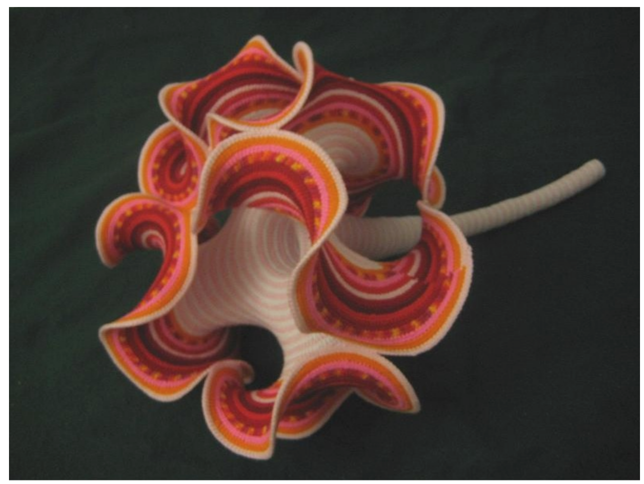
A Large Deviation Principle for non-linear log gases

(IHP, 2014)



Robert Berman

Chalmers University of Technology/U. of Gothenburg

Outline

- Motivation: a non-linear log gas in complex geometry

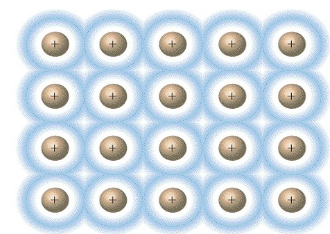
- A general LDP for Gibbs measures associated to singular Hamiltonians

- Geometric analysis on the $N-$particle space and Wasserstein geometry

1. Non-linear log gases in complex geometry

There is a "non-linear" generalization of the Coulomb gas on $X := \mathbb{C}$ to $\mathbb{C}^n$ where the role of the linear Coulomb interaction

$$E(z_1, ., , , z_N) := -\sum_{1 \leq i,j \leq N} \log |z_i - z_j|^2$$



is played by a *non-linear* version of $E(z_1, ., , , z_N)$.

(i.e. $E(z_1, ., , , z_N)$ is *not* the sum of 2-point functions).

- Recall that the large $N-$limit of the Coulomb gas in $\mathbb{C}$ is described in terms of the Laplace operator

- Indeed, the two point function $g(z,w) := \log |z - w|$ is a Green function for Δ

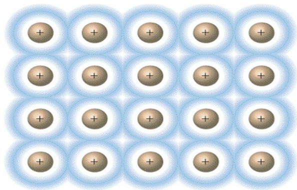
For the "non-linear log gas" in $\mathbb{C}^n$ the role of the Laplace operator will be played by the complex Monge-Ampère operator MA

$$MA(\phi) := \det(\partial\bar{\partial}\phi), \quad \partial\bar{\partial}\phi := \left(\frac{\partial^2\phi}{\partial z_i \partial \bar{z}_i}\right)$$

- This is a fully non-linear PDO
- A hint: it satisfies, with $g(z) := \log |z|^2$,

$$MA(g) = c_n \delta_0$$

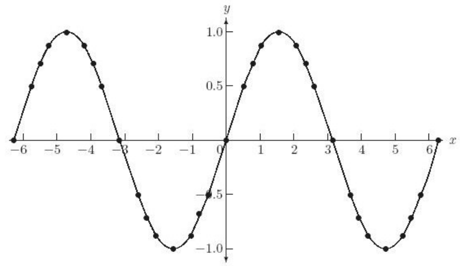
More precisely, in the case we will be interested in the higher dimensional analoge of the "mean field" scaling of the Coulomb energy:



$$E^{(N)}(z_1,.,,,z_N) := -\frac{1}{N-1}\sum_{1\leq i,j\leq N}\log|z_i-z_j|^2$$

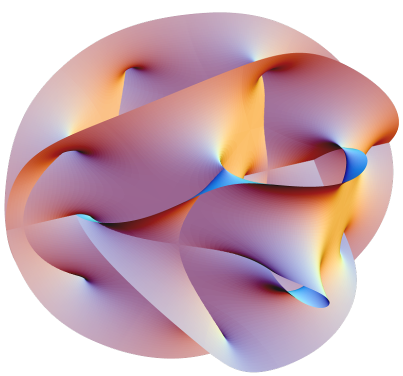
- The case of $\mathbb{C}$ has been studied by Caglioti-Lions-Marchioro-Pulvirenti ('92), Kiessling ('93) ($T>0$ fixed) and Ben Arous-Zeitouni ('98) ($T\to\infty$),....

