p. $m$ random vectors induce a tessellation with $\delta$-small cells.
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- **Sparse case:** Let $K_s$ be the set of $s$-sparse vectors in $\mathbb{S}^d$. If $m \gtrapprox \delta^{-2} s \log_+ \frac{d}{s}$, then for a random set $Z$ of $m$ points in $\mathbb{S}^d$ w.h.p. we have 
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- **One-bit Johnson-Lindenstrauss lemma:** If $K$ is finite and $m \gtrsim \delta^{-2} \log(\#K)$, then there exists a $\delta$-isometry between $K \subset \mathbb{S}^d$ and the Hamming cube $\mathcal{H}^m$. 
Tessellations and discrepancy

\[ H_x = \{ z : \langle z, x \rangle > 0 \} \]

\[ W_{xy} = H_x \triangle H_y \]
\[ = \{ z \in S^d : \text{sign} \langle z, x \rangle \neq \text{sign} \langle z, y \rangle \} \]
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\[ \Delta_Z(x, y) = d_H(x, y) - d(x, y) = \frac{\#(Z \cap W_{xy})}{N} - \sigma(W_{xy}) \]

\[ D_{\text{wedge}}(Z) = \| \Delta_Z(x, y) \|_\infty = \sup_{x, y \in S^d} \left| \frac{\#(Z \cap W_{xy})}{N} - \sigma(W_{xy}) \right| \].
Lemma

There exists an \( N \)-point set \( Z \subset \mathbb{S}^d \) with

\[
D_{\text{wedge}}(Z) \leq C_d N^{-\frac{1}{2}} - \frac{1}{2d} \sqrt{\log N}.
\]
Lemma

There exists an $N$-point set $Z \subset \mathbb{S}^d$ with

$$D_{\text{wedge}}(Z) \leq C_d N^{-\frac{1}{2} - \frac{1}{2d}} \sqrt{\log N}.$$ 

Corollary

This implies that for $\delta > 0$ there exists a $\delta$-uniform tessellation of $\mathbb{S}^d$ by $N$ hyperplanes with

$$N \leq C'_d \delta^{-2 + \frac{2}{d+1}} \cdot \left( \log \frac{1}{\delta} \right)^{\frac{d}{d+1}}.$$
Lemma (Blümlinger, 1991)

For any $N$-point set $Z \subset S^d$

$$D_{\text{slice}}(Z) \gtrsim N^{-\frac{1}{2}} - \frac{1}{2d},$$

where $D_{\text{slice}}$ is the spherical discrepancy with respect to “slices” $S_{xy} = \{ z \in S^d : \langle z, x \rangle > 0 \ \& \ \langle z, y \rangle > 0 \}$. 
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- Symmetrization can adapt this result to wedges $W_{xy}$, i.e. to $D_{\text{wedge}}(Z)$.

Corollary

This implies that for any $\delta > 0$, if there exists a $\delta$-uniform tessellation of $S^d$ by $N$ hyperplanes, then

$$N \geq c_d \delta^{-2+\frac{2}{d+1}}.$$
There exist constants $c_d, C_d$, such that the following discrepancy bounds hold:

$$c_d N^{-\frac{1}{2} - \frac{1}{2d}} \leq \inf_{Z \subseteq S^d: \#Z = N} \Delta(Z) \leq C_d N^{-\frac{1}{2} - \frac{1}{2d}} \sqrt{\log N}.$$  

Inverting this we find that the optimal value of $N$ satisfies

$$\delta^{-2 - \frac{2}{d+1}} \leq N \leq \delta^{-2 - \frac{2}{d+1}} \left( \log \frac{1}{\delta} \right)^{\frac{d}{d+1}}.$$
Define the $L^2$ discrepancy for wedges

$$[D_{L^2,\text{wedge}}(Z)]^2 = \int_{S^d} \int_{S^d} \left( \frac{1}{N} \sum_{k=1}^{N} 1_{W_{xy}(z_k)} - \sigma(W_{xy}) \right)^2 d\sigma(x) d\sigma(y)$$
Stolarsky principle for wedge discrepancy

