On the Central Limit Theorem for the linear eigenvalue statistics of the sum of independent matrices of rank one

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January 7, 2015

Intro. General Settings

$$M_n = M_{m,n} = \sum_{\alpha=1}^{m} \tau_\alpha \mathbf{y}_\alpha \mathbf{y}_\alpha{}^T$$

- $m = m_n : \; m_n/n \to c \in (0, \infty), \quad n \to \infty$
- $\tau_\alpha > 0, \; \sigma_m(\Delta) = \sharp\{\tau_\alpha \in \Delta\}/m$: $\quad \sigma_m \to \sigma$ weakly as $n \to \infty$
- $\mathbf{y}_\alpha = (y_{\alpha 1}, .., y_{\alpha n})^T \in \mathbb{R}^n$, $\alpha = 1, .., m$ – i.i.d. random vectors

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$$M_n = B_{m,n} D_m B_{m,n}^T$$

$$B_{m,n} = (\mathbf{y}_1 \;\; \mathbf{y}_2 \;\; \dots \;\; \mathbf{y}_m), \quad D_m = \{\tau_\alpha \delta_{\alpha\beta}\}_{\alpha,\beta=1}^{m}$$

Intro. Some Definitions

Normalized Counting Measure (NCM) of eigenvalues $\{\lambda_i^{(n)}\}_{i=1}^n$ of M_n:

$$N_n(\Delta) = \sharp\{\lambda_i^{(n)} \in \Delta\}/n$$

Linear Eigenvalue Statistic for a given test-function $\varphi : \mathbb{R} \to \mathbb{C}$:

$$\mathcal{N}_n[\varphi] := \sum_{j=1}^{n} \varphi(\lambda_j^{(n)}) = \mathrm{Tr}\varphi(M_n)$$

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Stieltjes transform of N_n:

$$s_n(z) = \int \frac{N_n(d\lambda)}{\lambda - z}, \quad \mathrm{Im}\, z \neq 0$$

Let us note that

$$s_n(z) = n^{-1}\mathrm{Tr}G(z),$$

where $G(z) = (M_n - zI)^{-1}$ is the resolvent of M_n.

Intro. Convergence of NCM

Theorem (A. Marchenko, L. Pastur (1967)) Let $M_n = \sum_{\alpha=1}^{m} \tau_\alpha \mathbf{y}_\alpha \mathbf{y}_\alpha{}^T$, where

- $m = m_n : \; m_n/n \to c \in (0, \infty), \quad n \to \infty,$
- $\tau_\alpha > 0, \; \sigma_m(\Delta) = \sharp\{\tau_\alpha \in \Delta\}/m$: $\sigma_m \to \sigma$ weakly as $n \to \infty$,
- $\mathbf{y}_\alpha \in \mathbb{R}^n$, $\alpha = 1, .., m$ are independent uniformly distributed on the unit sphere random vectors.

Then there exists a non-random probability measure N s.t. we have convergence in probability

$$\lim_{n\to\infty} N_n(\Delta) = N(\Delta), \quad \forall \Delta \subset \mathbb{R}.$$

The Stieltjes transform of N,

$$s(z) = \int \frac{N(d\lambda)}{\lambda - z}, \quad \mathrm{Im}\, z \neq 0,$$

is uniquely determined by the functional equation

$$zs(z) = c - 1 - c \int (1 + \tau s(z))^{-1} \sigma(d\tau)$$

considered in the class of functions analytic in $\mathbb{C}\backslash\mathbb{R}$ and such that $\mathrm{Im}\, z \, \mathrm{Im}\, f(z) \geq 0$, $\mathrm{Im}\, z \neq 0$.

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Theorem (A. Pajor, L. Pastur (2007)) The Marchenko-Pastur theorem remains valid if $M_n = \sum_{\alpha=1}^{m} \tau_\alpha \mathbf{y}_\alpha {\mathbf{y}_\alpha}^T$, where $\{\mathbf{y}_\alpha\}_{\alpha=1}^{m}$ are i.i.d. copies of $\mathbf{y}$:

- $$\mathbf{E}\{y_j\} = 0, \quad \mathbf{E}\{y_i y_j\} = \delta_{ij} n^{-1}, \tag{1}$$
- for any $n \times n$ complex matrix A_n, which does not depend on $\mathbf{y}$,
$$\mathbf{Var}\{(A_n\mathbf{y}, \mathbf{y})\} \leq ||A_n||^2 \delta_n, \quad \delta_n = o(1),\ n \to \infty. \tag{2}$$

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Remark Condition (2) can be replaced with

$$\mathbf{E}\{|(A_n \mathbf{y}, \mathbf{y})^\circ|\} \leq ||A_n|| \delta_n, \quad \delta_n = o(1),\ n \to \infty,$$

where $\xi^\circ = \xi - \mathbf{E}\{\xi\}$.

