Large complex correlated Wishart matrices: Fluctuations and asymptotic independence at the edges

Joint work with W. Hachem and J. Najim.

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Plan

1. Introduction and statement of the results
2. More precisions
3. Beyond universality
1) The matrix model

Complex correlated Wishart matrix:

\[ M_{N \times N} = X_{N \times n} \Sigma_{n \times n} X_{n \times N}^* \]

where \( X_{N \times n} \) is an \( N \times n \) matrix with independent \( \mathcal{N}(0, 1) \) entries, \( \Sigma_{n \times n} \) is an \( n \times n \) symmetric positive definite matrix.

Let \( x_1 \leq \cdots \leq x_N \) be the eigenvalues of \( M_{N \times N} \) (main characters), and \( \lambda_1 \leq \cdots \leq \lambda_n \) be the eigenvalues of \( \Sigma_{n \times n} \) (parameters).
1) The matrix model

Complex correlated Wishart matrix:

\[ \mathbf{M}_N = \frac{1}{N} \mathbf{X}_N \Sigma_N \mathbf{X}_N^* \]

where

- \( \mathbf{X}_N \) is an \( N \times n \) matrix with independent \( \mathcal{N}_\mathbb{C}(0, 1) \) entries
- \( \Sigma_N \) is an \( n \times n \) symmetric positive definite matrix
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Let \( x_1 \leq \cdots \leq x_N \) be the eigenvalues of \( \mathbf{M}_N \) (main characters), and \( \lambda_1 \leq \cdots \leq \lambda_n \) be the eigenvalues of \( \mathbf{\Sigma}_N \) (parameters).
1) Global behavior

Asymptotic regime:

\[ N, n \to \infty, \quad \frac{n}{N} \to \gamma \in (0, \infty), \]

\[ \nu_N := \frac{1}{n} \sum_{j=1}^{n} \delta_{\lambda_j} \xrightarrow{N \to \infty} \nu \quad \text{with compact support}. \]
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Global behavior (Marčenko-Pastur,67):

There exists \( \mu(\nu, \gamma) \) only depending on \( \nu, \gamma \) such that

\[ \frac{1}{N} \sum_{j=1}^{N} \delta_{x_j} \xrightarrow{N \to \infty} \mu(\nu, \gamma) \quad \text{a.s.} \]

[The Stieltjes transform of \( \mu(\nu, \gamma) \) satisfies a fixed-point equation]
Remark (to keep in mind for later):

At fixed finite $N$,

a good approximation for the distribution of $x_1, \ldots, x_N$

is the deterministic equivalent $\mu(\nu_N, \frac{n}{N})$. 
1) Global behavior

Examples: \( \nu = \frac{1}{3}(\delta_1 + \delta_3 + \delta_7) \) and \( \gamma = \frac{1}{10} \)
1) Local behavior

**Question:** Fluctuations of the extremal eigenvalues at each edge?

More precisely,

- Can we identify the *extremal eigenvalues*?
- Law of the *fluctuations*?
- Given several extremal eigenvalues, *asymptotic independence* of the fluctuations?
1) Local behavior, $\Sigma_N = I_N$

Example: The non-correlated case $\Sigma_N = I_N \implies \nu = \delta_1$
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Example: The non-correlated case $\Sigma_N = I_N \implies \nu = \delta_1$

- Limiting support (ignoring the Dirac mass at zero):

  \[ \text{Supp } \mu(\delta_1, \gamma) = [a, b], \quad \begin{cases} 
  a &= (1 - \sqrt{\gamma})^2 \\
  b &= (1 + \sqrt{\gamma})^2 
\end{cases} \]
1) Local behavior, $\Sigma_N = I_N$

**Example:** The non-correlated case $\Sigma_N = I_N \implies \nu = \delta_1$

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  \end{cases}
  \]

- (Geman, 80/Bai-Yin, 93)
  \[
  x_{\min} \xrightarrow{a.s.} a, \quad x_{\max} \xrightarrow{a.s.} b
  \]
  \[
  N \to \infty \\
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  \]
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Fluctuations:
1) Local behavior, $\Sigma_N = I_N$

