Yang Mills, unitary Brownian bridge and potential theory under constraint

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Outline of the talk
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▶ A very brief introduction to the physical context
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- Unitary Brownian motion : asymptotics
- Unitary Brownian bridge : shape of the dominant representation
- Some concluding remarks
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A lot of results concerning Yang-Mills on a cylinder or a sphere (Douglas-Kazakov, Gross-Matytsin (circa 1995)), in particular

Some properties of large \( N \) two-dimensional Yang-Mills theory
Unitary Brownian motion

One can define a Brownian motion on the unit circle $U := \{ z \in \mathbb{C} / |z| = 1 \}$, as follows:

$$U_1(t) = e^{iB(t)}$$

where $B$ is a standard Brownian motion on $\mathbb{R}$.

Otherwise stated, $U_1$ is a solution of the following very simple SDE:

$$dU_1(t) = idB(t)U_1(t) - \frac{1}{2}U_1(t)dt.$$  

For $N \geq 1$, this can be generalized as follows:

$$dU_N(t) = dK_N(t)U_N(t) - \frac{1}{2}U_N(t)dt,$$

with $K_N$ a Brownian motion on $\mathbb{C}$ equipped with $(X, Y)$ where $X Y^* = N \text{Tr}(X^* Y)$. 


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$$dU_N(t) = dK_N(t)U_N(t) - \frac{1}{2}U_N(t)dt,$$

with $K_N$ a Brownian motion on $\mathfrak{u}(N)$ equipped with
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Poisson summation formula: if $\tilde{f}(x) = \int_{\mathbb{R}} e^{iux} f(u) du$,

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In dimension $N$,

$$Q_{N,t}(U) = \sum_{\alpha \in \mathbb{Z}^N_\downarrow} e^{-\frac{c_2(\alpha)t}{2N}} s_{\alpha}(l_N)s_{\alpha}(U),$$
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In dimension $N$,

$$Q_{N,t}(U) = \sum_{\alpha \in \mathbb{Z}_N^\perp} e^{-\frac{c_2(\alpha)t}{2N}} s_\alpha(l_N) s_\alpha(U), \quad \text{with} \quad \Delta s_\alpha = -c_2(\alpha) s_\alpha$$
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\mathbb{E}[F(W_{N,T}(t_1), \ldots, W_{N,T}(t_n))] = \int_{\mathcal{U}(N)^n} F(U_1, U_2, \ldots, U_n) Q_{N,t_1} Q_{N,t_2-t_1} (U_1^{-1} U_2) \ldots 
\cdots Q_{N,t_n-t_{n-1}} (U_{n-1}^{-1} U_n) Q_{N,T-t_n} (U_n^{-1}) \frac{dU_1 \ldots dU_n}{Z_{N,T}}.
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For any $t \in (0, T)$, the density $Q_{N, t, T}^*: \mathcal{U}(N) \rightarrow \mathbb{R}$ of the distribution of $W_{N, T}(t)$ is given by

$$
Q_{N, t, T}^*(U) = \frac{Q_{N, t}(U) Q_{N, T-t}(U^{-1})}{Z_{N, T}},
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It is obtained by conditioning the Brownian motion to go back to the identity matrix at time $T$:

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\mathbb{E}[F(W_N, T(t_1), \ldots, W_N, T(t_n))] = \int_{\mathcal{U}(N)^n} F(U_1, U_2, \ldots, U_n) Q_{N, t_1} (U_1) Q_{N, t_2-t_1} (U_1^{-1} U_2) \ldots \ldots Q_{N, t_n-t_{n-1}} (U_{n-1}^{-1} U_n) Q_{N, T-t_n} (U_{n-1}^{-1}) \frac{dU_1 \ldots dU_n}{Z_{N, T}}.
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with

$$
Z_{N, T} := \int_{\mathcal{U}(N)} Q_{N, t}(U) Q_{N, T-t} (U^{-1}) dU = Q_{N, T}(I_N) = \sum_{\lambda \in \mathbb{Z}_N^T} e^{-\frac{c_2(\alpha)}{2N} T s_\alpha(I_N)^2}.
$$
Convergence of the u.B.m in large dimension (Biane, 97)
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If \( \hat{\mu}_N := \frac{1}{N} \sum_{i=1}^{N} \delta_{\lambda_i,N(t)} \), then \( \int_{\mathbb{U}} x^n d\hat{\mu}_N(x) = \frac{1}{N} \sum_{i=1}^{N} \lambda_{i,N}(t)^n = \frac{1}{N} \text{Tr}(U_N(t)^n) \).
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We are seeking for