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Remark If $\mathbf{y}$ satisfies (1) and A_n does not depend on $\mathbf{y}$, then

$$\mathbf{E}\{(A_n\mathbf{y}, \mathbf{y})\} = n^{-1}\mathrm{Tr}A_n.$$

Intro. Main steps of the proof

$$M_n = \sum_{\alpha=1}^{m} \tau_\alpha \mathbf{y}_\alpha {\mathbf{y}_\alpha}^T, \qquad G(z) = (M_n - zI)^{-1}, \quad s_n(z) = n^{-1} \mathrm{Tr} G(z),$$

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Some ingredients of the proofs: (i) rank-one perturbation formula

$$G - G^\alpha = -\frac{\tau_\alpha G^\alpha \mathbf{y}_\alpha {\mathbf{y}_\alpha}^T G^\alpha}{1 + \tau_\alpha (G^\alpha \mathbf{y}_\alpha, \mathbf{y}_\alpha)}$$

(ii) the mean and variance of a bilinear form

$$\mathbf{E}\{(G^\alpha \mathbf{y}_\alpha, \mathbf{y}_\alpha)\} = \mathbf{E}\{s_n^\alpha(z)\} = \mathbf{E}\{s_n(z)\} + O(n^{-1}),$$
$$\mathbf{Var}\{(G^\alpha \mathbf{y}_\alpha, \mathbf{y}_\alpha)\} \leq |\operatorname{Im} z|^{-2}\delta_n, \quad \delta_n = o(1),\ n \to \infty.$$

(iii) certain martingale-differences technique used to estimate vanishing terms.

$$\mathbf{E}\{s_n(z)\} = z^{-1}\big(mn^{-1} - 1\big) - \frac{1}{nz}\sum_{\alpha=1}^{m} \mathbf{E}\Big\{\frac{1}{1 + \tau_\alpha(G^\alpha \mathbf{y}_\alpha, \mathbf{y}_\alpha)}\Big\}$$

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$$= z^{-1}\big(mn^{-1} - 1\big) - \frac{1}{nz}\sum_{\alpha=1}^{m} \frac{1}{1 + \tau_\alpha \mathbf{E}\{s_n^\alpha(z)\}}(1 + r_{\alpha n}),$$

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$$r_{\alpha n} = \mathbf{E}\Big\{\frac{\tau_\alpha\big(G^\alpha \mathbf{y}_\alpha, \mathbf{y}_\alpha\big)^\circ}{1+\tau_\alpha\big(G^\alpha \mathbf{y}_\alpha, \mathbf{y}_\alpha\big)}\Big\} = o(1), \quad n \to \infty.$$

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$$s(z) = z^{-1}(c - 1) - cz^{-1}\int (1 + \tau s(z))^{-1}\sigma(d\tau)$$

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Remark

$$\frac{1}{A} = \frac{1}{\mathbf{E}\{A\}} - \frac{A^\circ}{A\mathbf{E}\{A\}}, \quad A^\circ = A - \mathbf{E}\{A\}$$

$$\frac{1}{A} = \frac{1}{\mathbf{E}\{A\}} - \frac{A^\circ}{\mathbf{E}\{A\}^2} + \frac{(A^\circ)^2}{\mathbf{E}\{A\}^3} - \dots + (-1)^t \frac{(A^\circ)^t}{A\mathbf{E}\{A\}^t}.$$

Intro. An analog of the Law of Large Numbers

If

$$n^{-1}\mathcal{N}_n[\varphi] = \int \varphi(\lambda) N_n(d\lambda) = \sum_{j=1}^{n} \varphi(\lambda_j^{(n)})$$

is a normalized linear eigenvalue statistic corresponding to a bounded continuous test-function $\varphi : \mathbb{R} \to \mathbb{C}$, then under conditions of the previous theorem we have in probability:

$$\lim_{m,n\to\infty, m/n\to c} n^{-1}\mathcal{N}_n[\varphi] = \int \varphi(\lambda) N(d\lambda).$$

