Order Determination of Large Dimensional Dynamic Factor Model

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Joint work with
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Outline

Background

Consider the framework of a large dimensional dynamic k-factor model with lag q

$$\mathbf{R}_t = \sum_{i=0}^{q} \mathbf{\Lambda}_i \mathbf{F}_{t-i} + \mathbf{e}_t, \quad t = 1, ..., T$$

- $\mathbf{\Lambda}_i$: $n \times k$ non-random matrices with full rank
- $\mathbf{F}_t$: $k \times 1$ iid standard complex random vector
- $\mathbf{e}_t$: $n \times 1$ iid complex, mean zero, variance σ^2, independent of $\mathbf{F}_t$
- a **information-plus-noise** type model (Dozier & Silverstein, 2007a, b; Bai & Silverstein, 2012)
- $n, T \to \infty$, with $\frac{n}{T} \to c > 0$
- k, q small and fixed but unknown

Under this high dimensional setting, an important statistical problem is to estimate k and q (Bai & Ng, 2002; Harding, 2012).

Notations

For fixed τ, define

$$\begin{aligned}
\mathbf{\Phi}_n(\tau) &= \frac{1}{2T}\sum_{j=1}^{T}(\mathbf{R}_j\mathbf{R}_{j+\tau}^* + \mathbf{R}_{j+\tau}\mathbf{R}_j^*) \\
&= \frac{1}{2T}\left\{\mathbf{\Lambda}\left(\mathbf{F}_0\mathbf{F}_\tau' + \mathbf{F}_\tau\mathbf{F}_0'\right)\mathbf{\Lambda}'\right\} + \\
&\quad \frac{1}{2T}\left\{\left(\mathbf{E}_0\mathbf{F}_\tau'\mathbf{\Lambda}' + \mathbf{\Lambda}\mathbf{F}_\tau\mathbf{E}_0'\right) + \left(\mathbf{E}_\tau\mathbf{F}_0'\mathbf{\Lambda}' + \mathbf{\Lambda}\mathbf{F}_0\mathbf{E}_\tau'\right)\right\} + \\
&\quad \frac{1}{2T}\left(\mathbf{E}_0\mathbf{E}_\tau' + \mathbf{E}_\tau\mathbf{E}_0'\right),
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&\quad \frac{1}{2T}\left(\mathbf{E}_0\mathbf{E}'_\tau + \mathbf{E}_\tau\mathbf{E}'_0\right),
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\mathbf{M}_n(\tau) &= \frac{1}{2T}\sum_{j=1}^{T}(\mathbf{e}_j\mathbf{e}^*_{j+\tau} + \mathbf{e}_{j+\tau}\mathbf{e}^*_j) \\
&= \frac{1}{2T}\left(\mathbf{E}_0\mathbf{E}'_\tau + \mathbf{E}_\tau\mathbf{E}'_0\right).
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Notations

Here,

$$\begin{aligned}
\mathbf{\Lambda} &= (\mathbf{\Lambda}_0, \mathbf{\Lambda}_1, \cdots, \mathbf{\Lambda}_q)_{n \times k(q+1)}, \\
\mathbf{F}_\tau &= \begin{pmatrix} \mathbf{F}_{T+\tau} & \mathbf{F}_{T+\tau-1} & \cdots & \mathbf{F}_{\tau+1} \\ \mathbf{F}_{T+\tau-1} & \mathbf{F}_{T+\tau-2} & \cdots & \mathbf{F}_{\tau} \\ \vdots & \vdots & \vdots & \vdots \\ \mathbf{F}_{T+\tau-q} & \mathbf{F}_{T+\tau-1-q} & \cdots & \mathbf{F}_{\tau+1-q} \end{pmatrix}_{k(q+1) \times T}, \\
\mathbf{E}_\tau &= (\mathbf{e}_{T+\tau}, \mathbf{e}_{T+\tau-1}, \cdots, \mathbf{e}_{\tau+1})_{n \times T}.
\end{aligned}$$

Case $\tau = 0$

Fact 1:

$$\mathbf{M}_n(0) = \frac{1}{T}\sum_{j=1}^{T} \mathbf{e}_j \mathbf{e}_j^*$$

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- the LSD is MP law (Marčenko and Pastur, 1967) with density

$$f_c(x) = \frac{1}{2\pi c x}\sqrt{(b_c - x)(x - a_c)}, x \in [a_c, b_c]$$

and a point mass $1 - 1/c$ at the origin if $c > 1$.
Here $c = \lim_{n\to\infty} \frac{n}{T}$, $a_c = (1 - \sqrt{c})^2$ and $b_c = (1 + \sqrt{c})^2$.

