Recent advances on log gases, IHP
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Large-N asymptotic expansions in 1-d repulsive particle systems

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based on joint works with
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1. Model and results

2. Schwinger-Dyson equations

3. Sketch of proof of the main result

4. Conclusion
The $\beta$ ensembles

- Probability measure on $A^N \subseteq \mathbb{R}^N$

$$d\mu_N^A = \frac{1}{Z_N^A} \exp \left( N \sum_{i=1}^{N} T(\lambda_i) \right) \prod_{1 \leq i < j \leq N} |\lambda_i - \lambda_j|^\beta \prod_{i=1}^{N} 1_A(\lambda_i) d\lambda_i \quad \beta > 0$$

- It is the measure induced on eigenvalues of a random matrix $M$

$$dM e^{N \text{Tr} T(M)} \begin{cases} 
\beta = 1 & \text{real symmetric matrices} \\
\beta = 2 & \text{hermitian matrices} \\
\beta = 4 & \text{quaternionic self-dual matrices}
\end{cases}$$

Wigner, Dyson, Mehta (50s-60s)

$M = \text{triagonal}$ all $\beta > 0$, $T$ polynomial of even degree

Dumitriu, Edelman ’02
Krishnapur, Rider, Virág ’13
Mean-field models

- Probability measure on $A^N \subseteq \mathbb{R}^N$

$$d\mu_N = \frac{1}{Z_N} \exp \left( N^2 \mathcal{T}_0(\lambda_N) \right) \prod_{1 \leq i < j \leq N} |\lambda_i - \lambda_j|^{\beta} \prod_{i=1}^N 1_{A}(\lambda_i) d\lambda_i \quad \beta > 0$$

where $L_N^{(\lambda)} = \frac{1}{N} \sum_{i=1}^N \delta_{\lambda_i}$ is the (random) empirical measure

- Exemples

  in Chern-Simons theory

  $$\mathcal{T}_0(\mu) = \int \int d\mu(x) d\mu(y) \sum_m \beta_m \ln \left| \frac{\sinh[\alpha_m(x-y)]}{\alpha_m(x-y)} \right|$$

  O(n) model on random lattices

  $$\mathcal{T}_0(\mu) = -\frac{n}{2} \int \int d\mu(x) d\mu(y) \ln |x+y|$$

- Here, we take

  $$\mathcal{T}_0(\mu) = \int T(x_1, \ldots, x_r) \prod_{i=1}^r d\mu(x_i)$$

  $T$ real-analytic on $A^r$
We would like to study when $N \to \infty$ ...

- the (random) empirical measure $L_N^{(\lambda)} = \frac{1}{N} \sum_{i=1}^{N} \delta_{\lambda_i}$

- what kind of random variable is $\sum_{i=1}^{N} f(\lambda_i) = N \int f(\xi) dL_N^{(\lambda)}(\xi)$?

- the partition function

$$Z_N = \int_{A^N} \exp \left( N^2 \mathcal{T}_0(L_N^{(\lambda)}) \right) \prod_{1 \leq i < j \leq N} |\lambda_i - \lambda_j|^\beta \prod_{i=1}^{N} d\lambda_i$$

- the k-point correlators

$$W_k(x_1, \ldots, x_k) = \text{Cumulant} \left( \int \frac{N \, dL_N^{(\lambda)}(\xi_1)}{x_1 - \xi_1}, \ldots, \int \frac{N \, dL_N^{(\lambda)}(\xi_k)}{x_k - \xi_k} \right)$$
The leading order ... is given by a continuous approximation

- Define the energy functional on a proba. measure $\mu$

$$\mathcal{T}(\mu) = \int \left[ \prod_{i=1}^{r} d\mu(x_i) \right] T(x_1, \ldots, x_r) + \frac{\beta}{2} \iint d\mu(x_1) d\mu(x_2) \ln |x_1 - x_2|$$