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What about the limiting probability law for the fluctuations

$$\mathcal{N}_n^{\circ}[\varphi] = \mathcal{N}_n[\varphi] - \mathbf{E}\{\mathcal{N}_n[\varphi]\}?$$

$$\mathbf{y}_\alpha - ? : \ \mathcal{N}_n^{\circ}[\varphi] \underset{m,n\to\infty, m/n\to c}{\longrightarrow} ? \textit{ in distribution}$$

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The bound for the variance of LES

Lemma Let $M_n = \sum_{\alpha=1}^{m} \tau_\alpha \mathbf{y}_\alpha {\mathbf{y}_\alpha}^T$, where $\{\mathbf{y}_\alpha\}_{\alpha=1}^m$ are independent copies of $\mathbf{y} = \{y_i\}_{j=1}^n$ having an unconditional distribution, and

- $\mathbf{E}\{y_j\} = 0, \quad \mathbf{E}\{y_i y_j\} = \delta_{ij} n^{-1}$,
- $a_{2,2} := \mathbf{E}\{y_i^2 y_j^2\} = n^{-2} + O(n^{-3}),\ i \neq j,\ \ \kappa_4 := \mathbf{E}\{y_j^4\} - 3a_{2,2} = O(n^{-2})$.

Then there exists $C > 0$ s.t.

$$\mathbf{Var}\{\mathcal{N}_n[\varphi]\} \le C||\varphi||_{2+\delta}^2,$$

where $\varphi \in H_{2+\delta} = \{\psi : ||\psi||_{2+\delta}^2 = \int (1 + 2|\xi|)^{2(2+\delta)} |\widehat{\psi}(\xi)|^2 d\xi < \infty\}$.

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Proposition (M. Shcherbina)

$$\mathbf{Var}\{\mathcal{N}_n[\varphi]\} \leq C_q ||\varphi||_q^2 \int_0^\infty d\eta e^{-\eta} \eta^{2q-1} \int_{-\infty}^{\infty} \mathbf{Var}\{s_n(x + i\eta)\} dx.$$

Definition. We say that a random vector $\mathbf{y} \in \mathbb{R}^n$ is a **CLT-vector** if

$$\mathbf{E}\{y_j\} = 0, \quad \mathbf{E}\{y_i y_j\} = \delta_{ij} n^{-1},$$

its distribution is unconditional, there exist n-independent $a, b \in \mathbb{R}$, s.t.

$$\begin{aligned} a_{2,2} &= \mathbf{E}\{y_i^2 y_j^2\} = n^{-2} + an^{-3} + o(n^{-3}), \quad i \neq j, n \to \infty, \\ \kappa_4 &= \mathbf{E}\{y_j^4\} - 3a_{2,2} = bn^{-2} + o(n^{-2}), \quad n \to \infty, \end{aligned}$$

and

$$\mathbf{E}\{|(A_n \mathbf{y}, \mathbf{y})^\circ|^4\} \leq ||A_n||^4 \widetilde{\delta}_n, \quad \widetilde{\delta}_n = O(n^{-2}), \ n \to \infty$$

for any $n \times n$ complex matrix A_n, which does not depend on $\mathbf{y}$.

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for any $n \times n$ complex matrix A_n, which does not depend on $\mathbf{y}$.

For example, if $\mathbf{x}$ is uniformly distributed on $B_p^n = \{\mathbf{x} \in \mathbb{R}^n : \; ||\mathbf{x}||_p = (\sum |x_j|^p)^{1/p} \leq 1\}$, then

$$\mathbf{y} = \left(\frac{1}{n} \frac{B(1/p, 2/p)}{B(n/p+1, 2/p)} \right)^{1/2} \mathbf{x}$$

is a CLT-vector, and

$$a = -4p^{-1}, \quad b = \Gamma(1/p)\Gamma(5/p)\Gamma(3/p)^{-2} - 3.$$

CLT (O. Guédon, AL, A. Pajor, L. Pastur (2013))

Theorem Let $M_n = \sum_{\alpha=1}^m \tau_\alpha \mathbf{y}_\alpha \mathbf{y}_\alpha{}^T$, where

- $\{\mathbf{y}_\alpha\}_{\alpha=1}^m$ are CLT-vectors,
- $\int \tau^4 d\sigma(\tau) < \infty$.