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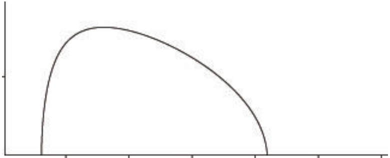

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$$\Rightarrow \mathrm{Cov}\mathbf{R}_t = \sigma^2\mathbf{I} + \mathbf{\Lambda}\mathbf{\Lambda}^* \sim \begin{pmatrix} \sigma^2\mathbf{I} + \mathbf{\Lambda}^*\mathbf{\Lambda} & \mathbf{0} \\ \mathbf{0} & \sigma^2\mathbf{I} \end{pmatrix}$$

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- a **spiked population model** (Johnstone, 2001; Baik & Silverstein, 2006; Bai & Yao, 2008) with population eigenvalue $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{k(q+1)} > \sigma^2 = \cdots = \sigma^2$.

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- when $\boldsymbol{\Lambda}^*\boldsymbol{\Lambda}$ is "not small", sample eigenvalue $\hat{\lambda}_1 \geq \hat{\lambda}_2 \geq \cdots \geq \hat{\lambda}_{k(q+1)} > (\sigma^2 b_c) \geq \hat{\lambda}_{k(q+1)+1} \cdots \geq \hat{\lambda}_n$.

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$\Rightarrow$ Can estimate $k(q+1)$ by counting the number of eigenvalues $> \sigma^2 b_c$.

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To do so, we need to investigate the case for at least one $\tau \geq 1$.

Outline

Main Result

Theorem 1 (Jin et al. (2014))

Assume:

- (a) $\tau \geq 1$ is a fixed integer.
- (b) $\mathbf{e}_k = (\varepsilon_{1k}, \cdots, \varepsilon_{nk})'$, $k = 1, 2, ..., T+\tau$, are n dimensional vectors of independent standard complex components with $\sup_{1\leq i\leq n, 1\leq t\leq T+\tau} \mathrm{E}|\varepsilon_{it}|^{2+\delta} \leq M < \infty$ for some $\delta \in (0, 2]$, and for any $\eta > 0$,

$$\frac{1}{\eta^{2+\delta} nT} \sum_{i=1}^{n} \sum_{t=1}^{T+\tau} \mathrm{E}(|\varepsilon_{it}|^{2+\delta} I(|\varepsilon_{it}| \geq \eta T^{1/(2+\delta)})) = o(1). \tag{1}$$

- (c) $n/(T+\tau) \to c > 0$ as $n, T \to \infty$.
- (d) $\mathbf{M}_n = \sum_{k=1}^{T} (\gamma_k \gamma_{k+\tau}^* + \gamma_{k+\tau} \gamma_k^*)$, where $\gamma_k = \frac{1}{\sqrt{2T}} \mathbf{e}_k$.

Theorem 1 (Jin et al. (2014)) (cont'd)

Then as $n, T \to \infty$, $F^{\mathbf{M}_n} \xrightarrow{D} F_\tau$ a.s. and F_τ has a density function

$$\phi_c(x) = \frac{1}{2c\pi}\sqrt{\frac{y_0^2}{1+y_0} - \left(\frac{1-c}{x} + \frac{1}{\sqrt{1+y_0}}\right)^2}, \ |x| \le d_c,$$

where

$$d_c = \begin{cases} \frac{(1-c)\sqrt{1+y_1}}{y_1-1}, & c \neq 1, \\ 2, & c = 1, \end{cases}$$

y_0 is the largest real root of the equation:
$y^3 - \frac{(1-c)^2-x^2}{x^2}y^2 - \frac{4}{x^2}y - \frac{4}{x^2} = 0$;
and y_1 is the only real root of the equation:
$((1-c)^2 - 1)y^3 + y^2 + y - 1 = 0$
such that $y_1 > 1$ if $c < 1$ and $y_1 \in (0,1)$ if $c > 1$.
Further, if $c > 1$, then F_τ has a point mass $1 - 1/c$ at the origin.

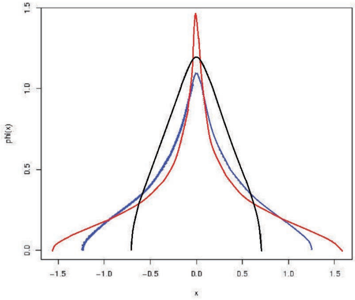

Figure 1 : $\phi_c(x)$ with $c = 0.2$ (black), 0.5 (blue) and 0.7 (red).

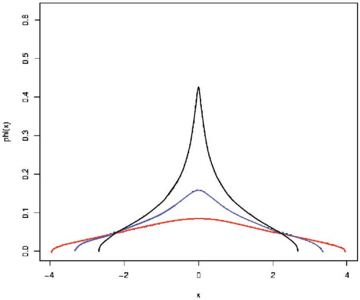

Figure 2 : $\phi_c(x)$ with $c = 1.5$ (black), 2 (blue) and 2.5 (red). The area under each curve is $1/c$.