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$$= \lim_{N \to \infty} \frac{1}{N} \sum_{\alpha \in \mathbb{Z}_N} e^{-\frac{c_2(\alpha)}{2N} t} s_\alpha(I_N) \int_{U(N)} \overline{s_\alpha(U)} \text{Tr}(U^n) dm_N(U).$$
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\]

\[
p_n(x_1, \ldots, x_N) := \sum_{i=1}^N x_i^n = \sum_{r=0}^{n-1} (-1)^r s_{(n-r,1,1,\ldots,1,0,\ldots,0)}(x_1, \ldots, x_N).
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\]

Ex : \( p_2 := \sum x_i^2 = \sum_{i \leq j} x_i x_j - \sum_{i < j} x_i x_j = s_2 - s_{(1,1)} \)
Convergence of the u.B.m in large dimension
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\[
\begin{align*}
c_n(t) := \lim_{N \to \infty} \mathbb{E} \left[ \frac{1}{N} \text{Tr}((U_N(t))^n) \right] \\
= \lim_{N \to \infty} \frac{1}{N} \sum_{\alpha \in \mathbb{Z}_N} e^{-\frac{c_2(\alpha)t}{2N}} s_\alpha(I_N) \int_{\mathcal{U}(N)} \overline{s_\alpha(U)} \text{Tr}(U^n) dm_N(U).
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\]

and \( \int_{\mathcal{U}(N)} \overline{s_\alpha(U)} s_\beta(U) dm_N(U) = \delta_{\alpha,\beta} \mathbf{1}_{\ell(\alpha) \leq N} \).
For $\alpha(n, r, N) := (n - r, 1, 1, \ldots, 1, 0, \ldots, 0)$ (with $r < n \leq N$),
For \( \alpha(n, r, N) := (n - r, 1, 1, \ldots, 1, 0, \ldots, 0) \) (with \( r < n \leq N \)), one can explicitely compute

\[
c_2(\alpha(n, r, N)) = Nn + n^2 - (2r + 1)n
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$$c_2(\alpha(n, r, N)) = Nn + n^2 - (2r + 1)n$$

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$$s_{\alpha(n, r, N)}(I_N) = \frac{(N + n - r - 1)!}{(N - r - 1)!r!n(n - r - 1)!}$$
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to obtain

**Proposition** (Biane, 97)

$$c_n(t) = e^{-\frac{nt}{2}} \sum_{k=0}^{n-1} (-1)^k \frac{t^k}{k!} n^{k-1} \binom{n}{k+1} = e^{-\frac{nt}{2}} \frac{1}{n} L_{n-1}(nt).$$
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For any $t > 0$, we denote by $\nu_t$ the probability measure on $\mathbb{U}$ such that, for all $n \geq 0$, $\int z^{-n} d\nu_t(z) = \int z^n d\nu_t(z) = c_n(t)$. 
Unitary Brownian bridge: shape of the dominant representation
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\[ Z_{N,T} = \sum_{\alpha \in Z_N^T} e^{-\frac{c_2(\alpha)}{2N}} T s_\alpha(I_N)^2. \]
Unitary Brownian bridge: shape of the dominant representation

\[ Z_{N,T} = \sum_{\alpha \in \mathbb{Z}_N} e^{-\frac{c_2(\alpha)}{2N}} T s_\alpha (I_N)^2. \]

\[ \hat{\mu}_\ell := \frac{1}{N} \sum_{i=1}^{N} \delta_{\frac{\alpha_i + N - i}{N}}. \]
Unitary Brownian bridge: shape of the dominant representation

\[ Z_{N,T} = \sum_{\alpha \in \mathbb{Z}^N_{\downarrow}} e^{-\frac{c_2(\alpha)}{2N} T} s_\alpha (l_N)^2. \]