- Assumption 1: uniqueness of maximizer $\mu_{eq}$

- Characterization: exists a constant $C$ such that $\mathcal{T}'(\mu_{eq})[\delta_x] \leq C$
  
  for $x \in A$ $\mu_{eq}$-everywhere

- Assumption 2: local strict concavity at $\mu_{eq}$

  for any $\nu = \text{finite signed measure of mass } 0$

  $$-\mathcal{T}''(\mu_{eq})[\nu, \nu] = \mathfrak{D}^2[\nu] \in [0, +\infty]$$

  and $= 0$ iff $\nu = 0$
The leading order ... is given by a continuous approximation

- Define the energy functional on a proba measure $\mu$

$$\mathcal{T}(\mu) = \int \left[ \prod_{i=1}^{r} d\mu(x_i) \right] T(x_1, \ldots, x_r) + \frac{\beta}{2} \iint d\mu(x_1)d\mu(x_2) \ln |x_1 - x_2|$$

- Assumption 1: uniqueness of maximizer $\mu_{eq}$
- Assumption 2: local strict concavity at $\mu_{eq}$

**Lemma**

$$L_N^{(\lambda)} \rightarrow \mu_{eq} \quad \text{almost surely and in expectation}$$

$$Z_N = \exp \left\{ N^2 \left( \mathcal{T}(\mu_{eq}) + o(1) \right) \right\}$$
A particle at position $x$ feels the effective potential

$$J(x) = \mathcal{T}'(\mu_{eq})[\delta_x] - \sup_{\xi \in A} \mathcal{T}'(\mu_{eq})[\delta_{\xi}]$$

**Lemma**

For any closed $F \subseteq A$

$$\mathbb{P}[\exists i, \lambda_i \in F] \leq \exp\left\{ N \left( \sup_{x \in F} J(x) + o(1) \right) \right\}$$

One can restrict to a compact $B \subseteq A$ neighborhood of $\{J(x) = 0\}$

$$Z_N^B = Z_N^A \left( 1 + o(e^{-cN}) \right)$$
Large deviations of empirical measure

- Natural “distance” \(-\mathcal{T}''(\mu_{eq})[\nu, \nu] = \mathcal{D}^2[\nu] \in [0, +\infty]\) but \(\mathcal{D}[L_N^{(\lambda)} - \mu_{eq}] = +\infty\) because of atoms and log singularity

- Let us pick a nice regularization \(\text{idea from Maïda, Maurel-Segala}\)

\[
L_N^{(\lambda)} = \frac{1}{N} \sum_{i=1}^{N} \delta_{\lambda_i} \sim \widetilde{L}_N^{(\lambda)}
\]

**Lemma**

If \(T\) is smooth, we have for \(N\) large enough

\[
\mathbb{P}_N [\mathcal{D}[\tilde{L}_N^{(\lambda)} - \mu_{eq}] > t] \leq \exp \left( N \ln N - N^2 t^2 / 2 \right)
\]
The equilibrium measure

- $T$ real-analytic $\implies \left\{ \begin{array}{l} \mu_{eq} \text{ is supported on a finite number of segments} \\
S = \bigcup_{h=0}^{g} [a_h, b_h] \end{array} \right.$

- $\alpha \in \partial S$ is a hard edge if $\alpha \in \partial A$, is a soft edge otherwise

\[
d\mu_{eq}(x) = \frac{1_{S(x)}dx}{2\pi} M(x) \prod_{\alpha \text{ soft}} |x - \alpha|^{1/2} \prod_{\alpha \text{ hard}} |x - \alpha|^{-1/2}
\]

- We say that $\mu_{eq}$ is off-critical when $M(x) > 0$ on $A$
Finite size corrections: we assume ...

