The spectrum of random graphs in free probability theory

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Purpose of the talk :

1. State a "weak asymptotic freeness theorem" for random matrices invariant in law by conjugation by permutation matrices and two criterion to compare with classical "asymptotic freeness".
2. Application to adjacency of random graphs.
3. Idea of the proofs.

A weak notion of asymptotic freeness

The (mean) empirical spectral distribution (e.s.d.) of A_N :

$$\mathcal{L}_{A_N} = \mathbb{E}\Big[\frac{1}{N}\sum_{i=1}^{N}\delta_{\lambda_i}\Big],$$

where $\lambda_1, \dots, \lambda_N$ are the eigenvalues of A_N.

Thanks to free probability, one can study the possible limiting e.s.d. of Hermitian matrices of the form

$$H_N = P(A_1, \dots, A_L)$$

where

1. P is a fixed *-polynomial (non commutative polynomial in the matrices and their adjoint)
2. $A_1, \dots, A_L$ are independent random matrices whose eigenvectors are "sufficiently" uniformly distributed.

Definition

A family or random matrices $\mathbf{A}_N = (A_j)_{j \in J}$ *is unitarily invariant whenever* $\mathbf{A}_N \stackrel{\mathcal{L}}{=} (UA_jU^*)_{j\in J}$ *for any unitary matrix* U*.*

Example : G.U.E. matrices, unitary matrices distributed according to the Haar measure.

Definition

The **-distribution of a family* $\mathbf{A}_N$ *is the map* $\Phi_{\mathbf{A}_N} : P \mapsto \mathbb{E}\big[\frac{1}{N}\mathrm{Tr} P(\mathbf{A}_N)\big]$

If $H_N = P(A_1, \ldots, A_L)$, then

$$\mathcal{L}_{H_N}(Q) = \mathbb{E}\frac{1}{N}\mathrm{Tr}\, Q\big(P(A_1, \ldots, A_L)\big) = \Phi_{\mathbf{A}_N}\big(Q(P)\big)$$

So point wise convergence of $\Phi_{\mathbf{A}_N}$ implies convergence in moments of any $H_N = P(A_1, \ldots, A_L)$.

Theorem (Voiculescu (91), Collins and Śniady (04))

$\mathbf{A}_N^{(1)}, \dots, \mathbf{A}_N^{(L)}$ *independent families of random matrices such that*

1. *each family (except possibly one) is unitarily invariant,*
2. *each family converges in* **-distribution,*
3. $+$ *technical condition, say* $\mathbf{A}_N^{(j)} = U \tilde{\mathbf{A}}_N^{(j)} U^*$ *with* $\tilde{\mathbf{A}}_N^{(j)}$ *deterministic.*

Then the collection of all families converges in **-distribution :*

$$\Phi(P) := \lim_{N\to\infty} \mathbb{E}\Big[\frac{1}{N} \mathrm{Tr} P(\mathbf{A}_N^{(1)}, \dots, \mathbf{A}_N^{(L)}) \Big] \text{ exists}$$

and depends only on the limiting **-distributions of* $\mathbf{A}_N^{(1)}, \dots, \mathbf{A}_N^{(L)}$.

The families are said to be asymptotically free. Explicit formula for $\Phi(P)$ and algorithmes are known for approximations of the limiting e.s.d. of any $H_N = P(\mathbf{A}_N^{(1)}, \dots, \mathbf{A}_N^{(L)})$.

Definition

A family or random matrices $\mathbf{A}_N = (A_j)_{j \in J}$ is permutation invariant whenever $\mathbf{A}_N \overset{\mathcal{L}}{=} (UA_jU^*)_{j\in J}$ for any permutation matrix U.

Example : Adjacency matrices of random graphs whose distribution is invariant by relabeling of vertices, Wigner matrices, diagonal matrices with i.i.d. entries.

