About topological expansions

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Outline

About topological expansions

Topological expansions

Asymptotic expansions

Ideas of proofs

The GUE : topological expansion

Let $X^N = (X_1^N, \ldots, X_d^N)$ be independent GUE matrices. Then if q is a monomial

$$\mathbb{E}[\frac{1}{N}\mathrm{Tr}(q(X_1^N, \ldots, X_d^N))] = \sum_{g \geq 0} \frac{1}{N^{2g}} M(q, g)$$

with $M(q, g)$ the number of maps with genus g build over a star of type q.
Proof. Gaussian computation.

Maps

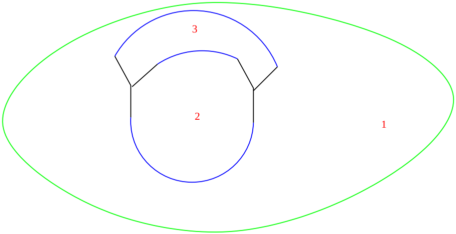
A map is a connected graph which is properly embedded into a surface, that is so that its edges do not cross and the faces (obtained by cutting the surface along the edges of the graph) are homeomorphic to disks. The genus of a map is the genus of this surface.

By Euler formula,

$$2 - 2g = \#\{vertices\}$$

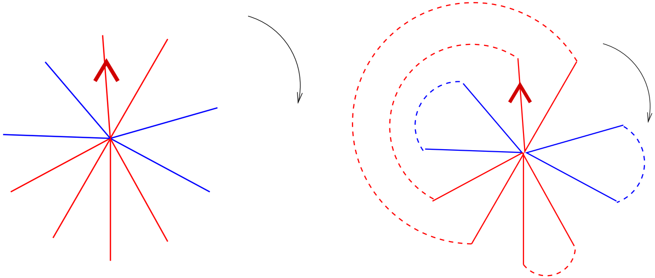
$$-\#\{edges\} + \#\{faces\}.$$

$$= 2 + 3 - 3$$



Enumeration of colored maps

Consider vertices with colored half-edges and enumerate maps build by matching half-edges of the same color.
Let $q(X_1, \ldots, X_d) = X_{i_1} X_{i_2} \cdots X_{i_p}$. A "star of type q" is the vertex with first half-edge of color i_1, the second color i_2 etc until the last which has color i_p.



$M((q_i, k_i)_{1 \leq i \leq m}; g)$ denotes the number of maps with genus g build on k_i stars of type q_i, $1 \leq i \leq m$.

Matrix models ('t Hooft ; Brézin, Itzykson, Parisi and Zuber)

Let $V_t = \frac{1}{2}\sum X_i^2 - \sum_{i=1}^{\ell} t_i q_i$ be a polynomial in d non-commutative indeterminates.

$$d\mathbb{P}_N^{V_t}(X_1^N,\ldots,X_d^N) = \frac{1}{Z_{V_t}^N} e^{-N\mathrm{Tr}(V_t(X_1^N,\ldots,X_d^N))} dX_i^N$$

Then for any monomial P in d non-commuting variables, we have the *formal* expansion

$$\mathbb{E}_{\mathbb{P}_N^{V_t}}[\frac{1}{N}\mathrm{Tr}(q(X_1^N,\ldots,X_d^N))] = \sum_{g=0}^{\infty} \frac{1}{N^{2g}} \tau_g^t(q), \quad \frac{1}{N^2}\log\frac{Z_{V_t}^N}{Z_{V_0}^N} = \sum_{g=0}^{\infty} F_g^t$$

where

$$\tau_g^t(q) = \sum \prod_i \frac{t_i^{k_i}}{k_i!} M((k_i,q_i)_{1\le i\le \ell},(1,q);g)$$

if $M((k_i,q_i)_{1\le i\le\ell},(1,q);g)$ is the number of maps with genus g build over k_i stars of type q_i, $1\le i\le \ell$ and one of type q.

This talk : Obtain such expansions asymptotically

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- CLT : Expansion of the partition function up to $o(1)$ gives estimation of :

$$\mathbb{P}_N^V(e^{\sum f(\lambda_i)}) = \frac{Z^N_{V-\frac{1}{N}f}}{Z^N_V}$$

for smooth enough f.

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- Universality : Would like to do double scale limits to get universality at the edge. Does not work yet. But can use similar arguments based on transport maps (Shcherbina 13', Bekerman-Figalli-G 13').

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- Beyond universality : Edelman-G-Peche 14' : Let W be $N \times (N+\nu)$ with i.i.d centered entries with covariance a and fourth cumulant κ_4, G be i.i.d standard Gaussian $N \times (N+\nu)$

$$P(\lambda_{min}((W+G)(W+G)^*) \geq s/n) = F_n(s) + \frac{sF_n'(s)\kappa_4}{(a^2+1)^2 n} + o(\frac{1}{n})$$

where $F_n(s) = P((1+a^2)\lambda_{min}(GG^*) \geq s/n)$.