**Fluctuations:**
Consider the support of the deterministic equivalent

$$\text{Supp } \mu(\delta_1, \frac{n}{\bar{N}}) = [a_N, b_N],$$

$$\begin{cases} a_N &= (1 - \sqrt{\frac{n}{\bar{N}}})^2 \\ b_N &= (1 + \sqrt{\frac{n}{\bar{N}}})^2. \end{cases}$$
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    a_N &= (1 - \sqrt{\frac{n}{N}})^2 \\
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\end{align*}
\]

Then, for some bounded sequence $(\sigma_N)$ **[varying from line to line]**,

- (Johansson, 00),

$$N^{2/3} \sigma_N (x_{\text{max}} - b_N) \xrightarrow{L} \frac{\mathcal{L}}{N \to \infty} \text{Tracy-Widom}$$

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- (Borodin-Forrester, 03),

If $\gamma \neq 1$, 

$$N^{2/3} \sigma_N (a_N - x_{\text{min}}) \xrightarrow{\mathcal{L}} \frac{\mathcal{L}}{N \to \infty} \text{Tracy-Widom}$$
Assume now $n = N + \alpha$ with $\alpha \in \mathbb{N}$ fixed.

Thus $\frac{n}{N} \to \gamma = 1$, $\chi_{\min} \xrightarrow{N \to \infty} \alpha = 0$ (hard edge)
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Thus \( \frac{n}{N} \to \gamma = 1 \), \( x_{\min} \overset{\text{a.s.}}{\to} \alpha = 0 \) (hard edge)

(Forrester, 93),

\[
N^2 \sigma_N x_{\min} \overset{\mathcal{L}}{\to} \text{Bessel}(\alpha)
\]
1) Local behavior, $\Sigma_N = l_N + \text{finite rank}$

Finite rank perturbation (Baik-Ben Arous-Péché,05):

$$\Sigma_N = \text{diag}(1 + \varepsilon, \ldots, 1 + \varepsilon, 1, \ldots, 1), \quad k \text{ fixed.}$$
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  $$x_{\text{max}} \xrightarrow{N \to \infty} b, \quad \text{Tracy-Widom fluctuations}$$
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- **If** $\varepsilon > \varepsilon_c$,
  $$x_{\max} \xrightarrow{a.s. N \to \infty} b_{\text{jump}} > b, \quad \text{GUE}(k) \text{ behavior}$$
1) Local behavior, $\Sigma_N = l_N + \text{finite rank}$

**Conclusion:** Local behaviors are sensitive to the convergence

$$\nu_N = \frac{1}{n} \sum_{j=1}^{n} \delta \lambda_j \xrightarrow{N \to \infty} \nu$$

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Large complex correlated Wishart matrices
1) Local behavior, General $\Sigma_N$

*General* $\Sigma_N$, with eigenvalues $\lambda_1 \leq \cdots \leq \lambda_n$
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**General** $\Sigma_N$, with eigenvalues $\lambda_1 \leq \cdots \leq \lambda_n$

- Assume

\[
\nu_N = \frac{1}{n} \sum_{j=1}^{n} \delta_{\lambda_j} \xrightarrow{N \to \infty} \nu \quad \text{with compact support}
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**General $\Sigma_N$, with eigenvalues $\lambda_1 \leq \cdots \leq \lambda_n$**

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0 < \liminf_{N \to \infty} \lambda_1, \quad \limsup_{N \to \infty} \lambda_n < +\infty
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- **Fact:** The limiting support $\text{Supp} \mu(\nu, \gamma)$ is compact, but not necessarily connected.
1) Local behavior, General $\Sigma_N$

**General $\Sigma_N$, with eigenvalues $\lambda_1 \leq \cdots \leq \lambda_n$**

- Assume

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\nu_N = \frac{1}{n} \sum_{j=1}^{n} \delta_{\lambda_j} \xrightarrow{N \to \infty} \nu \quad \text{with compact support}
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and

$$
0 < \liminf_{N \to \infty} \lambda_1, \quad \limsup_{N \to \infty} \lambda_n < +\infty
$$

- **Fact**: The limiting support $\text{Supp} \mu(\nu, \gamma)$ is **compact**, but not necessarily connected.