- Uniqueness of maximizer $\mu_{eq}$
- Local strict concavity at $\mu_{eq}$

\[ V = V_0 + (1/N)V_1 + \cdots \]
\[
\begin{cases} 
  V_0 & \text{real analytic on } A \\
  V_1 & \text{complex analytic on } A 
\end{cases}
\]

- Control of large deviations $J(x) < 0$ for $x \in A \setminus S$
- $\mu_{eq}$ is off-critical
- $f$ = test function, analytic on $A$
Result in the 1-cut regime

- **1/N asymptotic expansion**

\[
Z_N = N^{\gamma N + \gamma'} \exp \left[ \sum_{m \geq -2} N^{-m} F[m] + O(N^{-\infty}) \right]
\]

\[\gamma, \gamma' \text{ depend only on } \beta \text{ and the nature of the edges}\]

- **Central limit theorem**

\[
\left( \sum_{i=1}^{N} f(\lambda_i) - N \int_{A} f(\xi) d\mu_{eq}(\xi) \right) \longrightarrow \text{ (non-centered) gaussian}
\]
Result in the \((g + 1)\)-cuts regime

- Oscillatory asymptotic expansion

\[
Z_N = N^{\gamma N + \gamma'} (\mathcal{D}_N \Theta_{-N\epsilon_{eq}})(F^{[-1]'}|F^{[2]''}) \exp \left[ \sum_{m \geq -2} N^{-m} F^m + O(N^{-\infty}) \right]
\]

where \(\mathcal{D}_N = \sum_{p \geq 0} \frac{1}{p!} \sum_{\ell_1, \ldots, \ell_p \geq 1 \atop m_1, \ldots, m_p \geq -2 \atop \sum_i (m_i + \ell_i) > 0} N^{-\sum_i (m_i + \ell_i)} \prod_{i=1}^p \frac{F_{\epsilon_{eq}|i}(\ell_i)}{\ell_i!} \cdot \nabla \otimes \ell_i \)

acts as a differential operator on the Siegel theta function

\[
\Theta_{\mu}(w|Q) = \sum_{m \in \mathbb{Z}^g} e^{w \cdot (m+\mu) + \frac{1}{2} (m+\mu) \cdot Q \cdot (m+\mu)}
\]

- (Pseudo)-periodicity come from \(\mu = -N\epsilon_{eq} \mod \mathbb{Z}^g\)
Result in the \((g + 1)\)-cuts regime

- No central limit theorem in general ...

\[
\mathbb{E}\left[e^{is\left(\sum_{i=1}^{N} f(\lambda_i) - N \int f(x) d\mu_{eq}(x)\right)}\right] \sim e^{is m_1[f] - m_2[f] s^2/2} \frac{\Theta - N\epsilon_{eq}(F^{[-1]'} + is v[f] | F^{[-2]''})}{\Theta - N\epsilon_{eq}(F^{[-1]'} | F^{[-2]''})}
\]

(non-centered) gaussian

+ discrete Gaussian, centered at \(\mu = -N\epsilon_{eq} \mod \mathbb{Z}^g\)

step \(v[f] \propto \left(\int_{S} \frac{f(x) x^i dx}{\prod_{\alpha} |x - \alpha|^{1/2}}\right)_{0 \leq i \leq g-1}\)

**Corollary**

\[
\left(\sum_{i=1}^{N} f(\lambda_i) - N \int_{A} f(\xi) d\mu_{eq}(\xi)\right)
\]

converges in law along subsequences
History of $\beta$ ensembles : 1-cut regime

$\beta = 2$
- If $1/N$ expansion exists, then
  \[ Z_N = N^{\gamma N + \gamma'} \exp \left[ \sum_{m \geq -1} N^{-2m} F^{\{m\}} \right] \]
  and $F^{\{m\}}$ can be computed by the moment method
  \textit{Ambjørn, Chekhov, Kristjansen, Makeenko, 90s}

- Rewriting of $F^{\{m\}}$ in terms of a universal topological recursion
  \textit{Eynard, ’04}

- Existence of $1/N$ expansion by
  - analysis of SD equations \textit{Albeverio, Pastur, Shcherbina ’01}
  - RH techniques \textit{Ercolani, McLaughlin ’02}
  - analysis of int. system \textit{Bleher, Its, ’05}
History of $\beta$ ensembles: 1-cut regime

$\beta > 0$  
- If $1/N$ expansion exists, then $Z_N = N^{\gamma N + \gamma'} \exp \left[ \sum_{m \geq -2} N^{-m} F[m] \right]$  
  and $F[m]$ computed by a $\beta$-topological recursion  
  *Chekhov, Eynard ’06*

