Atoms in the limiting spectrum of sparse graphs

Justin Salez (lpma)
A graph $G = (V, E)$ can be represented by its adjacency matrix:

$$A_{ij} = \begin{cases} 1 & \text{if } \{i, j\} \in E \\ 0 & \text{otherwise} \end{cases}$$

Eigenvalues $\lambda_1 \geq \ldots \geq \lambda_{|V|}$ capture essential information about $G$.

$$\mu_G := \frac{1}{|V|} \sum_{k=1}^{\delta} \lambda_k$$

**Question:** How does $\mu_G$ typically look when $G$ is large?
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**Question:** How does $\mu_G$ **typically** look when $G$ is large?
SPECTRUM OF A RANDOM GRAPH ON 10000 NODES
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THE SEMI-CIRCLE LAW

Erdős-Rényi model: \( n \) nodes, edges present with proba \( p \).

Theorem (Wigner, 50's): if \( np \rightarrow \infty \),

\[
\mu_{G_n}(\sqrt{np}(1-p/n) d\lambda) \longrightarrow_{n \to \infty} \sqrt{4 - \lambda^2} 2\pi 1(|\lambda| \leq 2) d\lambda.
\]

Uniformly chosen random \( d \) -regular graph on \( n \) nodes.

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In both cases, graphs are required to be dense: \(|E| \gg |V|\).

What about sparse graphs: \(|E| \approx |V|\)?
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- In both cases, graphs are required to be **dense**: \( |E| \gg |V| \)
- What about **sparse graphs**: \( |E| \propto |V| \)?
GRAPH WITH AVERAGE DEGREE 3 ON 1000 NODES
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RANDOM 3-REGULAR GRAPH ON 10000 NODES
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Along many sequences \( \{G_n\}_{n \geq 1} \) of sparse graphs, the spectrum \( \mu \) converges to a deterministic, model-dependent limit \( \mu \):

\[
\mu_{G_n} \rightarrow \mu \quad \text{as} \quad n \to \infty
\]

- Random \( d \)-regular graph on \( n \) nodes (McKay, 1981)
- Erdős-Rényi graph with edge probability \( p \sim cn \) (Khorunzhy-Shcherbina-Vengerovsky 2004)
- Uniform random tree on \( n \) vertices (Bhamidi-Evans-Sen 2009)

Actually, this phenomenon is just one of the many consequences of the fact that the underlying local geometry converges.
Along many sequences \( \{ G_n \}_{n \geq 1} \) of sparse graphs, the spectrum \( \mu_{G_n} \) approaches a deterministic, \textbf{model-dependent} limit \( \mu \):
SPECTRA OF SPARSE GRAPHS

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LOCAL WEAK CONVERGENCE (Benjamini-Schramm)
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\[ G_n \xrightarrow{\text{loc.}} \xrightarrow{n \to \infty} \mathcal{L} \]

\[ \sum_{o \in V_n} 1 \{ B_R(G_n, o) \equiv \ast \} \xrightarrow{n \to \infty} \mathcal{L}(B_R(G, o) \equiv \ast) \]

\( \Delta L \) describes the local geometry of \( G_n \) around a random node.
**LOCAL WEAK CONVERGENCE (Benjamini-Schramm)**

\[ G_n \xrightarrow{\text{loc.}}_{n \to \infty} \mathcal{L} \]

\[ \forall R \in \mathbb{N}, 1 \leq |V_n| \sum_{o \in V_n} 1 \{ BR(G, o) \equiv \bullet \} \xrightarrow{n \to \infty} \mathcal{L}(BR(G, o) \equiv \bullet). \]

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LOCAL WEAK CONVERGENCE (BENJAMINI-SCHRAMM)

\[ G_n \xrightarrow{\text{loc.}}_{n \to \infty} \mathcal{L} \]