To understand the limiting distribution of independent permutation families of random matrices, one needs more than their limiting *-distribution : if A_N and B_N are two independent diagonal permutation invariant matrices, U_N Haar unitary matrix

$$\mathcal{L}_{A_N+B_N} = \mathcal{L}_{A_N} * \mathcal{L}_{B_N} \text{ but } \lim_{N\to\infty} \mathcal{L}_{A_N+U_NB_NU_N^*} = \lim_{N\to\infty} \mathcal{L}_{A_N} \boxplus \lim_{N\to\infty} \mathcal{L}_{B_N}$$

Given $\mathbf{A}_N = (A_j)_{j\in J}$ define a new matrix $t(\mathbf{A}_N)$

Definition (Generalization of *-polynomials)

Let $T = (V, E)$ be a finite connected graph, $\gamma : E \to J$, $\varepsilon : E \to \{1, *\}$ (so that for an edge $e \in E$ corresponds the matrix $A_{\gamma(e)}^{\varepsilon(e)}$) and two distinguished vertices "in","out" $\in V$. Call graph polynomial the data $t = (T, \gamma, \varepsilon, in, out)$ and define the matrix $t(\mathbf{A}_N)$

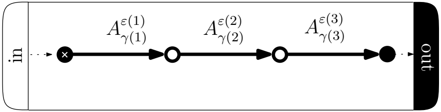
$$t(\mathbf{A}_N)(i,j) = \sum_{\substack{\phi: V\to[N] \\ \phi(in)=i,\phi(out)=j}} \prod_{e=(v,w)\in E} A_{\gamma(e)}^{\varepsilon(e)}\big(\phi(v),\phi(w)\,\big).$$

Definition (Generalization of the *-distribution)

The distribution of traffics of $\mathbf{A}_N$ is the map $t \mapsto \mathbb{E}\big[\frac{1}{N}\mathrm{Tr}\, t(\mathbf{A}_N)\big]$.

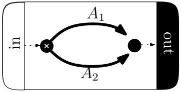
$$t(\mathbf{A}_N)(i,j) = \sum_{\substack{\phi:V\to[N]\\ \phi(in)=i,\phi(out)=j}} \prod_{e=(v,w)\in E} A_{\gamma(e)}^{\varepsilon(e)}\big(\phi(v),\phi(w)\,\big).$$

The distribution of traffics contains the *-distribution

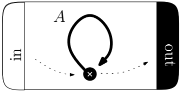


$= A_{\gamma(1)}^{\varepsilon(1)} \times A_{\gamma(2)}^{\varepsilon(2)} \times A_{\gamma(3)}^{\varepsilon(3)}$

but also more quantities



$= A_1 \circ A_2$



$= diag(A(i,i))_{i=1,\dots,N}$

(where $\circ$ denotes the entry wise product of matrices)

Main general theorem

Theorem (M. 12)

$\mathbf{A}_N^{(1)}, \dots, \mathbf{A}_N^{(L)}$ *independent families of random matrices such that*

1. *each family (except possibly one) is* **permutation** *invariant,*
2. *each family converges in* **distribution of traffics**,
3. *+ technical condition, say the deccorelation property : for any ℓ and any $t_1, \dots, t_p$*

$$\lim_{N\to\infty} \mathbb{E}\Big[\prod_{i=1}^{p} \frac{1}{N}\mathrm{Tr}\, t_i(\mathbf{A}_N^{(\ell)})\Big] = \prod_{i=1}^{p} \lim_{N\to\infty} \mathbb{E}\Big[\frac{1}{N}\mathrm{Tr}\, t_i(\mathbf{A}_N^{(\ell)})\Big].$$

Then the collection of all families converges in distribution of traffics and so in *-distribution

The families are said to be asymptotically **traffic-free**. Explicit formula for $\Phi(P)$ but no hope for a general analytical theory in general

Two practical criterions

Proposition (Rigidity of freeness)

If A_N and B_N are asymptotically traffic-free and A_N has the same limiting distribution of traffics as a unitary invariant random matrix, then A_N and B_N are asymptotically free.

Example : if A_N as the same limit as a GUE matrix.

Proposition (Criterion for the lack of freeness)

If A_N and B_N are asymptotically traffic-free but, with $\Phi_N := \mathbb{E}\big[\frac{1}{N}\mathrm{Tr}\ \cdot\ \big]$

$$\lim_{N\to\infty} \Phi_N\big[\, P_1(A_N) \circ P_2(A_N)\big] \neq \lim_{N\to\infty} \Phi_N\big[P_1(A_N)\big] \times \Phi_N\big[P_2(A_N)\big]$$

$$\lim_{N\to\infty} \Phi_N\big[Q_1(B_N) \circ Q_2(B_N)\big] \neq \lim_{N\to\infty} \Phi_N\big[Q_1(B_N)\big] \times \lim_{N\to\infty} \Phi_N\big[Q_2(B_N)\big],$$

then A_N and B_N are not asymptotically free.