- For an edge $b$ of $\text{Supp} \mu(\nu, \gamma)$, we introduce a **regularity condition**.
Theorem (Right edges)

*Consider a regular right edge* \( b \). *Then,*
1) Local behavior, General $\Sigma_N$

Theorem (Right edges)

Consider a **regular** right edge $b$. Then,

- **(Existence of the extremal eigenvalue)**

There exists a deterministic sequence $(\Phi(N))$ such that

$$\lim_{N \to \infty} x_{\Phi(N)} \xrightarrow{a.s.} b, \quad \liminf_{N \to \infty} x_{\Phi(N)+1} > b \quad \text{a.s.}$$

- **(Tracy-Widom fluctuations)**

There exists a right edge $b_N$ of $\mu(\nu_N, n_N)$ such that

$$\frac{x_{\Phi(N)} - b_N}{\sigma_N} \xrightarrow{L} \text{Tracy-Widom},$$

for some explicit bounded sequence $(\sigma_N)$. 

---

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Theorem (Right edges)

Consider a **regular** right edge $b$. Then,

- **(Existence of the extremal eigenvalue)**
  
  There exists a deterministic sequence $(\Phi(N))$ such that
  
  $$\lim_{N \to \infty} x_{\Phi(N)} \rightarrow b, \quad \lim_{N \to \infty} \inf x_{\Phi(N)+1} > b \quad a.s.$$

- **(Tracy-Widom fluctuations)**
  
  There exists a right edge $b_N$ of $\mu(\nu_N, n_N)$ such that $b_N \rightarrow b$ and
  
  $$N^{2/3} \sigma_N (x_{\Phi(N)} - b_N) \xrightarrow{\mathcal{L}} Tracy-Widom,$$

  for some explicit bounded sequence $(\sigma_N)$. 

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Large complex correlated Wishart matrices
When \( b \) is the rightmost edge and there is no outliers, the Tracy-Widom fluctuations have already been obtained (El Karoui, 07) when \( \gamma \leq 1 \), and then extended to general \( \gamma \in (0, \infty) \) (Onatski, 08)
Consider a positive regular left edge $\alpha$. Then,

Theorem (Left soft edges)

There exists a deterministic sequence $(\Phi(N))$ such that $x_{\Phi(N)} \rightarrow \alpha$, $\lim_{N \rightarrow \infty} x_{\Phi(N)} - 1 < \alpha$. 

(Tracy-Widom fluctuations)

There exists a left edge $\alpha_N$ of $\mu(\nu_N, n_N)$ such that $\alpha_N \rightarrow \alpha$ and $N^{2/3} \sigma_N (\alpha_N - x_{\Phi(N)}) \overset{L}{\rightarrow} \text{Tracy-Widom}$, for some explicit bounded sequence $(\sigma_N)$. 

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Large complex correlated Wishart matrices
Consider a positive regular left edge $\alpha$. Then,

**Existence of the extremal eigenvalue**

There exists a deterministic sequence $(\Phi(N))$ such that

$$
x_{\Phi(N)} \xrightarrow{a.s.} \alpha, \quad \liminf_{N \to \infty} x_{\Phi(N)} - 1 < \alpha \quad a.s.
$$
Consider a **positive regular** left edge $a$. Then,

1. **(Existence of the extremal eigenvalue)**
   There exists a deterministic sequence $(\Phi(N))$ such that
   
   $$\lim_{N \to \infty} x_{\Phi(N)} = a, \quad \lim_{N \to \infty} x_{\Phi(N)} - 1 < a \quad \text{a.s.}$$

2. **(Tracy-Widom fluctuations)**
   There exists a left edge $a_N$ of $\mu(\nu_N, \frac{n}{N})$ such that $a_N \to a$ and
   
   $$N^{2/3} \sigma_N (a_N - x_{\Phi(N)}) \xrightarrow{\mathcal{L}} \text{Tracy-Widom},$$

   for some explicit bounded sequence $(\sigma_N)$.
Theorem (Asymptotic independence)

Given two finite families of positive **regular** left edges \((a_i)_{i \in I}\) and **regular** right edges \((b_j)_{j \in J}\), the associated fluctuations are asymptotically independent.
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Given two finite families of positive regular left edges \( (a_i)_{i \in I} \) and regular right edges \( (b_j)_{j \in J} \), the associated fluctuations are asymptotically independent.