This higher dimensional setting appears naturally in



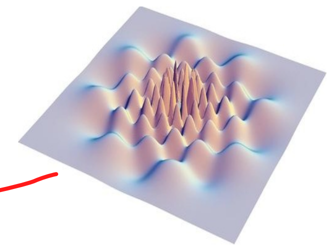
- Probabilistic approaces to finding optimal interpolation nodes in $X := \mathbb{C}^n$

- Probabilistic constructions of Kähler-Einstein metrics on a complex algebraic variety X



Let $X := \mathbb{C}^n$ and set

- $V_k := \{$polynomials $p_k(z)$ on $\mathbb{C}^n$ with total degree $\leq k\}$
- This is a vector space of dimension $N_k \sim k^n$



Let $D_k(z_1, z_2, ...z_{N_k})$ be the corresponding *Vandermonde determinants:*

$$D_k(z_1, z_2, ...z_{N_k}) := \det_{1 \leq i,j \leq N_k} (p_i(z_j))$$

for p_i a base of monomials in V_k. Set

$$E^{(N_k)}(z_1, ..., z_{N_k}) := -\frac{1}{k} \log |D_k(z_1, z_2, ...z_{N_k})|^2$$

on X^{N_k}, which is thus a "repulsive" interaction of "log type".

Given a function $\phi_0(z)$ on $X := \mathbb{C}^n$ of superlogarithmic growth (the "confining potential") we set

$$E^{(N_k)}_{\phi_0}(z_1, ..., z_{N_k}) := E^{(N_k)}(z_1, ..., z_{N_k}) + \sum_{i=1}^{N_k} \phi_0(z_i)$$

Now, given a positive number β (the "inverse temperature") consider the corresponding *Gibbs measures* on X^{N_k} :

$$\mu^{(N_k)}_\beta := \frac{e^{-\beta E^{(N_k)}_{\phi_0}(z_1,...,z_{N_k})} dV^{\otimes N_k},}{Z_{N,k}[\phi_0]}$$

where dV is the Euclidean volume form on X

In the general formalism of statistical mechanics the Gibbs measure

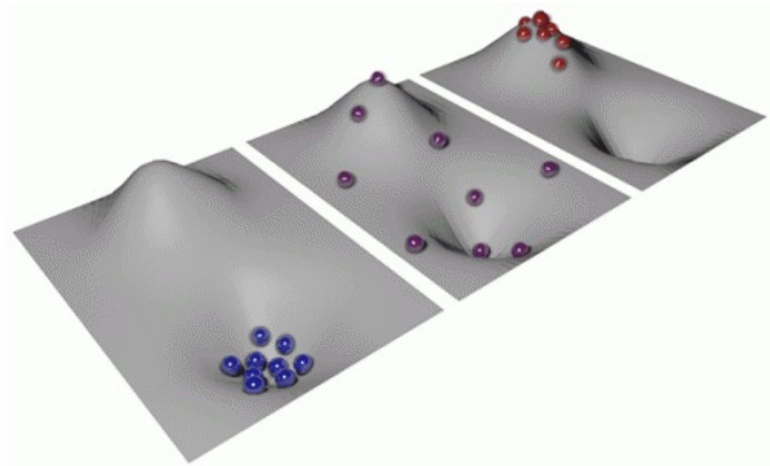
$$\mu_\beta^{(N)} := \frac{e^{-\beta E^{(N)}(x_1,\ldots,x_N)} dV^{\otimes N}}{Z_N}$$

gives the equilibrium description of particles interacting by $E^{(N)}(x_1, \ldots, x_N$ at a fixed temperature $T := 1/\beta$.

- More generally, it will be important to allow the inverse temperature to depend on N and set

$$\beta := \lim_{N\to\infty} \beta_N \in [0, \infty]$$

- Then $\beta = \infty$ is the "zero temperature regime"



- One then expects that the main contribution to $\mu_\beta^{(N)}$ comes from the minimizers of $E^{(N)}(x_1, ..., x_N)$ for N large.