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\( \mathcal{L} \): probability distribution over locally finite rooted graphs \((G, o)\).
Local weak convergence (Benjamini–Schramm)

\[ G_n \xrightarrow{\text{loc.}} \xrightarrow{n \to \infty} \mathcal{L} \]

\[ \forall R \in \mathbb{N}, \quad \frac{1}{|V_n|} \sum_{o \in V_n} 1_{\{B_R(G_n,o) \equiv \bullet\}} \xrightarrow{n \to \infty} \mathcal{L}(B_R(G,o) \equiv \bullet). \]

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\( \mathcal{L} \) describes the local geometry of \( G_n \) around a random node.
SOME SPARSE GRAPHS AND THEIR LOCAL LIMITS

- \( G_n = \) box of size \( n \times \cdots \times n \) in the lattice \( \mathbb{Z}^d \), \( L^d = \) Dirac at \( (\mathbb{Z}^d, 0) \)
- \( G_n = \) random \( d \)-regular graph on \( n \) nodes, \( L = \) Dirac at the \( d \)-regular infinite rooted tree
- \( G_n = \) Erdős-Rényi graph with \( p_n = c/n \) on \( n \) nodes, \( L = \) law of a Galton-Watson tree with degree Poisson \( c \)
- \( G_n = \) random graph with degree distribution \( \nu \) on \( n \) nodes, \( L = \) law of a Galton-Watson tree with degree distribution \( \nu \)
- \( G_n = \) uniform random tree on \( n \) nodes, \( L = \) Infinite Skeleton Tree (Grimmett, 1980)
- \( G_n = \) preferential attachment graph on \( n \) nodes, \( L = \) Polya-point graph (Berger-Borgs-Chayes-Sabery, 2009)
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Can we give a sense to $\mu_G = \frac{1}{|V|} \sum \delta_{\lambda_i}$ when $G$ is replaced by $L$?

If $G = (V, E)$ is a finite graph, we have for $z \in \mathbb{C} \setminus \mathbb{R}$

$$\int_{\mathbb{R}} 1 \lambda - z \mu_G(d\lambda) = \frac{1}{|V|} \sum_{o \in V} (A_G - z) - 1_{oo}.$$

If $L$ is the law of a random rooted graph $(G, o)$, define $\mu_L$ by

$$\int_{\mathbb{R}} 1 \lambda - z \mu_L(d\lambda) = \mathbb{E}\left[\langle e_o | (A_G - z) - 1 \rangle e_o \right].$$

Fact: $G_n \loc \rightarrow_{n \to \infty} L \Rightarrow \mu_{G_n} \rightarrow_{n \to \infty} \mu_L$.
Can we give a sense to \( \mu_G = \frac{1}{|V|} \sum_i \delta_{\lambda_i} \) when \( G \) is replaced by \( \mathcal{L} \)?
SPECTRAL CONVERGENCE REVISITED

Can we give a sense to $\mu_G = \frac{1}{|V|} \sum_i \delta_{\lambda_i}$ when $G$ is replaced by $\mathcal{L}$?

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\int \frac{1}{\lambda - z} \mu_G(d\lambda) = \frac{1}{|V|} \sum_{o \in V} (A_G - z)_{oo}^{-1}.
\]

If \( \mathcal{L} \) is the law of a random rooted graph \( (G, o) \), define \( \mu_{\mathcal{L}} \) by

\[
\int \frac{1}{\lambda - z} \mu_{\mathcal{L}}(d\lambda) = \mathbb{E} \left[ \langle e_o | (A_G - z)^{-1} e_o \rangle \right].
\]

Fact:

\[
G_n \xrightarrow{\text{loc.}} \mathcal{L} \quad \implies \quad \mu_{G_n} \xrightarrow{n \to \infty} \mu_{\mathcal{L}}
\]
RECURSION IN THE CASE OF TREES

\[ T = 1 \ 2 \ \cdots \ d \]

\[ T = \bigg( A T - z \bigg) - 1 = -z + \sum_{d} (A_{Ti} - z) - 1 \]

− Explicit resolution for infinite regular trees
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Let's keep things simple: $L$ is a GW-tree with degree Poisson($c$).

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\mu_L = \mu_{pp} + \mu_{sc} + \mu_{ac}
$$

Open problem: determine the support of each type of spectrum.