Theorem (Hard edge)

Assume \( n = N + \alpha \) with \( \alpha \in \mathbb{Z} \) fixed. Then

\[
N^2 \sigma_N x_{\min} \xrightarrow{\mathcal{L}} \frac{\mathcal{L}}{N \to \infty} \text{Bessel}(\alpha),
\]

for some explicit bounded sequence \( (\sigma_N) \).

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Large complex correlated Wishart matrices
Corollary (Study of the Condition number)

We obtain convergence and fluctuations for

$$\kappa_N = \frac{x_{\text{max}}}{x_{\text{min}}}$$

in different regimes.
Beyond the Gaussian case?
Beyond the Gaussian case?

Recall

\[ M_N = \frac{1}{N} X_N \Sigma_N X_N^* \]

where

- \( X_N \) is an \( N \times n \) matrix with independent \( \mathcal{N}_\mathbb{C}(0, 1) \) entries
- \( \Sigma_N \) is an \( n \times n \) symmetric positive definite matrix
Beyond the Gaussian case?

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**Local law (Knowles-Yin,14)**: One can drop the Gaussian assumption and still have independent Tracy-Widom fluctuations.
2) More precisions
2) Regularity condition

Characterization of $\text{Supp } \mu_{(\nu, \gamma)}$ (Silverstein-Choi, 95):

The Cauchy transform $m(z) = \int \mu_{(\nu, \gamma)}(d\nu, d\gamma) z - x$, $z \in \mathbb{H}$, has an inverse given by $g(m) = \frac{1}{m} + \frac{\gamma}{m} \int x^{-1 - mx} \nu(dx)$ and which analytically extends to $\text{Dom } = \{m \in \mathbb{R} : m \neq 0, \frac{1}{m} \in \text{Supp } \nu\}$ (and takes real values there).
Characterization of $\text{Supp } \mu_{(\nu, \gamma)}$ (Silverstein-Choi, 95):

The Cauchy transform

$$m(z) = \int \frac{\mu_{(\nu, \gamma)}(dx)}{z - x}, \quad z \in \mathbb{H},$$
2) Regularity condition

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The Cauchy transform

$$m(z) = \int \frac{\mu(\nu,\gamma)(dx)}{z-x}, \quad z \in \mathbb{H},$$

has an inverse given by

$$g(m) = \frac{1}{m} + \gamma \int \frac{x}{1 - mx} \nu(dx)$$
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and which analytically extends to

$$\text{Dom} = \left\{ m \in \mathbb{R} : m \neq 0, \frac{1}{m} \notin \text{Supp}(\nu) \right\}$$

(and takes real values there).
2) Regularity condition

**Characterization of** $\text{Supp} \, \mu(\nu, \gamma)$ (Silverstein-Choi, 95):

Consider every (maximal) intervals $I \subset \text{Dom}$ where $g$ decreases, and delete the $g(I)$'s from $\mathbb{R}$, what is left is $\text{Supp} \, \mu(\nu, \gamma)$ (but zero).

Example: $\nu = \frac{7}{10} \delta_1 + \frac{3}{10} \delta_3$ and $\gamma = \frac{1}{10}$. 

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2) Regularity condition

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Consider every (maximal) intervals $I \subset \text{Dom}$ where $g$ decreases, and delete the $g(I)$'s from $\mathbb{R}$, what is left is $\text{Supp} \mu(\nu, \gamma)$ (but zero).

**Example:** $\nu = \frac{7}{10} \delta_1 + \frac{3}{10} \delta_3$ and $\gamma = \frac{1}{10}$
Thus, if $b$ is an edge of $\text{Supp} \mu_{(\nu,\gamma)}$, there exists $\partial$ such that

$$b = g(\partial),$$
2) Regularity condition

Thus, if \( b \) is an edge of \( \text{Supp} \mu_{(\nu,\gamma)} \), there exists \( d \) such that

\[
b = g(d),
\]

where either

- \( d \in \text{Dom} \) is a local extremum for \( g \)
- or \( d \in \partial(\text{Dom}) \)

\text{Definition}

We say \( b \) is regular if

\[
\liminf_{N \to \infty} n \min_{j=1} \| d - 1 \lambda_j \| > 0.
\]