- Central limit theorem  
  *Johansson ’98*

- Existence of $1/N$ expansion (analysis of SD eqn)  
  *Borot, Guionnet ’11*
History of $\beta$ ensembles: multi-cut regime

$\beta = 2$
- numerous observations of oscillatory behavior
  physicists, ‘90s
- Riemann-Hilbert techniques up to o(1)
  Deift, Kriecherbauer, McLaughlin, Venakides, Zhou, ...
- heuristic derivation up to o(1)
  Bonnet, David, Eynard ’00
- generalization to all orders
  Eynard ’07
- observation of “no CLT”
  Pastur ’06

$\beta > 0$
- Proof of “no CLT” and asymptotics of $Z_N^A$ up to o(1)
  Shcherbina ’12
- General proof
  Borot, Guionnet ’13
History of mean-field models

\[ \text{d}\mu_N = \frac{1}{Z_N} \exp \left( N^2 \mathcal{T}_0(L_N^{(\lambda)}) \right) \prod_{1 \leq i < j \leq N} |\lambda_i - \lambda_j|^\beta \prod_{i=1}^{N} 1_A(\lambda_i) \text{d}\lambda_i \]

with r-body interaction \( \mathcal{T}_0(\mu) = \int T(x_1, \ldots, x_r) \prod_{i=1}^{r} \text{d}\mu(x_i) \)

- same results for mean field models
  Borot, Guionnet, Kozlowski ’13
- computation of expansion by topological recursion
  Borot, ’13
Large-N asymptotic expansions in 1-d repulsive particle systems

1. Model and results

2. Schwinger-Dyson equations

3. Sketch of proof of the main result

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What are Schwinger-Dyson equations?

= relations between expectation values from integration by parts

- In the model  
  \[ d\mu_N = \frac{1}{Z_N} \exp\left(N^2 T_0(L_N^{(\lambda)})\right) \prod_{1 \leq i < j \leq N} |\lambda_i - \lambda_j|^{\beta} \prod_{i=1}^{N} 1_A(\lambda_i) d\lambda_i \]

we find for any smooth test function \( h \) and smooth functional \( O \)

\[
\mathbb{E}\left[ \left( \sum_i N h(\lambda_i) T'_0(L_N^{(\lambda)})[\delta\lambda_i] + \beta \sum_{i<j} \frac{h(\lambda_i) - h(\lambda_j)}{\lambda_i - \lambda_j} + \sum_i h'(\lambda_i) \right) O(L_N^{(\lambda)}) \right] \\
+ \sum_i N^{-1} h(\lambda_i) O'(L_N^{(\lambda)})[\delta\lambda_i] \right] + \text{boundary} = 0
\]
What are Schwinger-Dyson equations?

- Remind the k-points correlators

\[ W_k(x_1, \ldots, x_k) = \text{Cumulant}(\int \frac{N \, dL^{(\lambda)}_N(\xi_1)}{x_1 - \xi_1}, \ldots, \int \frac{N \, dL^{(\lambda)}_N(\xi_k)}{x_k - \xi_k}) \]

- Choose \( h_z(x) = \frac{1}{z - x} \) and \( \mathcal{O}_{z_2, \ldots, z_k}(L^{(\lambda)}_N) = \prod_{i=2}^{k} \int \frac{dL^{(\lambda)}_N(\xi_i)}{z_i - \xi_i} \)

for \( z, z_i \in \mathbb{C} \setminus A \)

\( \rightarrow \) family of functional relations between \( W_1, \ldots, W_{r+k-1} \)

indexed by \( k \geq 1 \)
Decompose \( W_1(z) = N \left( W_{eq}(z) + \delta_{-1} W_1(z) \right) \) with \( W_{eq}(z) = \int \frac{d\mu_{eq}(\xi)}{z - \xi} \)