If $\varphi \in H_{5/2+\varepsilon}$, $\varepsilon > 0$, then $\mathcal{N}_n^\circ[\varphi] = \mathcal{N}_n[\varphi] - \mathbf{E}\{\mathcal{N}_n[\varphi]\}$ converges in distribution to the Gaussian random variable with zero mean and the variance

$$V[\varphi] = \lim_{\eta \downarrow 0} \frac{1}{2\pi^2} \int \varphi(\mu) d\mu \int \varphi(\lambda) d\lambda \, \mathrm{Re}\left[C(z_\mu, \overline{z_\lambda}) - C(z_\mu, z_\lambda)\right],$$

where $z_\lambda = \lambda + \eta$, $z_\mu = \mu + \eta$,

$$\begin{aligned} C(z_1, z_2) &= \lim_{n\to\infty} \mathbf{Cov}\{\mathrm{Tr} G(z_1),\ \mathrm{Tr} G(z_2)\} \\ &= \frac{\partial^2}{\partial z_1 \partial z_2}\left(-(a+b)s(z_1)s(z_2)\frac{\Delta z}{\Delta s} + 2\ln\frac{\Delta s}{\Delta z}\right). \end{aligned}$$

$$zs(z) = c - 1 - c\int (1+\tau s(z))^{-1}\sigma(d\tau), \quad \Delta s = s(z_1) - s(z_2),\ \Delta z = z_1 - z_2.$$

If $\tau_1 = ... = \tau_m = 1$, then $\mathcal{N}_n^\circ[\varphi] \to \mathcal{N}(0, V[\varphi])$ in distribution,

$$V[\varphi] = \frac{1}{2\pi^2} \int_{a_-}^{a_+} \int_{a_-}^{a_+} \left(\frac{\triangle\varphi}{\triangle\lambda} \right)^2 \frac{(4c - (\lambda_1 - a_m)(\lambda_2 - a_m)) d\lambda_1 d\lambda_2}{\sqrt{(a_+ - \lambda_1)(\lambda_1 - a_-)}\sqrt{(a_+ - \lambda_2)(\lambda_2 - a_-)}} + \frac{a+b}{4c\pi^2} \left(\int_{a_-}^{a_+} \varphi(\mu) \frac{\mu - a_m}{\sqrt{(a_+ - \mu)(\mu - a_-)}} d\mu \right)^2, \qquad (3)$$

where

$$a_\pm = (1 \pm \sqrt{c})^2, \quad a_m = 1 + c,$$

and as before a, b:

$$a_{2,2} = \mathbf{E}\{y_{\alpha}{}_i^2 y_{\alpha}{}_j^2\} = n^{-2} + an^{-3} + o(n^{-3}), \quad \mathbf{E}\{y_{\alpha}{}_j^4\} - 3a_{2,2} = bn^{-2} + o(n^{-2}).$$

The expression for $V[\varphi]$ coincides with the known one for the limiting variance of linear eigenvalue statistics of Sample Covariance matrices, in which $a = 0$. In particular, we can get (3) if we suppose that all $\{y_{\alpha j}\}_{\alpha,j=1}^{m,n}$ are i.i.d. random variables having finite forth moment and

$$\mathbf{E}\{y_{\alpha j}\} = 0, \quad \mathbf{E}\{y_{\alpha}{}_j^2\} = n^{-1} + an^{-2}/2 + o(n^{-3}), \quad b = \mathbf{E}\{y_{\alpha}{}_j^4\} - 3\mathbf{E}\{y_{\alpha}{}_j^2\}^2.$$

Vectors with a log-concave unconditional distribution

Isotropic random vectors with a log-concave unconditional distribution are almost CLT-vectors.

Lemma If $\mathbf{y} \in \mathbb{R}^n$ has a log-concave unconditional symmetric distribution and satisfies

$$\mathbf{E}\{y_j\} = 0, \quad \mathbf{E}\{y_i y_j\} = \delta_{ij} n^{-1},$$

then

$$a_{2,2} = \mathbf{E}\{y_i^2 y_j^2\} = n^{-2} + O(n^{-3}), \quad \kappa_4 = \mathbf{E}\{y_j^4\} - 3a_{2,2} = O(n^{-2}),$$
$$\mathbf{E}\{|(A_n \mathbf{y}, \mathbf{y})^\circ|^{2p}\} \leq C||A_n||^{2p} n^{-p}, \; n \to \infty$$

for any $n \times n$ complex matrix A_n, which does not depend on $\mathbf{y}$.