Theorem (Bordenave-Sen-Virag'13): $\mu_{pp}(R) < 1$ as soon as $c > 1$.

We will focus on the pure-point part, i.e. the atoms of $\mu_L$. This specific question was raised by Ben Arous (open problem 14, AMS workshop on random matrices, 2010).

Remark: every finite tree has positive probability under $L$.

$\forall$ tree eigenvalues are atoms of $\mu_L$ (e.g. 0, 1, $\sqrt{3}$, $2 \cos \frac{2\pi}{5}$,...)
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A theorem (Lück'02, Veselić'05, Abért-Thom-Virág'11). Fix $\lambda \in \mathbb{R}$.

\[ \sup_{A \in A} \left| \mu_A(\lambda - \varepsilon, \lambda + \varepsilon) - \mu_A(\{\lambda\}) \right| \xrightarrow{\varepsilon \to 0} 0. \]

**Corollary.** If $G_n \xrightarrow{\text{loc}} L \xrightarrow{n \to \infty}$, then not only $\mu_{G_n} \xrightarrow{n \to \infty} \mu_L$ but also $\forall \lambda \in \mathbb{R}, \mu_{G_n}(\{\lambda\}) \xrightarrow{n \to \infty} \mu_L(\{\lambda\})$.

In particular, $\mu_L(\{\lambda\}) = 0$ unless $\lambda$ is a totally real algebraic integer (= root of some real-rooted monic integer polynomial).
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We are left with the following (crude) inner and outer-bounds:

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\{ \text{tree eigenvalues} \} \subseteq \text{Atoms}(\mu_L) \subseteq \{ \text{totally real alg. integers} \}
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**Theorem (S. 2013):** the inner and outer-bounds coincide.

**Remark:** the weaker assertion that every totally real algebraic integer is an eigenvalue of some symmetric integer matrix is known as Hofmann's conjecture (1975). It was proved by Estes (1992).

**Corollary:** many graph limits have the set of totally real algebraic integers as atomic support. This includes all Galton-Watson trees with \( \text{supp}(\nu) = N \), as well as the Infinite Skeleton Tree.
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PROOF IDEA: RECURSIVE FORMULATION

To a rooted tree $T$ with root $o$, associate a rational function $f_T(x) := 1 - \Phi_T(x)\frac{x}{\Phi_T(x)}\Phi_T(o)(x)$ with $

\Phi_T(x) = \det(x - A_T)$. 

$\lambda \neq 0$ is a tree eigenvalue $\iff 1$ can be generated from 0 by repeated applications of $(x_1, \ldots, x_d) \mapsto x_2 \sum_{i=1}^d 1 - x_i(d \in \mathbb{N})$. 

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T = \begin{array}{l}
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Proof Idea: Recursive Formulation

To a rooted tree $T$ with root $o$, associate a rational function

$$f_T(x) := 1 - \frac{\Phi_T(x)}{x\Phi_{T\setminus o}(x)} \quad \text{with} \quad \Phi_T(x) = \det(x - A_T).$$
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Diagram of a rooted tree $T$ with a root node $o$ and children nodes $1$, $2$, $d$, $T_1$, $T_2$, $\ldots$, $T_d$. The tree is structured as a hierarchical arrangement with $o$ at the top and $1$, $2$, $d$ as its children, each branching further with $T_1$, $T_2$, $\ldots$, $T_d$. The recursive formulation is illustrated through this diagram, where each node represents a subtree that can be recursively formulated using the same rational function $f_T(x)$.
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($d \in \mathbb{N}$).
A SURPRISING STATEMENT

Fix a totally real algebraic integer \( \lambda \neq 0 \).

Consider the smallest set \( F \subseteq \mathbb{R} \) satisfying

1. \( 0 \in F \)
2. \( x, y \in F \Rightarrow x + y \in F \)
3. \( x \in F \setminus \{1\} \Rightarrow x \lambda^2 (1 - x) \in F \)

Theorem (S. 2013): \( F \) is the field generated by \( \lambda^2 \).

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