Schwinger-Dyson equations can be recast

\[(\mathcal{K} + \delta \mathcal{K})[\delta_{-1} W_1](z) = A_1 + \text{boundary}\]

\[(\mathcal{K} + \delta \mathcal{K})[W_n(\cdot, z_2, \ldots, z_n)](z) = A_n + \text{boundary}\]

with: \( \mathcal{K}[f](z) = 2W_{eq}(z)f(z) + \frac{2}{\beta} \mathcal{T}_0(\mu_{eq}) \left[ \frac{f(\lambda)d\lambda}{z - \lambda} \right] \)

\[\delta \mathcal{K}[f](z) = 2\delta_{-1} W_1(z)f(z) + N^{-1}(1 - 2/\beta) \partial_z f(z) + \cdots\]
Asymptotic analysis

- Introduce norms $\|f\|_\Gamma = \sup_{z \in \text{Ext}(\Gamma)} |f(z)|$

- Large deviations of empirical measure

\[
\|N\delta_{-1}W_1\|_{\Gamma_1} \leq C_1 (N \ln N)^{1/2} \\
\|W_k\|_{\Gamma_k} \leq C_k (N \ln N)^{k/2}
\]

- Large deviation of single eigenvalue: boundary effects $\in o(e^{-cN})$

- Rigidity of SD equations: if $\mathcal{K}$ invertible and $\|\mathcal{K}^{-1}[f]\|_{\Gamma_{i+1}} \leq c \|f\|_{\Gamma_i}$

\[
\left\{
\begin{array}{c}
\|N\delta_{-1}W_1\|_{\Gamma_{i_1}} \leq c_1 (\eta_N \kappa_N + 1) \\
\|W_k\|_{\Gamma_{i_k}} \leq c_k (\eta_N^k \kappa_N + N^{2-k})
\end{array}
\right.
\]

\[
\downarrow
\]

\[
\left\{
\begin{array}{c}
\|N\delta_{-1}W_1\|_{\Gamma_{i_1+2}} \leq c_1' (\eta_N (\eta_N/N) \kappa_N + 1) \\
\|W_k\|_{\Gamma_{i_k+2}} \leq c_k' (\eta_N^k (\eta_N/N) \kappa_N + N^{2-k})
\end{array}
\right.
\]
Asymptotic analysis

Large deviations of empirical measure
+ Rigidity of SD equations

**Corollary**

If $\mathcal{K}$ invertible and $\|\mathcal{K}^{-1}[f]\|_{\Gamma_{i+1}} \leq c \|f\|_{\Gamma_i}$ we have, for any $M \geq 0$ an asymptotic expansion

$$W_k = \sum_{m=k-2}^{M-1} N^{-m} W_k^{[m]} + O(N^{-M}; \Gamma_{M,k})$$

- **Remark:**
  
  $(g + 1)$ cuts
  $c = \text{nb. critical conditions}$
  $\dim \ker \mathcal{K} = g + c$
Large-N asymptotic expansions in 1-d repulsive particle systems

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Scheme of the proof

Models with fixed filling fractions

1. Eq. measure and regularity (potential theory)
2. Invertibility of $\mathcal{K}$ (functional + cx analysis)
3. Expansion of correlators
4. Expansion of partition fn.
   interpolation

Initial model (multi-cut regime)

same large deviations estimates

same Schwinger-Dyson equations

5. Expansion of partition fn.
   series analysis
From large deviations on single eigenvalue: up to $o(e^{-cN})$, we can choose

We will study $\mu^{(A_0,\ldots,A_g)}(N_0,\ldots,N_g) = \mu^A_N$ conditioned to have \( \begin{cases} N_0 \text{ first } \lambda \text{'s in } A_0 \\ N_1 \text{ next } \lambda \text{'s in } A_1 \\ \text{etc.} \end{cases} \)

The partition function decomposes

$$Z_N^A = \sum_{N_0+\ldots+N_g=N} \frac{N!}{\prod_{h=0}^g N_h!} Z^{(A_0,\ldots,A_g)}(N_0,\ldots,N_g)$$

$\epsilon_h = N_h/N$ are the filling fractions
Equilibrium measures ...