In the present setting we thus have

$$\mu_{\beta_k}^{(N_k)} := \frac{\left| D_k(z_1, z_2, \ldots z_{N_k}) \right|^{2\frac{\beta_k}{k}} dV^{\otimes N_k}}{Z_{N,k}[\phi_0]}, \quad \beta := \lim_{k \to \infty} \beta_k \in [0, \infty]$$

- For $\beta_k = k$ ($\implies \beta = \infty$) this is a *determinantal point process* (wrt the N_k-dimensional vector space V_k)

- For $n = 1$ and $\beta_k \sim k$ ($\implies \beta = \infty$) this is the *standard log gas* in $\mathbb{C}$ at effectively zero temperature $T = 1/\infty$

To understand the case $n = 1$ (i.e. $X = \mathbb{C}$) note that $D_k(z_1, z_2, ...z_{N_k})$ has the following properties

- $D_k(z_1, z_2, ...z_{N_k})$ is a polynomial of degree $\leq k$ in each variable
- vanishes of $z_i = z_j$ for some pair (i, j) ("repulsion")

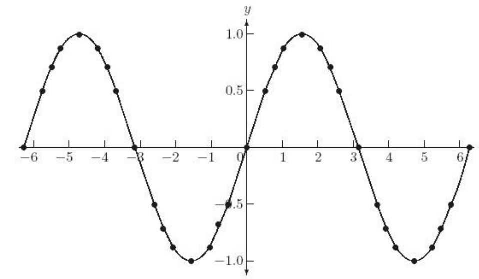
Hence, when $X = \mathbb{C}$ can factorize in each variable to get

$$D_k(z_1, z_2, ...z_{N_k}) = \prod_{1 \leq i,j \leq k} (z_i - z_j)$$

Accordingly, since $N_k = k + 1$, we indeed have

$$E^{(N)}(z_1, ., , , z_N) := -\frac{1}{N-1} \sum_{1 \leq i,j \leq N} \log |z_i - z_j|^2$$

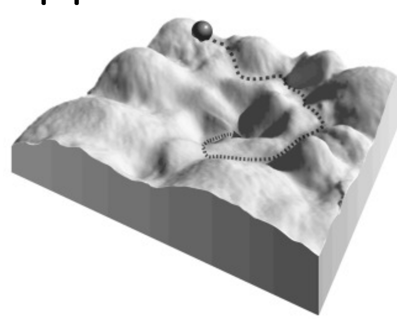
Relations to interpolation/sampling theory

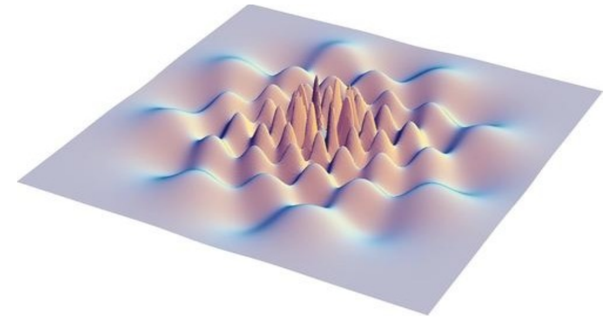


The maximizers of $\left|D_k(z_1, z_2, ...z_{N_k})\right|$, for z_i constrained to a compact subset $K \subset \mathbb{C}^n$ arise as *optimal nodes for interpolation* on K of polynomials of degree k

- Hence, the mininimizers of $E^{(N)}(z_1, ., , , z_N)$ are the natural higher dimensional complex analoge of *Fekete points.*

- The corresponding Gibbs measure gives a probabilistic approach to locating approximate mininimizers (in the case $\beta \approx \infty$)





Relations to mathematical physics

The Vandermonde determinant $D_k(z_1, z_2, ...z_{N_k})$ appears in Quantum Mecanics as the Slater determinant for N free fermions (electrons).

- The one particle fermions live on $\mathbb{C}^n$ and are represented by wave functions in the vector space V_k
- They are coupled to the magnetic two form $F_A := \partial\bar{\partial}\phi_0$ multiplied by k
- The probability of locating fermions at $z_1, ..., z_{N_k}$ is proportional to $|D_k(z_1, z_2, ...z_{N_k})|^2_{k\phi_0}$

Convergence results

The Gibbs measure $\mu_\beta^{(N_k)}$ defines a random point process on $X := \mathbb{C}^n$ (i.e. a symmetric probability measure on X^{N_k})