Definition. The distribution of random vector $\mathbf{y} \in \mathbb{R}^n$ is called *unconditional* if its components $\{y_j\}_{j=1}^n$ have the same joint distribution as $\{\pm y_j\}_{j=1}^n$ for any choice of signs. It is called *symmetric* if the components $\{y_j\}_{j=1}^n$ have the same joint distribution as $\{y_{\alpha(j)}\}_{j=1}^n$ for any choice of permutation α.

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Consider random vectors of the form

$$\mathbf{Y} = \mathbf{y}^{(1)} \otimes ... \otimes \mathbf{y}^{(k)} \in (\mathbb{R}^n)^{\otimes k},$$

where $\mathbf{y}^{(1)}$,..., $\mathbf{y}^{(k)}$ are i.i.d. copies of a random vector $\mathbf{y} \in \mathbb{R}^n$,

$$Y_{\mathbf{j}} = y^{(1)}_{j_1} \times ... \times y^{(k)}_{j_k}, \quad \mathbf{j} = \{j_1, ..., j_k\},\ 1 \leq j_l \leq n,$$

k is fixed.

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k is fixed. Let $\mathcal{M}_n$ be an $n^k \times n^k$ random matrix of the form

$$\mathcal{M}_n = \mathcal{M}_{m,n,k} = \sum_{\alpha=1}^{m} \tau_\alpha \mathbf{Y}_\alpha \mathbf{Y}_\alpha{}^T,$$

where

- $m/n^k \to c \in (0,\infty), \quad n \to \infty$
- $\tau_\alpha > 0,\ \sigma_m(\Delta) = \sharp\{\tau_\alpha \in \Delta\}/m$: $\quad \sigma_m \to \sigma$ weakly as $n \to \infty$
- $\mathbf{Y}_\alpha = \mathbf{y}^{(1)}_\alpha \otimes \dots \otimes \mathbf{y}^{(k)}_\alpha$, and $\{\mathbf{y}^{(p)}_\alpha\}_{\alpha,p=1}^{m,k}$ are i.i.d. random vectors in $\mathbb{R}^n$

Tensor Product Version of Sample Covariance Matrices

Consider random vectors of the form

$$\mathbf{Y} = \mathbf{y}^{(1)} \otimes \ldots \otimes \mathbf{y}^{(k)} \in (\mathbb{R}^n)^{\otimes k},$$

where $\mathbf{y}^{(1)}$,..., $\mathbf{y}^{(k)}$ are i.i.d. copies of a random vector $\mathbf{y} \in \mathbb{R}^n$,

$$Y_{\mathbf{j}} = y^{(1)}_{j_1} \times \ldots \times y^{(k)}_{j_k}, \quad \mathbf{j} = \{j_1, \ldots, j_k\},\ 1 \le j_l \le n,$$

k is fixed. Let $\mathcal{M}_n$ be an $n^k \times n^k$ random matrix of the form

$$\mathcal{M}_n = \mathcal{M}_{m,n,k} = \sum_{\alpha=1}^{m} \tau_\alpha \mathbf{Y}_\alpha \mathbf{Y}_\alpha{}^T,$$

where

- $m/n^k \to c \in (0, \infty), \quad n \to \infty$
- $\tau_\alpha > 0,\ \sigma_m(\Delta) = \sharp\{\tau_\alpha \in \Delta\}/m: \quad \sigma_m \to \sigma$ weakly as $n \to \infty$
- $\mathbf{Y}_\alpha = \mathbf{y}^{(1)}_\alpha \otimes \ldots \otimes \mathbf{y}^{(k)}_\alpha$, and $\{\mathbf{y}^{(p)}_\alpha\}_{\alpha,p=1}^{m,k}$ are i.i.d. random vectors in $\mathbb{R}^n$

$$(\mathcal{M}_n)_{\mathbf{jl}} = \sum_{\alpha=1}^{m} \tau_\alpha y^{(1)}_{j_1} \times \ldots \times y^{(k)}_{j_k} y^{(1)}_{l_1} \times \ldots \times y^{(k)}_{l_k}$$

$\mathcal{M}_n = \sum_{\alpha=1}^m \tau_\alpha \mathbf{Y}_\alpha {\mathbf{Y}_\alpha}^T, \quad \mathbf{Y}_\alpha = \mathbf{y}_\alpha^{(1)} \otimes \ldots \otimes \mathbf{y}_\alpha^{(k)}$

Studied by M. Hastings et al as a model of data hiding and correlation scheme of Quantum Informatics (Ambainis, A., Harrow, A. W., and Hastings, M. B. (2012). *Random tensor theory: extending random matrix theory to random product states.* Commun. Math. Phys., **310** 1, 25-74).