- Assumption 1: uniqueness of maximizer (= \( \mu_{eq} \)) of
  \[
  \mathcal{T}(\mu) = \int \left[ \prod_{i=1}^{r} d\mu(x_i) \right] T(x_1, \ldots, x_r) + \frac{\beta}{2} \iint d\mu(x_1)d\mu(x_2) \ln |x_1 - x_2|
  \]
  among all proba. measures

  Let \( \epsilon_{eq,h} = \mu_{eq}[A_h] \) be the equilibrium filling fraction

- Assumption 2: local strict concavity at \( \mu_{eq} \)

Lemma 1

For \( \epsilon \) close enough to \( \epsilon_{eq} \)

\( \mathcal{T} \) has a unique maximizer (= \( \mu_{eq,\epsilon} \)) over proba. measure with \( \mu[A_h] = \epsilon_h \)
Equilibrium measures ...

- Assumption 1: uniqueness of maximizer (= $\mu_{eq}$) of
  \[ \mathcal{T}(\mu) = \int \left[ \prod_{i=1}^{k} d\mu(x_i) \right] T(x_1, \ldots, x_k) + \frac{\beta}{2} \iint d\mu(x_1)d\mu(x_2) \ln |x_1 - x_2| \]
  among all proba. measures

  Let $\epsilon_{eq,h} = \mu_{eq}[A_h]$ be the equilibrium filling fraction

- Assumption 2: local strict concavity at $\mu_{eq}$
- Assumption 3: $T$ is analytic
- Assumption 4: $\mu_{eq}$ has ($g + 1$) cuts and is off-critical

**Lemma 2**

For $\epsilon$ close enough to $\epsilon_{eq}$

- $\mu_{eq;\epsilon}$ has ($g + 1$) cuts and is off-critical
- The edges depend smoothly on $\epsilon$
- The density of $\mu_{eq;\epsilon}$ depends smoothly on $\epsilon$ away from edges
Equilibrium measures ...

- Assumption 1: uniqueness of maximizer (= $\mu_{eq}$) of

$$\mathcal{T}(\mu) = \int \left[ \prod_{i=1}^{k} d\mu(x_i) \right] T(x_1, \ldots, x_k) + \frac{\beta}{2} \iint d\mu(x_1)d\mu(x_2) \ln |x_1 - x_2|$$

among all proba. measures

Let $\epsilon_{eq,h} = \mu_{eq}[A_h]$ be the equilibrium filling fraction

- Assumption 2: local strict concavity at $\mu_{eq}$

- Assumption 3: $T$ is analytic

- Assumption 4: $\mu_{eq}$ has $(g + 1)$ cuts and is off-critical

**Lemma 3**

For $\epsilon$ close enough to $\epsilon_{eq}$

the large deviation estimates also holds uniformly

in the conditioned model with filling fractions $\epsilon$
The return of the master operator

- The correlators $W_k$ in the initial model
  $W_{k;\epsilon}$ in the conditioned model

  satisfy the same Schwinger-Dyson equations

- We have
  \[ \int_{A_{h_1}} \cdots \int_{A_{h_k}} W_{k;\epsilon}(z_1, \ldots, z_k) \prod_{i=1}^{k} \frac{dz_i}{2i\pi} = \delta_{k,1} N_{\epsilon_{h_1}} \]

  we need the restriction $K_{0;\epsilon}$ of $K_{\epsilon}$ to the codim. $= g$ subspace

  \[ \{ f, \quad \forall h, \quad \int_{A_h} f(z)dz = 0 \} \]

**Lemma 4**

For $\epsilon$ close enough to $\epsilon_{eq}$

$K_{0;\epsilon}$ is continuously invertible, and $K_{0;\epsilon}^{-1}$ depends smoothly on $\epsilon$
Asymptotic expansion of correlators in the conditioned model