- The corresponding empirical measure

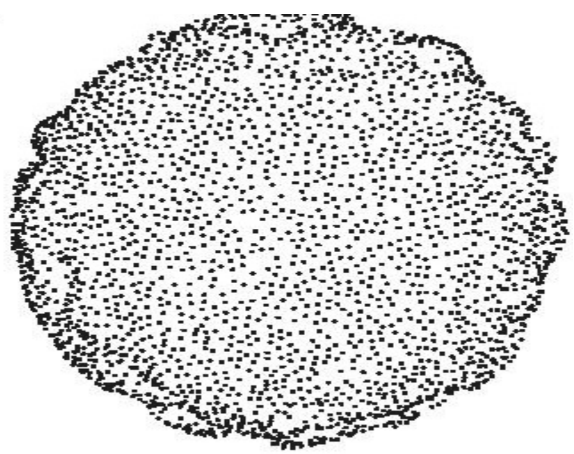
$$\delta_N := \frac{1}{N} \sum_{i=1}^{N} \delta_{z_i}$$

  defines a random measure

- When $N \to \infty$ the random measure δ_N converges, in probability, to a unique deterministic measure μ_β.

- More precisely, the laws of δ_N satisfy a Large Deviation principle (LDP):

$$\text{Prob}\left(\frac{1}{N_k}\sum_i \delta_{x_i} \in B_\epsilon(\mu)\right) \sim e^{-N\beta_N F_\beta(\mu)}, \qquad F_\beta = E + \frac{1}{\beta}H$$



Theorem (B. 13): Given a confining potential $\phi_0(z)$ on $X := \mathbb{C}^n$ the laws of the corresponding empirical measures δ_N (at "inverse temperature" β) satisfy a LDP with speed $\beta_{N_k} k$ and a good rate functional

$$F_\beta(\mu) = E_{\phi_0}(\mu) + \frac{1}{\beta} H(\mu),$$

where

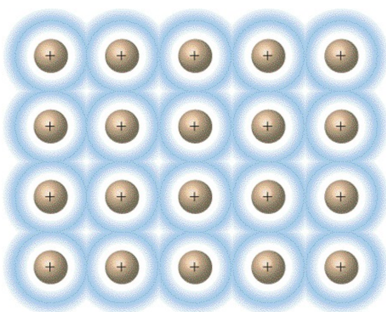
- $E(\mu)$ is the *pluricomplex energy* of the measure μ relative to ϕ_0

- $H(\mu)$ is the *entropy of* μ relative to dV

Here

$$E_{\phi_0}(\mu) = E(\mu) + \int \phi_0 \mu,$$

where $E(\mu)$ is a generalization of the standard log energy in $\mathbb{C}$



The rate functional F_β admits a unique minimizer μ_β which can be described in terms of the *complex Monge-Ampère operator* MA:

$$MA(\phi) := \det(\partial\bar{\partial}\phi), \quad \partial\bar{\partial}\phi := \left(\frac{\partial^2\phi}{\partial z_i \partial \bar{z}_i}\right)$$

for ϕ a "potential" i.e.

- ϕ is a function on $\mathbb{C}^n$ with logarithmic growth at ∞ and $\partial\bar{\partial}\phi \geq 0$

More precisely,

$$\mu_\beta = MA(\phi_\beta)dV,$$

where ϕ_β is the unique potential solving the following fully non-linear PDE in $\mathbb{C}^n$:

$$MA(\phi_\beta) = e^{\beta(\phi_\beta - \phi_0)}$$

- When $\beta \to \infty$ (the "zero-temperature case") $\phi_\beta \to \phi_\infty$, solving a free boundary value problem for the $MA-$operator, i.e.

$$\mu_\infty = MA(\phi_\infty) = 1_D MA(\phi_0)dV,$$

where D is a subset of $\mathbb{C}^n$ "the droplet" depending on ϕ_0

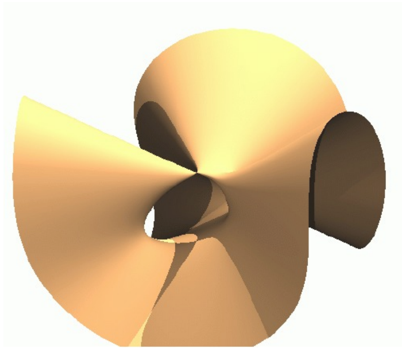
Applications in complex geometry ($\beta = 1$)

Replacing $\mathbb{C}^n$ with a complex algebraic variety X of "general type" the case $\beta = 1$ gives rise to a *canonical* random point process on X (i.e. without choosing a back-ground potential ϕ_0) such that, as $N \to \infty$,

$$\frac{1}{N} \sum_{i=1}^{N} \delta_{x_i} \to dV_{KE}$$

in law, where dV_{KE} is the normalized volume form of the unique *Kähler-Einstein* metric g_{KE} on X.