About topological expansions

Asymptotic topological expansion

Let V be a polynomial in d non-commutative indeterminates, put

$$d\mathbb{P}_N^V(X_1^N,\ldots,X_d^N) = \frac{1}{Z_V^N} e^{-\frac{N}{2}\mathrm{Tr}(V(X_1^N,\ldots,X_d^N))} \prod_i 1_{\|X_i^N\|\le R} dX_i^N$$

Theorem

$R \in (2,+\infty)$ and $k \in \mathbb{N}$. $V = V^* = \frac{1}{2}\sum X_i^2 - \sum t_i q_i$, t_i small.

- $$\frac{1}{N^2}\log Z_V^N = \sum_{g=0}^{k} \frac{1}{N^{2g}} F^g(t) + o\left(\frac{1}{N^{2k}}\right),$$
- $$\mathbb{E}_{\mathbb{P}_N^V}\left[\frac{1}{N}\mathrm{Tr}(q(X_1^N,\ldots,X_d^N))\right] = \sum_{g=0}^{k} \frac{1}{N^{2g}} \tau_g^t(q) + o\left(\frac{1}{N^{2k}}\right)$$

with τ_g^t the generating function for maps with genus g.

-$m = 1$: Ambjórn (95), Albeverio-Pastur-Shcherbina (01), Ercolani-McLaughlin (03)

-$m = 2$: G.-Maurel-Segala ($g \le 1$ (05)(06)),Maurel-Segala ($\forall g$ (06))

Corollary : CLT

Let V be a polynomial in d non-commutative indeterminates so that

$$dP_N^V(X_1^N, \ldots, X_d^N) = \frac{1}{Z_V^N} e^{-\frac{N}{2}\mathrm{Tr}(V(X_1^N,\ldots,X_d^N))} \prod_i 1_{\|X_i^N\| \le R} dX_i^N$$

Theorem

$R \in (2, +\infty)$ and $k \in \mathbb{N}$. Assume $V = V^* = \frac{1}{2}\sum X_i^2 - \sum t_i q_i$ with t_i small. The law of $\mathrm{Tr}(P(X_1^N, \ldots, X_d^N)) - N\tau_0^t(P)$ converges towards a centered Gaussian variable with covariance $C(P, Q)$ given by the generating function for the enumeration of planar maps with two fixed stars of type P and Q.

Away from perturbative results : β-ensembles

$$dP^N_{\beta,V} = \frac{1}{Z^N_{\beta,V}} \prod |\lambda_i - \lambda_j|^\beta e^{-N\sum_{i=1}^N V(\lambda_i)} \prod d\lambda_i$$

$L_N = \frac{1}{N}\sum \delta_{\lambda_i}$ converges weakly towards μ_V.

Theorem

Assume that V is off-critical. If μ_V has connected support then,

$$\tau^N_{\beta,V}(q) := \mathbb{E}_{P^N_{\beta,V}}[L_N(q)] = \sum_{g=0}^{k} \frac{1}{N^g} \tau^g_{\beta,V}(q) + o(\frac{1}{N^{2k}}),$$

A CLT holds for linear statistics.

- Albeverio, Pastur et Shcherbina (01), Ercolani-Mc Laughlin (03), Borot-G (11)
- CLT : Johansson (98)

β-ensembles with several cuts

$$dP^N_{\beta,V} = \frac{1}{Z^N_{\beta,V}} \prod |\lambda_i - \lambda_j|^\beta e^{-N\sum_{i=1}^N V(\lambda_i)} \prod d\lambda_i$$

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Theorem (Deift, Kriecherbauer, McLaughlin, Venakides, and Zhou (99), Eynard (11), Borot-G (13))

Assume that V is off-critical but the support of μ_V has $p > 1$ connected components. Then,

- *If condition on number of particles in each connected component, the same expansion as for 1 cut holds,*

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Theorem (Deift, Kriecherbauer, McLaughlin, Venakides, and Zhou (99), Eynard (11), Borot-G (13))

Assume that V is off-critical but the support of μ_V has $p > 1$ connected components. Then,

- *If condition on number of particles in each connected component, the same expansion as for 1 cut holds,*
- *There is an optimal feeling fraction $\epsilon_1, \ldots, \epsilon_p$ so that*

$$\begin{aligned} Z^N_{\beta,V} &= \sum_{\sum n_i = N} Z^{n_1,\ldots,n_p}_{\beta,V} = \sum e^{N^2 F_0(\frac{n_i}{N}) + N F_1(\frac{n_i}{N}) + \cdots} \\ &= e^{N^2 F_0^* + N F_1^*} \sum_{\sum n_i = N} e^{\sum D_{i,j} F_0(n_i - \epsilon_i N, n_j - \epsilon_j N) + \sum D_i F_1(n_i - \epsilon_i N) + \ldots} \end{aligned}$$

β-ensembles with several cuts

$$dP^N_{\beta,V} = \frac{1}{Z^N_{\beta,V}} \prod |\lambda_i - \lambda_j|^\beta e^{-N\sum_{i=1}^N V(\lambda_i)} \prod d\lambda_i$$

$L_N = \frac{1}{N}\sum \delta_{\lambda_i}$ converges weakly towards μ_V.