**Corollary**

For \( \epsilon \) close enough to \( \epsilon_{\text{eq}} \)

we have, for any \( M \geq 0 \), an asymptotic expansion

\[
W_{k; \epsilon} = \sum_{m=k-2}^{M-1} W^{[m]}_{k; \epsilon} + O(N^{-M}; \Gamma_{M,k})
\]

depending smoothly on \( \epsilon \), with remainder uniform in \( \epsilon \)
Partition function of the conditioned model

\[
\frac{Z_{N;\epsilon}^{(T_1)}}{Z_{N;\epsilon}^{(T_0)}} = \exp \left( N^{2-r} \int \partial_t T_t(x_1, \ldots, x_r) \prod_{i=1}^r dL_{N;\epsilon}^{(\lambda), T_t(x_i)} \right)
\]

can be expressed in terms of \( W_{T_t}^{j;\epsilon} \) for the model with interaction \( T_t \)

- If we can find a interpolating family \( (T_t)_{t \in [0,1]} \)
  - respecting uniformly our assumptions
  - for which \( Z_{N;\epsilon}^{(T_0)} \) is known

we deduce an expansion

\[
Z_{N;\epsilon}^{(T_1)} = Z_{N;\epsilon}^{(T_0)} \times \exp \left( \sum_{m=-2}^{M-1} N^{-m} F_{\epsilon}^{[m]} + O(N^{-M}) \right)
\]

- Idea: interpolate in the space of equilibrium measures

\((\mu_{eq;\epsilon}^t)_{t \in [0,1]} \leftarrow (T_t)_{t \in [0,1]}\)
An interpolation path ...

Convex linear combination with semi-circles

Squeezing the supports

\[
Z_{N;\epsilon}^{(T_t)} \overset{t \to 0}{\sim} \prod_{0 \leq h < h' \leq g} \left| \frac{a_h + b_h - a_{h'} - b_{h'}}{2} \right|^{N^2 \epsilon_h \epsilon_{h'} \beta} \prod_{h=0}^{g} \left( \text{Selberg } \beta-\text{Gaussian integral over } \mathbb{R}^{N_h} \right)
\]
We initially wanted to compute
\[ Z_N = \sum_{N_0 + \cdots + N_g = N} \frac{N!}{\prod_{h=0}^{g} N_h!} Z_{N;(N_0/N, \ldots, N_g/N)} \]

- From large deviations of empirical measures:

\[ Z_N = \left( \sum_{|N - N \epsilon^*| \leq \ln N} \frac{N!}{\prod_{h=0}^{g} N_h!} Z_{N;N/N} \right) \left(1 + O(e^{-cN})\right) \]

- For \( N - N \epsilon^* \in o(N) \), we just proved, with \( \epsilon = (N_h/N)_{1 \leq h \leq g} \)

\[ \frac{N!}{\prod_{h=0}^{g} N_h!} Z_{N;\epsilon} = N^{\gamma N + \gamma'} \exp \left[ \sum_{m=-2}^{M-1} N^{-m} F^{[m]}_{\epsilon} + O(N^{-M}) \right] \]

where \( F^{[m]}_{\epsilon} \) depend smoothly on \( \epsilon \approx \epsilon_{eq} \)

- Extra lemma: \( (\nabla_{\epsilon} F^{[-2]})_{\epsilon_{eq}} = 0 \) and \( (\nabla_{\epsilon} \nabla_{\epsilon} F^{[-2]})_{\epsilon_{eq}} < 0 \)
We plug the asymptotic formula and use a Taylor expansion at \( \epsilon \approx \epsilon_{eq} \)

- E.g. up to \( o(1) \):

\[
Z_N = N^{\gamma N + \gamma'} e^{N^2 F_{eq}[[-2]]} + NF_{eq}[[-1]] + F_{eq}[0] \\
\times \left( \sum_{|N - N\epsilon_{eq}| \leq \ln N} e^{\frac{1}{2} (\nabla \otimes^2 F[[-2]])_{eq} \cdot (N - N\epsilon_{eq}) \otimes^2 + (\nabla F[[-1]])_{eq} \cdot (N - N\epsilon_{eq})} \right) (1 + O(e^{-c'(\ln N)^3 / N}))
\]