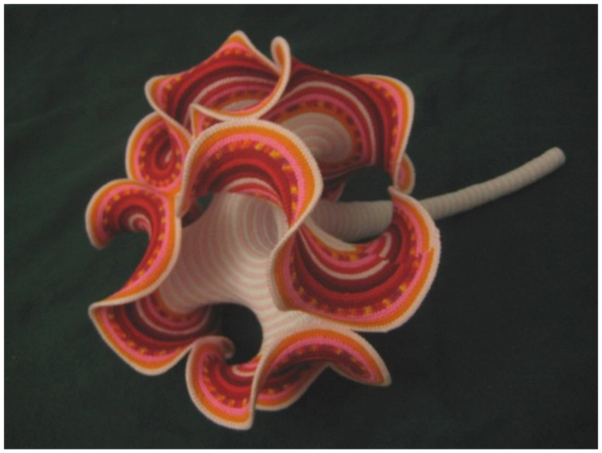
- The existence of g_{KE} was established by Aubin and Yau (independently) in the late 70's



In other words, g_{KE} is the unique Riemannian metric g solving the *Einstein equation*

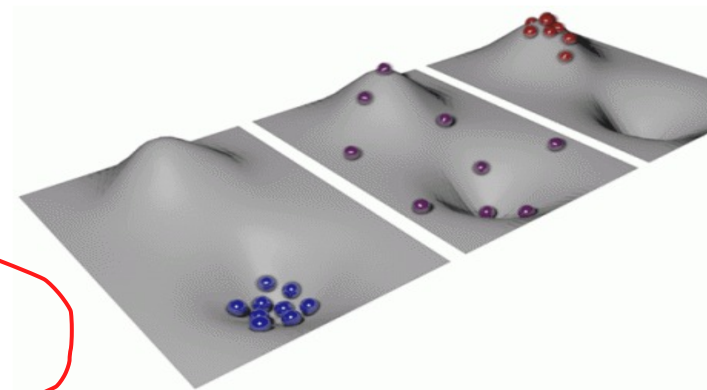
$$\mathrm{Ric}\ g = -g$$

and such that g is *Kähler*, i.e. compatible with the complex structure on X.



A LDP for Gibbs measures associated to singular Hamiltonians

Consider a general "one-particle space" (X, dV) (say compact) and let $E^{(N)}$ be a symmetric function on the "$N-$particle space" X^N (say bounded from below).



The corresponding Gibbs measure on X^N is

$$\mu_{\beta_N}^{(N)} := \frac{e^{-\beta_N E^{(N)}} dV^{\otimes N}}{Z_N}$$

for a given positive sequence β_N of "inverse temperatures".

The general problem

Assume that the law of the empirical measure δ_N defined wrt to

$$\mu_{\beta_N}^{(N)} := \frac{e^{-\beta_N E^{(N)}} dV^{\otimes N}}{Z_N}$$

satisfies an LDP in the "zero-temperature" limit, i.e. when $\beta_N \to \infty$ with rate functional $E(\mu)$ and speed $\beta_N N$, i.e.

$$\text{Prob}\ \left(\frac{1}{N_k}\sum_i \delta_{x_i} \in B_\epsilon(\mu)\right) \sim e^{-N\beta_N E(\mu)}$$

The general problem: Is there an LDP for a *fixed* $\beta > 0$?

Heuristically (by thermodynamics) if this is the case then the rate functional for $\beta > \infty$ should be the following *free energy type functional*

$$F_\beta(\mu) = E(\mu) + H_{dV}(\mu)/\beta$$

where $H_{dV}(\mu)$ is the *entropy of* μ *relative to* μ_0

- In general, there is no LDP for $\beta > 0$ if $E^{(N)}(x_1, ..., x_N)$ is too wild!