Application to random graphs

Two random undirected graphs with set of vertices $[N] = \{1, \dots, N\}$.

Erdös-Rényi random graph $G(\alpha_N)$ of parameter α_N : each edge is drawn independently with probability α_N. Denoting $A(\alpha_N)$ its adjacency matrix, denote

$$M(\alpha_N) = \frac{A(\alpha_N) - \alpha_N J_N}{\sqrt{d_N(1-\alpha_N)}}, \quad \alpha_N = \frac{d_N}{N}, \quad J_N = \text{matrix full of ones}$$

When $\alpha_N \sim \alpha \in]0,1[$, it is a Wigner matrix. Two regimes for the limit of the e.s.d.

Proposition

1. *[Wigner] if $d_N \underset{N\to\infty}{\longrightarrow} \infty$ then $\mathcal{L}_{M(\alpha_N)}$ converges to the semicircular law with radius 2.*
2. *[Khorunzhy, Shcherbina, Vengerovsky (04)] if $d_N \lim d$ then $\mathcal{L}_{M(\alpha_N)}$ converges to a distribution with unbounded support, depending on d for which few is known (see J. Salez's talk Friday).*

Uniform regular random graph G_{d_N} of parameter d_N : chosen uniformly on the set of simple graphs (no loops nor multiple edges) whose degree of each vertex is $d_N \in \{0, \dots, N-1\}$. Denoting A_{d_N} its adjacency matrix, denote

$$M_{d_N} = \frac{A_{d_N} - \alpha_N J_N}{\sqrt{d_N(1-\alpha_N)}}, \quad \alpha_N = \frac{d_N}{N}, \quad J_N = \text{matrix full of ones}$$

Proposition

1. *[McKay (81)] if $d_N \underset{N\to\infty}{\longrightarrow} d$ then A_{d_N} converges to the distribution*

$$d\pi_d(x) = \frac{d\sqrt{4(d-1)-x^2}}{2\pi(d^2-x^2)} 1_{|x|\leqslant 2\sqrt{d-1}} dx.$$

2. *[Tran, Vu, Wang (12), Dumitriu, Pal (12)] if $d_N \underset{N\to\infty}{\longrightarrow} \infty$, $N - d_N \underset{N\to\infty}{\longrightarrow} \infty$ then M_{d_N} converges to the semicircular law with radius 2.*

What happen from the point of view of free probability ?

Given $\mathbf{M}_N = (M_1, \dots, M_L)$ independent copies of the normalized adjacency matrices of the random graphs, are the matrices of $\mathbf{M}_N$ asymptotically free ? Given $\mathbf{A}_N$ a family of deterministic matrices converging in *-distribution, does $\mathbf{M}_N$ asymptotically free from $\mathbf{A}_N$?

asymp freeness of $\mathbf{M}_N$	$d_N \to d$	$d_N \to \infty$
$Erdos - Renyi$	No	Yes
$Regular$	Yes	Yes *

asy. free. of $\mathbf{M}_N$ and $\mathbf{A}_N$	$d_N \to d$	$d_N \to \infty$
$Erdos - Renyi$	No	Yes
$Regular$	No	Yes *

* : in collaboration with S. Péché with a slight assumption on d_N.

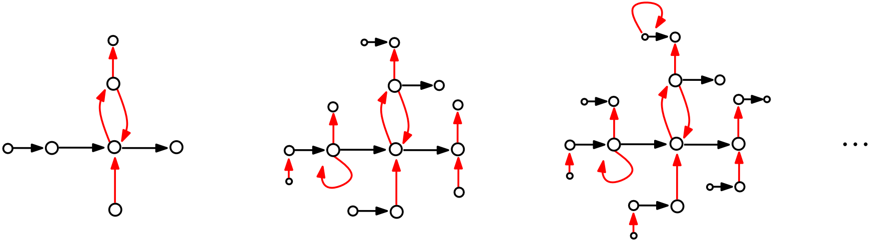
Case $d_N \underset{N\to\infty}{\longrightarrow} d$

Proposition

Let G_N be a graph and A_N its adjacency matrix. Assume $\mathbb{E}\big[\text{degree}^k\big] \leqslant a_k$ uniformly in N. Then A_N converges in distribution of traffics if and only if G_N converges in weak local topology : i.e. choosing uniformly at random a vertex ρ_N of G_N, for any $p \geq 1$ the subgraph of G_N consisting of vertices at distance less than p of ρ_N converges.