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Lemma Let $\mathbf{Y} = \mathbf{y}^{(1)} \otimes \ldots \otimes \mathbf{y}^{(k)}$, where $\{\mathbf{y}^{(p)}\}_{p=1}^{k}$ are i.i.d. copies of random vector $\mathbf{y} \in \mathbb{R}^n$ s.t.

$$\mathbf{E}\{y_j\} = 0, \quad \mathbf{E}\{y_i y_j\} = \delta_{ij} n^{-1},$$

and for any $n \times n$ complex matrix A_n, which does not depend on $\mathbf{y}$,

$$\mathbf{E}\{|(A_n \mathbf{y}, \mathbf{y})^\circ|^{2t}\} \leq ||A_n||^{2t} \delta_n, \quad \delta_n = O(n^{-\alpha}), \; n \to \infty.$$

Then for any $n^k \times n^K$ complex matrix $\mathcal{A}_n$, which does not depend on $\mathbf{y}$

$$\mathbf{E}\{|(\mathcal{A}_n \mathbf{Y}, \mathbf{Y})^\circ|^{2t}\} \leq C_t k^t ||\mathcal{A}_n||^{2t} \delta_n.$$

Theorem (Convergence of NCM) If $\{\mathbf{y}_\alpha^{(p)}\}_{\alpha,p=1}^{m,k}$ are i.i.d. copies of random vector $\mathbf{y} \in \mathbb{R}^n$ s.t.

$$\mathbf{E}\{y_j\} = 0, \quad \mathbf{E}\{y_i y_j\} = \delta_{ij} n^{-1},$$

and for any $n \times n$ complex matrix A_n, which does not depend on $\mathbf{y}$,

$$\mathbf{Var}\{(A_n\mathbf{y}, \mathbf{y})\} \leq ||A_n||^2 \delta_n, \quad \delta_n = o(1), \ n \to \infty,$$

then the NCM of $\mathcal{M}_n$, $N_n(\Delta) = \sharp\{\lambda_i^{(n)} \in \Delta\}/n^k$, converges in probability to the non-random probability measure N defined in the Marchenko-Pastur theorem.

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Lemma If $\mathcal{N}_n[\varphi] = \mathrm{Tr}\varphi(\mathcal{M}_n)$, $\varphi \in H_{5/2+\varepsilon}$, $\varepsilon > 0$, and $\{\mathbf{y}_\alpha^{(p)}\}_{\alpha,p=1}^{m,k}$ are i.i.d. copies of random vector $\mathbf{y} \in \mathbb{R}^n$ having an unconditional distribution, and

$$\mathbf{E}\{y_j\} = 0, \quad \mathbf{E}\{y_i y_j\} = \delta_{ij} n^{-1},$$
$$\mathbf{Var}\{(A_n\mathbf{y}, \mathbf{y})\} \leq C||A_n||^2 n^{-1},$$

then

$$\mathbf{Var}\{\mathcal{N}_n[\varphi]\} = O(kn^{k-1}),\ n \to \infty.$$

$\mathcal{M}_n = \sum_{\alpha=1}^m \tau_\alpha \mathbf{Y}_\alpha \mathbf{Y}_\alpha{}^T, \quad \mathbf{Y}_\alpha = \mathbf{y}_\alpha^{(1)} \otimes \ldots \otimes \mathbf{y}_\alpha^{(k)}$

Theorem (CLT) If $\varphi \in H_{5/2+\varepsilon}$, $\varepsilon > 0$, $\int \tau^4 d\sigma(\tau) < \infty$, and $\{\mathbf{y}_\alpha^{(p)}\}_{\alpha,p=1}^{m,k}$ are i.i.d. copies of random vector $\mathbf{y} \in \mathbb{R}^n$ having an unconditional distribution,

$$\begin{aligned}
&\mathbf{E}\{y_j\} = 0, \quad \mathbf{E}\{y_i y_j\} = \delta_{ij} n^{-1},\\
&a_{2,2} = \mathbf{E}\{y_i^2 y_j^2\} = n^{-2} + an^{-3} + o(n^{-3}),\\
&\kappa_4 = \mathbf{E}\{y_j^4\} - 3a_{2,2} = bn^{-2} + o(n^{-2}),\\
&\mathbf{E}\{|(A_n\mathbf{y}, \mathbf{y})^\circ|^{2(k+1)}\} \le C||A_n||^{2(k+1)} n^{-(k+1)},
\end{aligned}$$