Theorem

Assume that V is off-critical but the support of μ_V has $p > 1$ connected components then, take ϕ smooth.

$$P^N_{\beta,V}(e^{is(\sum \phi(\lambda_i) - N\int \phi d\mu_{eq})}) \simeq e^{iM(\phi) - \frac{1}{2}Q_*(\phi)} F(u(\phi))$$

where $F(0) = 1$ and oscillating otherwise,

$$u(\phi) = \frac{\beta}{2}\left((\partial_{\epsilon_h} - \partial_{\epsilon_0})\int \phi(x) d\mu_{eq,\epsilon}\right)_{1\le h\le p}$$

-Cf Pastur (06), Kriecherbauer-Shcherbina (11).

-Generalization to non-linear V [Borot-Kozlowski-G (14')]

The sinsh-model

In quantum integrable models, quantities such as :

$$z_N = \int_{\mathbb{R}^N} \prod_{a<b} \Big\{ \sinh[\pi\omega_1(y_a - y_b)] \sinh[\pi\omega_2(y_a - y_b)] \Big\}^{\beta} \cdot \prod_{a=1}^{N} e^{-W(y_a)} \cdot \mathbf{d}^N y$$

or

$$Z_N[V] = \int_{\mathbb{R}^N} \prod_{a<b} \Big\{ \prod_{i=1,2} \sinh[\pi\omega_i T_N(\lambda_a - \lambda_b)] \Big\}^{\beta} \cdot \prod_{a=1}^{N} e^{-NT_N V_N(\lambda_a)} \cdot \mathbf{d}^N \lambda \,.$$

show up (with e.g. $V(x) = w\cosh(x)$, $T_N = \log N$). The interaction presents the same singularity as β-models. Can we study their large N expansions, in particular derive the term of order 1 ?

Results on the Sinsh-model

$$Z_N[V] = \int_{\mathbb{R}^N} \prod_{a<b} \Big\{ \prod_{i=1,2} \sinh[\pi\omega_i T_N(\lambda_a - \lambda_b)] \Big\}^{\beta} \cdot \prod_{a=1}^{N} e^{-NT_N V(\lambda_a)} \cdot d^N\lambda \,.$$

Theorem (Borot-Kozlowski-G (14))

Assume $T_N = N^{\alpha}, \alpha < 1/6$, V strictly convex and smooth (not analytic), ($\beta = 1$)

$$\ln\left(\frac{Z_N[V]}{Z_N[V_{G;N}]}\right) = -N^{2+\alpha} \sum_{p=0}^{\lfloor 2/\alpha \rfloor + 1} \frac{\mathcal{Z}_p[V]}{N^{\alpha p}} + N^{\alpha} A[V] + B[V] + o(1) \,.$$

$Z_N[V_{G;N}]$ can be computed exactly.

About topological expansions

Schwinger-Dyson equations

All proofs rely on equations, called Schwinger-Dyson (or loop) equations, which are derived by integration by parts. Let us consider the Coulomb gas interacting particles models :

$$dP_V^N(\lambda_1,\ldots,\lambda_N) = \frac{1}{Z_V^N} \prod_{i<j} |\lambda_i - \lambda_j|^\beta e^{-N\sum V(\lambda_i)} \prod d\lambda_i$$

The empirical measure $L_N = \frac{1}{N}\sum_{i=1}^N \delta_{\lambda_i}$ satisfies

$$\int \left(\frac{\beta}{2} \iint \partial f(x,y) dL_N(x) dL_N(y) - \int V'(x) f(x) dL_N(x) \right) dP_V^N$$

$$= \frac{1}{N}(\frac{\beta}{2} - 1) \int \int f'(x) dL_N(x) dP_V^N .$$

Analysis of Schwinger-Dyson equations

For β models,

- when V is analytic, one deduces from the Schwinger-Dyson's equations, equations for the correlators

$$W^k(z_1, \ldots, z_k) = \partial_{\varepsilon_1} \cdots \partial_{\varepsilon_k} \log Z^N_{V+\frac{1}{N}\sum \frac{\epsilon_k}{z_k - .}}|_{\epsilon_i=0}$$

These equations can be linearized around their limit and, up to invert some linear operator and using a priori concentration inequalities, solved asymptotically.