E-R : converges to the Galton-Watson tree with poisson offspring,
d_N-regular graph converges to the d-ary regular tree.

Moreover, if two adjacency matrices of graphs A_N and B_N are asymptotically traffic free, the limit (A_N, B_N) can be understood thanks to a "random free product" of the limiting graphs of A_N and B_N.



Case $d_N \underset{N\to\infty}{\longrightarrow} d$

Proposition

Let $G_N^{(1)}$ and $G_N^{(2)}$ be two asymptotically traffic free graphs.

1. *If both the limits of the $G_N^{(1)}$ and $G_N^{(2)}$ are not regular graphs, then $G_N^{(1)}$ and $G_N^{(2)}$ are not asymptotically free.*
2. *[Woess (86), Cartwright, Soardi (86)] If both the limits of the $G_N^{(1)}$ and $G_N^{(2)}$ are deterministic graphs, then $G_N^{(1)}$ and $G_N^{(2)}$ are asymptotically free.*

Case $d_N \underset{N\to\infty}{\longrightarrow} \infty$

Theorem (M., Péché (14))

Let G_N be a random graph on $[N]$ invariant in law by relabeling of its vertices. Denote $d_N = \mathbb{E}[\sum_{i=1}^N A(i,j)]$ the mean degree of any vertex. Given a finite simple graph T with edges $e_1, \ldots, e_n$ denote $e_i(G_N) = 1_{e_i \subset G_N}$. Assume that

$$\mathbb{E}\Big[\prod_{i=1}^n \big(e_i(G_N) - \frac{d_N}{N}\big)\Big] = \frac{d_N}{N} \times \varepsilon_N(T)$$

where $\varepsilon_N(T) = O(d_N^{-\frac{n}{2}})$. Then $\mathcal{M}_N$ converges to the semicircular law with radius two and is asymptotically free from copies of itself and deterministic matrices $\mathbf{A}_N$.

application for the regular graphs

Theorem (M., Péché (14))

Assume $d_N, N - d_N \underset{N\to\infty}{\longrightarrow} \infty$ and there exists $\eta > 0$ such that $|\frac{N}{2} - d_N - \eta\sqrt{d_N}| \underset{N\to\infty}{\longrightarrow} \infty$. Then the above estimates holds for the d_N regular graph.

Generalization : for random weighted random graphs Potentially : for stochastic block models

Idea of the proofs

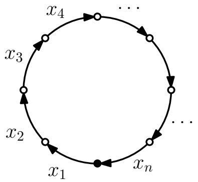
One main tool : the injective trace.
Recall the moment method :

$$\mathbb{E}\frac{1}{N}\mathrm{Tr}\, A_1 \dots A_n = \mathbb{E}\frac{1}{N}\sum_{i_1,\dots,i_n=1}^{N} A_1(i_1,i_2)\dots A_n(i_n,i_1)$$

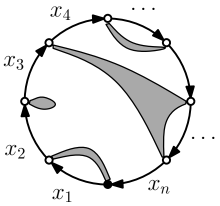
Then

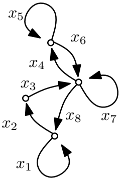
$$\mathbb{E}\frac{1}{N}\mathrm{Tr}\, A_1 \dots A_n = \sum_{\pi\in\mathcal{P}(n)} \mathbb{E}\frac{1}{N}\sum_{\substack{i_1,\dots,i_n \\ \ker \mathbf{i}=\pi}} A_1(i_1,i_2)\dots A_n(i_n,i_1)$$

where $\ker i$ is the partition such that $p \sim q$ iff $i_p = i_q$. Consider the following graph $T = (V, E, \gamma, \varepsilon)$



Given $T = (V, E, \gamma, \varepsilon)$ and $\pi \in \mathcal{P}(V)$ denote T^π the induced labelled graph