then $n^{(1-k)/2}\mathcal{N}_n^\circ[\varphi]$ converges in distribution to the Gaussian random variable with zero mean and the variance

$$V[\varphi] = \lim_{\eta \downarrow 0} \frac{1}{2\pi^2} \int \varphi(\mu) d\mu \int \varphi(\lambda) d\lambda \,\mathrm{Re}\left[C(z_\mu, \overline{z_\lambda}) - C(z_\mu, z_\lambda)\right],$$

where $z_\lambda = \lambda + \eta$, $z_\mu = \mu + \eta$,

$$C(z_1, z_2) = -k(a+b+2) \frac{\partial^2}{\partial z_1 \partial z_2} \left(s(z_1) s(z_2) \frac{\Delta z}{\Delta s} \right),$$

$$zs(z) = c - 1 - c \int (1 + \tau s(z))^{-1} \sigma(d\tau).$$

Theorem (CLT) If $\varphi \in H_{5/2+\varepsilon}$, $\varepsilon > 0$, $\int \tau^4 d\sigma(\tau) < \infty$, and $\{\mathbf{y}_\alpha^{(p)}\}_{\alpha,p=1}^{m,k}$ are i.i.d. copies of random vector $\mathbf{y} \in \mathbb{R}^n$ having an unconditional distribution,

$$\mathbf{E}\{y_j\} = 0, \quad \mathbf{E}\{y_i y_j\} = \delta_{ij} n^{-1},$$
$$a_{2,2} = \mathbf{E}\{y_i^2 y_j^2\} = n^{-2} + an^{-3} + o(n^{-3}),$$
$$\kappa_4 = \mathbf{E}\{y_j^4\} - 3a_{2,2} = bn^{-2} + o(n^{-2}), \quad a + b + 2 \neq 0,$$
$$\mathbf{E}\{|(A_n\mathbf{y}, \mathbf{y})^\circ|^{2(k+1)}\} \leq C||A_n||^{2(k+1)} n^{-(k+1)},$$

then $n^{(1-k)/2}\mathcal{N}_n^\circ[\varphi]$ converges in distribution to the Gaussian random variable with zero mean and the variance

$$V[\varphi] = \lim_{\eta \downarrow 0} \frac{1}{2\pi^2} \int \varphi(\mu) d\mu \int \varphi(\lambda) d\lambda \, \mathrm{Re} \left[C(z_\mu, \overline{z_\lambda}) - C(z_\mu, z_\lambda) \right],$$

where $z_\lambda = \lambda + \eta$, $z_\mu = \mu + \eta$,

$$C(z_1, z_2) = -k(a + b + 2) \frac{\partial^2}{\partial z_1 \partial z_2} \left(s(z_1) s(z_2) \frac{\Delta z}{\Delta s} \right),$$
$$zs(z) = c - 1 - c \int (1 + \tau s(z))^{-1} \sigma(d\tau).$$

If $\{\mathbf{y}_\alpha^{(p)}\}_{\alpha,p=1}^{m,k}$ are i.i.d. copies of random vector $\mathbf{y}$ uniformly distributed on the unit sphere in $\mathbb{R}^n$, then

$$a = -2, \quad b = 0.$$

After renormalization we get: $n^{(2-k)/2}\mathcal{N}_n^\circ[\varphi]$ converges in distribution to the Gaussian random variable with zero mean and the variance

$$V[\varphi] = \lim_{\eta \downarrow 0} \frac{1}{2\pi^2} \int \varphi(\mu) d\mu \int \varphi(\lambda) d\lambda \, \mathrm{Re} \left[C(z_\mu, \overline{z_\lambda}) - C(z_\mu, z_\lambda) \right],$$

where $z_\lambda = \lambda + \eta$, $z_\mu = \mu + \eta$,

$$C(z_1, z_2) = \frac{\partial^2}{\partial z_1 \partial z_2} \left(3k(k-1) s(z_1) s(z_2) \frac{\Delta z}{\Delta s} + 2\delta_{k2} \ln \frac{\Delta s}{\Delta z} \right).$$

Thank you for your attention