Theorem (B.13): Assume that, for some sequence $\beta_N \to \infty$, $e^{-\beta_N E^{(N)}} dV^{\otimes N}$ satisfies and LDP with a "strictly convex" rate functional $E(\mu)$. Then, if

$$\Delta_{x_1} E^{(N)}(x_1, ..., x_N) \leq C$$

there is an LDP for any fixed $\beta > 0$ with rate functional $F_\beta = E + H_{dV}/\beta$

- The strict convexity of E ensures that F_β has a unique minimizer μ_β for $\beta \geq 0$

The existence and strict convexity assumption on $E(\mu)$ is equivalent to (by the Gärtner-Ellis theorem):

$$-\lim_{N\to\infty} \frac{1}{N\beta_N} \log Z_{N,\beta_N}[\phi] = \mathcal{E}(\phi),$$

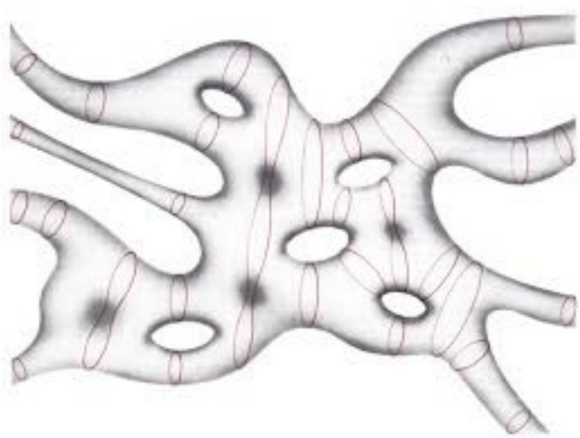
for a Gateaux differentiable functional $\mathcal{E}(\phi)$ (the Legendre transform of E)

- By the quasi-superharmoncity assumption the limit above is equivalent to

$$\lim_{N\to\infty} \frac{1}{N\beta_N} \inf_{X^N} \left(E^{(N)}(x_1,...,x_N) + \sum_{i=1}^{N} \phi(x_i) \right) = \mathcal{E}(\phi)$$

(indicating the Legendre duality in question)

- In the Vandermonde setting the functional $\mathcal{E}(\phi)$ can be explicitely determined (it generalizes the "Liouville action" in RMT and 2D Quantum Gravity)



This theorem applies to the setting of *Vandermonde determinants* with $X = \mathbb{C}^n$, where

$$-E_{\phi_0}^{(N_k)}(z_1, ..., z_{N_k}) := \frac{1}{k} \log |D_k(z_1, z_2, ...z_{N_k})|^2 - \phi_0(z_1) + ...$$

for D_k the corresponding sequence of Vandermonde determinants.

- The property of $\Delta_{x_1}(-E_{\phi_0}^{(N_k)}) \geq -C$ follows from $\Delta \log |p|^2 \geq 0$ if $p(z)$ a polynomial on $\mathbb{C}^n$.

- Taking $\beta_{N_k} = k$ ($\implies \beta = \infty$) gives

$$\mu_{\beta_N}^{(N)} := \left| D_{k\phi_0}(z_1, z_2, ...z_{N_k}) \right|^2 / Z_N$$

i.e. a determinantal point-process, which can be studied using L^2-methods

- Then the case $\beta > 0$ follows from the previous theorem!

The proof in the zero temperature regime ($\beta = \infty$) appears in

- B.: *Determinantal Point Processes and Fermions on Complex Manifolds: Large Deviations and Bosonization*, Comm. in Math. Phys. (2014),

The proof builds on the joint work

- *B*.;Boucksom: *Growth of balls of holomorphic sections and energy at equilibrium.* 42 pages, Invent. Math. (2010)

- *B.*; Boucksom, D. Witt Nystrom: *Fekete points and convergence towards equilibrium measures on complex manifolds,* 26 pages, Acta Math. Vol. 207, Issue 1 (2011), 1-27, It is closely related to the variational approach to MA-equations developed in the joint work

- B; Boucksom; Guedj; Zeriahi: *A variational approach to complex Monge-Ampere equations,* Publications Mathématiques de l'IHÉS (2012):

The proof in the case $\beta < \infty$ appears in

- B.: *Kähler-Einstein metrics, canonical random point processes and birational geometry.* arXiv:1307.3634