$T^\pi =$ 

$=$ 

$$\mathbb{E}\frac{1}{N}\mathrm{Tr}\, A_1 \dots A_n = \sum_{\pi \in \mathcal{P}(n)} \tau_N^0\big[T^\pi(\mathbf{A}_N)\big]$$

where

$$\tau_N^0\big[T(\mathbf{A}_N)\big] = \sum_{\substack{\phi : V \to [N] \\ \text{injective}}} \prod_{e=(v,w)\in E} A_{\gamma(e)}^{\varepsilon(e)}\big(\phi(v), \phi(w)\big).$$

$$\tau_N^0\big[T(\mathbf{A}_N)\big] = \sum_{\substack{\phi: V\to[N]\\ \text{injective}}} \prod_{e=(v,w)\in E} A_{\gamma(e)}^{\varepsilon(e)}\big(\phi(v),\phi(w)\big).$$

By an inclusion/exclusion principle,

Proposition

The convergence in distribution of traffics of $\mathbf{A}_N$ is equivalent to the point wise convergence of $T \mapsto \tau_N^0\big[T(\mathbf{A}_N)\big]$.

Example : for a Wigner matrix A_N, one has

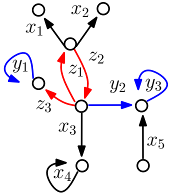
$$\tau_N^0\big[T(A_N)\big] \underset{N\to\infty}{\longrightarrow} \begin{cases} 1 & \text{if } T \text{ is a double tree} \\ 0 & \text{otherwise} \end{cases}$$

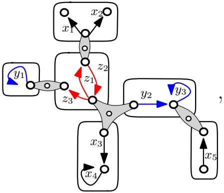
Definition of traffic-freeness : if $\mathbf{A}_N^{(1)}, \dots, \mathbf{A}_N^{(L)}$ are independent and permutation invariant, then $\tau_N^0\big[T(\mathbf{A}_N^{(1)}, \dots, \mathbf{A}_N^{(L)})\big]$ can be written as a product of some quantities $\tau_N^0\big[\tilde T(\mathbf{A}_N^{(\ell)})\big]$, times a normalizing constant. This yields

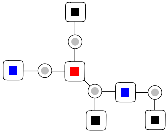
Definition

Families $\mathbf{A}_N^{(1)}, \dots, \mathbf{A}_N^{(L)}$ are asymptotically traffic free iff

$$\tau_N^0\big[T(\mathbf{A}_N^{(1)}, \dots, \mathbf{A}_N^{(L)})\big] \underset{N\to\infty}{\longrightarrow} \begin{cases} \prod_{\tilde T} \lim_{N\to\infty} \tau_N^0\big[\tilde T(\mathbf{A}_N^{(\ell)})\big] & \textit{if } T \textit{ is as below} \\ 0 & \textit{otherwise} \end{cases}$$

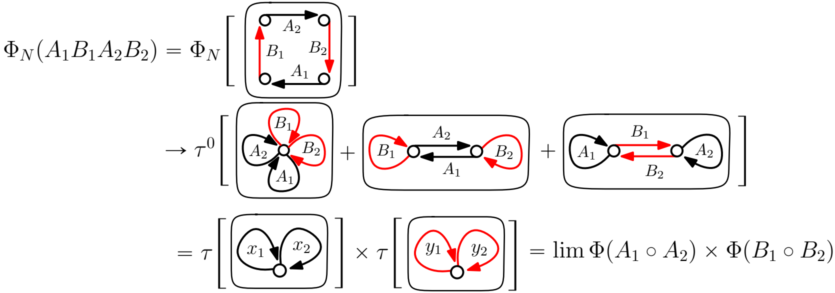
$T =$ 

$=$ 

, $\bar{T} =$ 

Example : proof of the criterion of non asymptotic freeness : (A_1, A_2) and (B_1, B_2) asymptotically traffic free. $\Phi_N = \mathbb{E}\frac{1}{N}\mathrm{Tr}$ and assume $\Phi_N(A_i), \Phi_N(B_i) \underset{N\to\infty}{\longrightarrow} 0$.

If (A_1, A_2) and (B_1, B_2) are asymptotically free than $\Phi_N(A_1B_1A_2B_2) \underset{N\to\infty}{\longrightarrow} 0$.



Thank you for your attention !