- For several matrix models, nor for the Sinsh model, this is not possible. We detail the approach for the β-ensembles below : It yields equations rather at the level of measures.

Ideas of proofs : β-models

$$dP_V^N(\lambda_1,\ldots,\lambda_N) = \frac{1}{Z_V^N}\prod_{i<j}|\lambda_i-\lambda_j|^\beta e^{-N\sum V(\lambda_i)}\prod d\lambda_i$$

- By Large deviation (or saddle point) methods, there exists a unique measure μ_V so that :

$$\lim_{N\to\infty} L_N = \mu_V \quad a.s$$

- Rewrite the SD equation in terms of $\tilde{L}_N = N(L_N - \mu_V)$

$$P_V^N\left(\tilde{L}_N(\Xi f)\right) = \frac{\beta}{2N}P_V^N\left(\int\frac{f(x)-f(y)}{x-y}d\tilde{L}_N(x)d\tilde{L}_N(y)\right)$$

$$+(\frac{\beta}{2}-1)\int\int f'(x)dL_N(x)dP_V^N$$

$$\Xi f(x) = \beta\int\frac{f(x)-f(y)}{x-y}d\mu_V(y) - V'(x)f(x)$$

$$P_V^N\left(\tilde{L}_N(\Xi f)\right) = \frac{\beta}{2N} P_V^N\left(\int \frac{f(x)-f(y)}{x-y} d\tilde{L}_N(x) d\tilde{L}_N(y)\right)$$

$$+(\frac{\beta}{2}-1)\int\int f'(x) dL_N(x) dP_V^N$$

Show that Ξ is invertible. Then

- $\lim_{N\to\infty} \int f(x) d\tilde{L}_N(x) = (\frac{\beta}{2}-1)\int (\Xi^{-1}f)'(x) d\mu_V(x)$
- $\delta_N^2(f) = N[\int f(x) d\tilde{L}_N(x) - (\frac{\beta}{2}-1)\int (\Xi^{-1}f)'(x) d\mu_V(x)] =$

$$P_V^N\left(\frac{\beta}{2}\int \frac{\Xi^{-1}f(x)-\Xi^{-1}f(y)}{x-y} d\tilde{L}_N(x) d\tilde{L}_N(y) + (\frac{\beta}{2}-1)\int (\Xi^{-1}f)' d\tilde{L}_N\right)$$

To get the limit of the right hand side, one needs to get the limit of the correlator

$$C_N(f,g) = \mathbb{E}[\tilde{L}_N(f)\tilde{L}_N(g)].$$

Asymptotic of the correlators

Making an infinitesimal change of variables $V \to V + \epsilon g$ in the SD equation we get

$$P_V^N\left(\tilde{L}_N(\Xi f)\tilde{L}_N(g)\right) = \mathbb{E}[L_N(g'f)] + (\frac{\beta}{2} - 1)\mathbb{E}[\int(\Xi^{-1}f)'(x)d\tilde{L}_N(x)]$$

$$+\frac{\beta}{2N}P_V^N\left(\tilde{L}_N(g)\left(\int \frac{f(x)-f(y)}{x-y}d\tilde{L}_N(x)d\tilde{L}_N(y)\right)\right)$$

Use concentration of measure to see the last term is neglectable, and invert Ξ to conclude that

$$C(f,g) = \lim_{N\to\infty} C_N(f,g) = \mu_V(g\Xi^{-1}f)$$

and finally, plugging back into previous equation, get δ_N^2. Continue to the next orders...

New ideas for Sinsh model

$$Z_N[V] = \int_{\mathbb{R}^N} \prod_{a<b} \Big\{ \prod_{i=1,2} \sinh[\pi\omega_i T_N(\lambda_a - \lambda_b)] \Big\}^{\beta} \cdot \prod_{a=1}^{N} e^{-NT_N V(\lambda_a)} \cdot d^N\lambda \, .$$

- Deal with N-dependent equilibrium measure whose analysis is based on a $2d$ Riemann-Hilbert problem (square root vanishing compared with step boundary behavior)
- 2 scales N and N^{α}.
- The interaction possesses a tower of poles so that the approach by correlators is not effective. Need to control inverse of master operator Ξ in good spaces.

Conclusion

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- Systems asymptotically driven by the free difference quotient can be shown to have topological expansions. The key point is to invert a Master operator and use concentration of measure.
- This extend to Sinsh model : what else ? get expansion for the cosh potential ?
- This approach can be generalized to prove topological expansion for several matrix models and uniform measure on the unitary or orthogonal groups [G-Novak 14'] in perturbative settings.