It is the general term of a super-exponentially fast converging series:

\[
Z_N = N^{\gamma N + \gamma'} e^{N^2 F_{eq}[[-2]]} + NF_{eq}[[-1]] + F_{eq}[0] \\
\times \left( \sum_{N \in \mathbb{Z}^g} e^{\frac{1}{2} (\nabla \otimes^2 F[[-2]])_{eq} \cdot (N - N\epsilon_{eq}) \otimes^2 + (\nabla F[[-1]])_{eq} \cdot (N - N\epsilon_{eq})} \right) (1 + O(e^{-c''(\ln N)^3 / N}))
\]

- We recognize \( \Theta - N\epsilon_{eq} \left( (\nabla F[[-1]])_{eq} \left| (\nabla \otimes^2 F[[-2]])_{eq} \right. \right) \)
Including higher orders yields terms of the form

\[
\sum_{\mathbf{N} \in \mathbb{Z}_g} \frac{1}{p!} \left( \prod_{i=1}^{p} \frac{(\nabla \otimes \ell_i \, F[m_i])_{eq}}{\ell_i!} \right) \cdot (\mathbf{N} - N\epsilon_{eq}) \otimes (\sum_i \ell_i) \cdot e^{\frac{1}{2} \mathbf{Q} \cdot (\mathbf{N} - N\epsilon_{eq}) \otimes^2 + \mathbf{w} \cdot (\mathbf{N} - N\epsilon_{eq})}
\]

We recognize

\[
\sum_{\mathbf{N} \in \mathbb{Z}_g} \frac{1}{p!} \left( \prod_{i=1}^{p} \frac{(\nabla \otimes \ell_i \, F[m_i])_{eq}}{\ell_i!} \right) \cdot (\nabla_{\mathbf{w}} (\sum_i \ell_i) \Theta_{-N\epsilon_{eq}})(\mathbf{w} | \mathbf{Q})
\]

Here \( \mathbf{Q} = (\nabla \otimes^2 F[-2])_{eq} \) and \( \mathbf{w} = (\nabla F[-1])_{eq} \)

We justified step by step the heuristics of Bonnet, David, Eynard '00, Eynard '07
Oscillatory asymptotic expansion

\[ Z_N = N^{\gamma N + \gamma'} (D_N \Theta_{-N\epsilon_{eq}}) (\nabla F[-1]_{eq} | (\nabla \otimes^2 F[-2])_{eq}) \exp \left[ \sum_{m \geq -2} N^{-m} F[m] + O(N^{-\infty}) \right] \]

where \[ D_N = \sum_{p \geq 0} \frac{1}{p!} \sum_{\ell_1, \ldots, \ell_p \geq 1, m_1, \ldots, m_p \geq -2} N^{-\sum_i (m_i + \ell_i)} \prod_{i=1}^{p} \left( \nabla \otimes \ell_i F[m_i] \right)_{eq} \cdot \nabla \otimes \ell_i \]

acts as a differential operator on the Siegel theta function

\[ \Theta_\mu (w|Q) = \sum_{m \in \mathbb{Z}^g} e^{w \cdot (m+\mu) + \frac{1}{2} (m+\mu) \cdot Q \cdot (m+\mu)} \]

Moving characteristics

\[ \mu = -N\epsilon_{eq} \mod \mathbb{Z}^g \]

Quadratic form

\[ Q = -\text{Hessian}_{\epsilon=\epsilon_{eq}} [T(\mu_{eq};\epsilon)] \]
All order asymptotics for $\beta$-ensembles in the multi-cut regime

1. Beta-ensembles and random matrices
2. Applications to orthogonal polynomials
3. Sketch of the proof of the main result
4. Conclusion
In progress

- A toy model for XXZ spin correlation functions (two-scale problem)

\[ Z_N = \prod_{1 \leq i < j \leq N} \sinh[N^\alpha c_1(\lambda_i - \lambda_j)] \sinh[N^\alpha c_2(\lambda_i - \lambda_j)] \prod_{i=1}^{N} e^{-N^{1+\alpha} V(\lambda_i)} d\lambda_i \]

Open problems

- Same questions for \( \lambda_i \in \mathbb{Z} \)?
  
  no Schwinger-Dyson equations ...

- Same questions for multi-matrix models?
  
  more complicated Schwinger-Dyson equations and convexity issues ...

- Universality from Schwinger-Dyson equations?