The precise meaning of an LDP

The empirical measure $\delta_N := \frac{1}{N}\sum_{i=1}^{N}\delta_{x_i}$ defines a map

$$\delta_N : \ X^N \to \mathcal{P}(X), \quad (x_1, ., , , x_N) \mapsto \frac{1}{N}\sum_{i=1}^{N}\delta_{x_i}$$

of X^N into the space $\mathcal{P}(X)$ of all probability measures. By definition its *law* is the probability measure

$$\Gamma_N := (\delta_N)_*(\mu^{(N)})$$

on $\mathcal{P}(X)$. *Convergence in probability towards a deterministic measure* μ_X equivalently means *convergence in law,* i.e. as $N \to \infty$,

$$\Gamma_N \to \delta_{\mu_X}$$

weakly on $\mathcal{P}(X)$. A *LDP* for Γ_N at a rate $N\beta_N$ means, formally, that

$$\Gamma_N \sim e^{-N\beta_N F(\mu)}[D\mu]$$

More precisely, an LDP means that

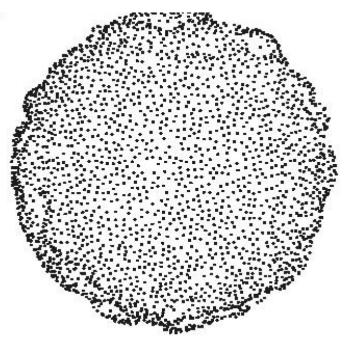
$$\lim_{\epsilon \to 0} \lim_{N \to \infty} \frac{1}{N\beta_N} \log \Gamma_N(B_\epsilon(\mu)) = -F(\mu)$$

where $B_\epsilon(\mu)$ is a ball in the metric space $(\mathcal{P}(X), d)$, where d is any metric compatible with the standard topology (say that X is compact).

In our case we can write the LDP as

$$-\lim_{N\to\infty}\frac{1}{N\beta_N}\log\int_{B_\epsilon(\mu)}e^{-\beta_N E^{(N)}}dV^{\otimes N}=E(\mu)+\frac{1}{\beta}H_{dV}(\mu)$$

where we used the map δ_N to identify $B_\epsilon(\mu)\subset\mathcal{P}(X)$ with a subset of X^N



- The proof of the LDP uses the Wasserstein 2-metric d_{W_2} on $\mathcal{P}(X)$ to defined the balls

- The point is that the map

$$\delta_N : \ (\frac{X^N, (g/N)^{\otimes N}}{\Sigma_N}) \to (\mathcal{P}(X), d_{W_2})$$

is an *isometry* (for a given Riemannian metric g on X)

- This gives a *Riemannian* meaning to the pulled-back balls in X^N.

The key point technical part of the proof is an asymptotically sharp *sub-mean inequality* on the (singular) Riemannian space

$$X^{(N)} := X^N/\Sigma_N$$

with the metric $g^{(N)}$ induced by the scaled Riemannian metric g/N on X :

$$\text{"}\, f(x_1, ..., x_N) \preceq e^{N\epsilon} \frac{\int_{B_\epsilon(x_1,....,x_N)} f dV_g}{\text{Vol } B_\epsilon(x_1, ..., x_N)} \text{"}$$

if $f \geq 0$ and $\Delta f \geq -\lambda f$ (here we take $f = e^{-\beta E}$)

- The entropy part in F_β then comes from the volume of the corresponding balls.

- The distortion factor $e^{\epsilon N}$ gets washed out at our logarithmic scale!

3. Relations to geometric analysis

If $-\kappa$ is a lower bound on the Ricci curvature on X^N then, by definition, we have on $X^{(N)}$ that $g^{(N)} := (g/N)^{\otimes N}$ has bounded diameter and

$$\operatorname{Ric} g^{(N)} \geq -(N\kappa)\, g^{(N)}$$

and hence there exists a constant C (independent of N) such that

$$\operatorname{Ric} g^{(N)} \geq -C\left(\dim(X^N) - 1\right) g^{(N)}$$

This means that we are in good shape to apply estimates from *comparison geometry/geometric analysis* to $(X^N, g^{(N)})$.

- More precisely, we have to work on the (singular) quotient $X^{(N)}$ of X^N under the Σ_N-action.

- But morally this should be fine since the previous curvature bound still holds in a weak sense on $X^{(N)}$ (in the sense of Alexandrov, Lott-Villanni,...)

The desired submean inequality is obtained by modifying the proof of Li-Schoen ('84) of a similar inequality.

Two new features:

- (1) need a dimension dependence which is sub-exponential on small balls

- (2) Need to work on (singular) Riemannian quotients

Thank you!!!


S = k log W
LVDWIG
BOLTZMANN
BOLTZMANN
ARTHUR
BOLTZMANN
LVDWIG
BOLTZMANN
HENRIETTE
BOLTZMANN