Remark: If \( b = g(d) \) is regular, then necessarily \( d / \in \partial(Dom) \).
2) Regularity condition

Thus, if $b$ is an edge of $\text{Supp} \mu_{(\nu,\gamma)}$, there exists $\vartheta$ such that

$$b = g(\vartheta),$$

where either

- $\vartheta \in \text{Dom}$ is a local extremum for $g$
- or $\vartheta \in \partial(\text{Dom})$

**Definition**

We say $b$ is **regular** if

$$\liminf_{N \to \infty} \min_{j=1}^{n} \left| \vartheta - \frac{1}{\lambda_j} \right| > 0.$$  

**Remark:** If $b = g(\vartheta)$ is **regular**, then necessarily $\vartheta \notin \partial(\text{Dom}).$
If $b$ is a **regular** edge, then
If $b$ is a regular edge, then

- **Exact separation** (Bai-Silverstein, 98, 99)

  $\implies$ Existence for the associated extremal eigenvalue
2) Existence for extremal eigenvalues

If $b$ is a regular edge, then

- **Exact separation (Bai-Silverstein, 98, 99)**
  \[ \implies \text{Existence for the associated extremal eigenvalue} \]

- **Complex analysis (Montel, Rouché,...)**
  \[ \implies \text{Existence of an edge } b_N \text{ of } \mu(\nu_N, \frac{n}{N}) \text{ such that } b_N \to b \]
2) Tracy-Widom fluctuations

Determinantal structure (Johansson/BBP, 05) ⇒ Repartition function ≃ Fredholm determinant, i.e.

\[ P \left[ \frac{N^2}{3} \sigma_N \left( x - \Phi(N) - b_N \right) \right] \leq s = \det (I - K_N)_{L^2(s, \epsilon N^2/3)} + o(1) \]

⇒ Enough to prove the convergence \( K_N \to K_{Ai} \) as \( N \to \infty \) (in an appropriate sense)

Complex integral representation for \( K_N \) (idem):

\[ K_N(x, y) = \frac{1}{(2i\pi)^2} \oint \oint F_N(x, y; z, w), \]

where \( F_N \) is explicit.
2) Tracy-Widom fluctuations

- Determinantal structure (Johansson/BBP, 05)
  ⇒ Repartition function $\sim$ Fredholm determinant, i.e.

$$P\left[ N^{2/3} \sigma_N (x_{\Phi(N)} - b_N) \leq s \right] = \det \left( I - K_N \right)_{L^2(s, \varepsilon N^{2/3})} + o(1)$$
2) Tracy-Widom fluctuations

- **Determinantal structure** (Johansson/BBP,05)

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  \mathbb{P}\left[N^{2/3}\sigma_N (x_{\Phi(N)} - b_N) \leq s\right] = \det (I - K_N)_{L^2(s,\varepsilon N^{2/3})} + o(1)
  \]

  \[ \Rightarrow \text{Enough to prove the convergence} \]

  \[ K_N \rightarrow K_{Ai} \quad \text{as} \quad N \rightarrow \infty \]

  (in an appropriate sense)
2) Tracy-Widom fluctuations

- **Determinantal structure** (Johansson/BBP,05)
  \[ \Rightarrow \text{Repartition function } \approx \text{Fredholm determinant, i.e.} \]
  \[
  \mathbb{P}\left[ N^{2/3} \sigma_N \left( x_{\Phi(N)} - b_N \right) \leq s \right] = \det \left( I - K_N \right)_{L^2(s, \varepsilon N^{2/3})} + o(1)
  \]
  \[ \Rightarrow \text{Enough to prove the convergence} \]
  \[ K_N \rightarrow K_{Ai} \quad \text{as} \quad N \rightarrow \infty \]
  (in an appropriate sense)

- **Complex integral representation for** \( K_N \) (idem):
  \[
  K_N(x, y) = \frac{1}{(2i\pi)^2} \oint_{\Gamma} \oint_{\Theta} dz \, dw \, F_N(x, y; z, w),
  \]
  where \( F_N \) is explicit.
2) Tracy-Widom fluctuations