Define the $L^2$ discrepancy for wedges

$$\left[ D_{L^2, \text{wedge}} (Z) \right]^2 = \int_{S^d} \int_{S^d} \left( \frac{1}{N} \sum_{k=1}^{N} 1_{W_{xy}} (z_k) - \sigma (W_{xy}) \right)^2 d\sigma(x) d\sigma(y)$$

Theorem (Stolarsky for wedges, DB, Lacey, ’15)

For any finite set $Z = \{z_1, \ldots, z_N\} \subset S^d$

$$\left[ D_{L^2, \text{wedge}} (Z) \right]^2 = \frac{1}{N^2} \sum_{i,j=1}^{N} \left( \frac{1}{2} - d(z_i, z_j) \right)^2 \int_{S^d} \int_{S^d} \left( \frac{1}{2} - d(x, y) \right)^2 d\sigma(x) d\sigma(y).$$
Frame potential

- \( Z = \{ z_1, \ldots, z_N \} \subset S^d \) is a frame in \( \mathbb{R}^d \) iff there exist \( c, C > 0 \) such that for any \( x \in \mathbb{R}^{d+1} \)

\[
c \| x \|^2 \leq \sum_k |\langle x, z_k \rangle|^2 \leq C \| x \|^2.
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Frame potential

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- $Z = \{z_1, \ldots, z_N\} \subset S^d$ is a tight frame iff there exists $A > 0$ such that for any $x \in \mathbb{R}^{d+1}$
  \[ \sum_k |\langle x, z_k \rangle|^2 = A\|x\|^2. \]
• $Z = \{z_1, \ldots, z_N\} \subset S^d$ is a frame in $\mathbb{R}^d$ iff there exist $c, C > 0$ such that for any $x \in \mathbb{R}^{d+1}$

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$$\sum_k |\langle x, z_k \rangle|^2 = A \|x\|^2.$$

**Theorem (Benedetto, Fickus)**

A set $Z = \{z_1, \ldots, z_N\} \subset S^d$ is a tight frame in $\mathbb{R}^{d+1}$ if and only if $Z$ is a local minimizer of the frame potential:

$$F(Z) = \sum_{i,j=1}^N |\langle z_i, z_j \rangle|^2.$$
Stolarsky principle for slices

Define the $L^2$ discrepancy for slices

$S_{xy} = \{ z \in \mathbb{S}^d : \langle z, x \rangle > 0 \ \& \ \langle z, y \rangle > 0 \}$

$$[D_{L^2, \text{slice}}(Z)]^2 = \int_{\mathbb{S}^d} \int_{\mathbb{S}^d} \left( \frac{1}{N} \sum_{k=1}^{N} 1_{S_{xy}}(z_k) - \sigma(S_{xy}) \right)^2 d\sigma(x) d\sigma(y)$$
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Define the $L^2$ discrepancy for slices

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S_{xy} = \{ z \in \mathbb{S}^d : \langle z, x \rangle > 0 \ \& \ \langle z, y \rangle > 0 \}
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\[
[D_{L^2, \text{slice}}(Z)]^2 = \int_{\mathbb{S}^d} \int_{\mathbb{S}^d} \left( \frac{1}{N} \sum_{k=1}^{N} 1_{S_{xy}}(z_k) - \sigma(S_{xy}) \right)^2 d\sigma(x) d\sigma(y)
\]

**Theorem (Stolarsky for slices, DB, ’16)**

*For any finite set $Z = \{z_1, \ldots, z_N\} \subset \mathbb{S}^d$*

\[
4[D_{L^2, \text{slice}}(Z)]^2 = 
\frac{1}{N^2} \sum_{i,j=1}^{N} (1 - d(z_i, z_j))^2 - \int_{\mathbb{S}^d} \int_{\mathbb{S}^d} (1 - d(x, y))^2 d\sigma(x) d\sigma(y).
\]