From harmonic analysis, we get that

\[ Z_{N,T} = C_{N,T} \sum_{\ell} e^{-N^2 I_T(\hat{\mu}_\ell)}, \]

with

\[ I_T(\mu) := - \int \int \ln |x - y| d\mu(x) d\mu(y) + \int \frac{T}{2} x^2 d\mu(x) \]

and

\[ \hat{\mu}_\ell := \frac{1}{N} \sum_{i=1}^{N} \delta_{\frac{\alpha_i + N - i}{N}}. \]
Proposition

For all $T > 0$,

$$\lim_{N \to \infty} \frac{1}{N^2} \ln Z_{N,T} = \frac{T}{24} + \frac{3}{2} - \inf I_T(\mu),$$

with

$$I_T(\mu) = -\int \int \ln |x - y| d\mu(x) d\mu(y) + \int \frac{T}{2} x^2 d\mu(x).$$

Tools: large deviations results.
Proposition

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\]

with\[
I_T(\mu) = - \iint \ln |x - y| d\mu(x) d\mu(y) + \int \frac{T}{2} x^2 d\mu(x).
\]

Tools: large deviations results.
Third order phase transition

Proposition

For any $T > 0$, there exists a unique minimizer of the functional $I_T$ over the set $L$, that we denote by $\mu^*_T$.

▶ If $T \leq \pi/2$, the density of $\mu^*_T$ with respect to Lebesgue measure is given by $d\mu^*_T(x) = T^2/\pi \sqrt{4T - x^2} [(-2\sqrt{T})^1, 2\sqrt{T}]^*(x)$.

▶ If $T > \pi/2$, the density of $\mu^*_T$ is described in terms of elliptic functions.

Consequence: The function $F$ is of class $C^2$ on $\mathbb{R}^* +$ and of class $C^\infty$ on $\mathbb{R}^* + \{\pi/2\}$. At $\pi/2$, $F(3)$ has a discontinuity of first kind.
Third order phase transition

Proposition (...)  
For any $T > 0$, there exists a unique minimizer of the functional $I_T$ over the set $\mathcal{L}$, that we denote by $\mu^*_T$. 

If $T \leq \frac{\pi}{2}$, the density of $\mu^*_T$ with respect to Lebesgue measure is given by 

$$d\mu^*_T(x) = \frac{T^2}{\pi} \frac{1}{\sqrt{4T - x^2}} 1_{[-2\sqrt{T}, 2\sqrt{T}]}(x),$$

If $T > \frac{\pi}{2}$, the density of $\mu^*_T$ is described in terms of elliptic functions.

Consequence: The function $F$ is of class $C^2$ on $\mathbb{R}^*_+$ and of class $C^\infty$ on $\mathbb{R}^*_+ \setminus \{\frac{\pi}{2}\}$. At $\frac{\pi}{2}$, $F(3)$ has a discontinuity of first kind.
Proposition (…)  
For any $T > 0$, there exists a unique minimizer of the functional $I_T$ over the set $\mathcal{L}$, that we denote by $\mu_T^*$.

- If $T \leq \pi^2$, the density of $\mu_T^*$ with respect to Lebesgue measure is given by

\[
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\]

- If $T > \pi^2$, the density of $\mu_T^*$ is described in terms of elliptic functions.

Consequence: The function $F$ is of class $C^2$ on $\mathbb{R}^*_{+}$ and of class $C^\infty$ on $\mathbb{R}^*_{+}\{\pi^2\}$. At $\pi^2$, $F(3)$ has a discontinuity of first kind.
Third order phase transition

Proposition (…) 

For any $T > 0$, there exists a unique minimizer of the functional $I_T$ over the set $\mathcal{L}$, that we denote by $\mu^*_T$.

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\[ U^\mu + Q \geq C \text{ outside the support} \]
\[ U^\mu + Q = C \text{ on the “free” part} \]
\[ U^\mu + Q \leq C \text{ where it saturates} \]
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- In a recent work of Liechty and Wang, $\mu^*_T$ appears as the equilibrium measure associated to orthogonal polynomials for a discrete gaussian measure (also linked with Unitary brownian bridge)

- For some parameters $(t, T)$, the asymptotic spectral measure of $uBb$ is known and related to the family $\mu^*_T$ in a way which is still to be understood in details (work in progress with T. Lévy).