Asymptotic analysis as \( N \to \infty \) for

\[
K_N(x, y) = \frac{1}{(2i\pi)^2} \oint_{\Gamma} dz \oint_{\Theta} dw \ F_N(x, y; z, w)
\]

- Local analysis around \( \partial \Rightarrow \) Airy kernel
  - Saddle point of order two, almost routine computation

- The remaining of the integral is negligible
  - Clever analytic deformation of \( \Gamma \) and \( \Theta \), this is the HARD part
2) Tracy-Widom fluctuations

Asymptotic analysis as $N \to \infty$ for

$$K_N(x, y) = \frac{1}{(2i\pi)^2} \oint_{\Gamma} \oint_{\Theta} d\zeta d\omega \ F_N(x, y; \zeta, \omega)$$
Existence of the steepest descent contours?
2) Tracy-Widom fluctuations

Existence of the steepest descent contours?

Non-constructive proof using the maximum principle for subharmonic functions
2) Tracy-Widom fluctuations

“Right edge” analytic landscape:

\[ \Delta_1 \Delta_2 \Delta_{-1} \Delta_{-2} \Omega_+ - \Omega_- + \leftarrow \text{radius } \eta \]

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Large complex correlated Wishart matrices
2) Tracy-Widom fluctuations

“Left edge (generic)” analytic landscape:

\[
\Omega_- - \Omega_+ + \Omega_+ c \Delta_0 - \Delta_1 - \Delta_2 - \Delta_3 - \Delta_{-1} - \Delta_{-2} - \Delta_{-3}
\]
2) Tracy-Widom fluctuations

“Left edge (singular)” analytic landscape:

\[
\begin{align*}
\Delta_1 & \quad \Delta_2 \\
\Delta_{-1} & \quad \Delta_{-2} \\
\Omega_+ & \\
\Omega_- & \\
\end{align*}
\]
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- Use regularized Fredholm determinant $\implies$ "Trace-class norm" $\rightarrow$ "Hilbert-Schmidt norm"

- Use the steepest descent paths from the TW analysis.

- Use Bleher-Kuijlaars representation for $K_N(x, y)$
2) Bessel fluctuations

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- **Key:** The critical point $\vartheta$ is at infinity!
- Appropriate complex integral representation for the Bessel kernel
- Asymptotic analysis (now easy)
3) Beyond Universality

What is happening around an edge which is not regular?

(Easy case) If $k$ of the $1/\lambda_i$'s equal $d$, but the rest satisfies the regularity condition, then the fluctuations are described by the $k$-deformed Tracy-Widom distribution of BBP.

(Mysterious case) What if $d \in \partial (\text{Dom})$ ?

Conjecture: Universality breaks down i.e. strongly depends on $\nu$ and the way $\nu \rightarrow \nu_N$.

Similar situations:
- Additive perturbations of Wigner matrices (Capitaine-Péché, 14)
- Random patterns on the Gelfand-Tsetlin cone (Duse-Metcalfe, 14)
What is happening around an edge $b$ which is **not regular**?
What is happening around an edge $\partial$ which is not regular?

**Easy case** If $k$ of the $1/\lambda_j$'s equal $\partial$, but the rest satisfies the regularity condition, then the fluctuations are described by the $k$-deformed Tracy-Widom distribution of BBP.

(Mysterious case) What if $\partial \in \partial(D)$? Conjecture: Universality breaks down i.e. strongly depends on $\nu$ and the way $\nu \to \nu_N$. Similar situations:
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3) Beyond Universality

What is happening around an edge $\partial$ which is not regular?

- **(Easy case)** If $k$ of the $1/\lambda_j$’s equal $\partial$, but the rest satisfies the regularity condition, then the fluctuations are described by the $k$-deformed Tracy-Widom distribution of BBP.

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What is happening around an edge $\mathfrak{b}$ which is \textbf{not regular}?

- **(Easy case)** If $k$ of the $1/\lambda_j$’s equal $\mathfrak{d}$, but the rest satisfies the regularity condition, then the fluctuations are described by the $k$-deformed Tracy-Widom distribution of BBP.

- **(Mysterious case)** What if $\mathfrak{d} \in \partial(Dom)$?

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  i.e. strongly depends on $\nu$ and the way $\nu_N \to \nu$. 

Adrien Hardy, KTH

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Thank you for your attention!