Outline

Motivation

Once the LSD of $\mathbf{M}_n(\tau)$ is derived, it is observed that the number of eigenvalues of $\mathbf{\Phi}_n(\tau)$ that lie outside the support of the LSD of $\mathbf{M}_n(\tau)$ at lags $1 \le \tau \le q$ is different from that at lags $\tau > q$. Thus, the estimates of k and q can be separated by counting the number of eigenvalues of $\mathbf{\Phi}_n(\tau)$ that lie outside the support of the LSD of $\mathbf{M}_n(\tau)$ from $\tau = 0, 1, 2, \cdots, q$, $q + 1, \cdots$.

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It is worth noting that for this method to work, we require that with probability 1, there is no eigenvalues outside the the support of the LSD of $\mathbf{M}_n(\tau)$ so that if an eigenvalue of $\mathbf{\Phi}_n(\tau)$ goes out of the support of the LSD of $\mathbf{M}_n(\tau)$, it must come from the information part.

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This motivates us to establish the limits of the largest and smallest eigenvalues of $\mathbf{M}_n(\tau)$, after showing that with probability 1 no eigenvalues exist outside the support of the LSD of $\mathbf{M}_n(\tau)$.

Main Results

Theorem 2

Assume:

- (a) $\tau \geq 1$ is a fixed integer.
- (b) $\mathbf{e}_k = (\varepsilon_{1k}, \cdots, \varepsilon_{nk})'$, $k = 1, 2, ..., T + \tau$, are n-vectors of independent standard complex components with $\sup_{i,t} \mathrm{E}|\varepsilon_{it}|^4 \leq M$ for some $M > 0$.
- (c) There exist $K > 0$ and a random variable X with finite fourth order moment such that, for any $x > 0$, for all n, T $\frac{1}{nT} \sum_{i=1}^{n} \sum_{t=1}^{T+\tau} \mathrm{P}(|\varepsilon_{it}| > x) \leq K\mathrm{P}(|X| > x)$.
- (d) $c_n \equiv n/T \to c > 0$ as $n \to \infty$.
- (e) $\mathbf{M}_n = \sum_{k=1}^{T} (\gamma_k \gamma_{k+\tau}^* + \gamma_{k+\tau} \gamma_k^*)$, where $\gamma_k = \frac{1}{\sqrt{2T}} \mathbf{e}_k$.
- (f) The interval [a,b] lies outside the support of F_τ.

Then P (no eigenvalues of $\mathbf{M}_n$ appear in $[a, b]$ for all large n) $= 1$.

Main Results

Theorem 3

Assuming conditions (a)–(e) in Theorem 2 hold, we have

$$\lim_{n\to\infty} \lambda_{\min}(\mathbf{M}_n) = -d_c \quad \text{a.s.} \quad \text{and} \quad \lim_{n\to\infty} \lambda_{\max}(\mathbf{M}_n) = d_c \quad \text{a.s.}$$

Here, $-d_c$ and d_c are the left and right boundary points of the support of the LSD of $\mathbf{M}_n$, as defined in Theorem 1.

Simulation

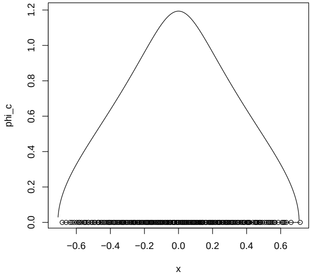

Figure 3 : $\phi_c(x)$ and plot of sample eigenvalues with $\tau = 1, c = 0.2$ $(n = 200, T = 1000)$.

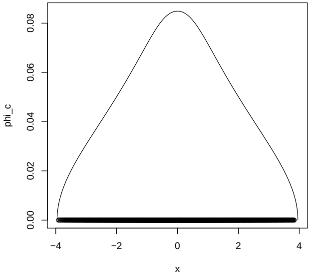

Figure 4 : $\phi_c(x)$ and plot of sample eigenvalues with $\tau = 1, c = 2.5$ $(n = 2500, T = 1000)$.

Outline

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$$\begin{aligned}\mathbf{\Phi}_n(\tau) &= \frac{1}{2T}\left\{\mathbf{\Lambda}\left(\mathbf{F}_0\mathbf{F}_\tau' + \mathbf{F}_\tau\mathbf{F}_0'\right)\mathbf{\Lambda}'\right\} + \\ &\quad \frac{1}{2T}\left\{\left(\mathbf{E}_0\mathbf{F}_\tau'\mathbf{\Lambda}' + \mathbf{\Lambda}\mathbf{F}_\tau\mathbf{E}_0'\right) + \left(\mathbf{E}_\tau\mathbf{F}_0'\mathbf{\Lambda}' + \mathbf{\Lambda}\mathbf{F}_0\mathbf{E}_\tau'\right)\right\} + \\ &\quad \mathbf{M}_n,\end{aligned}$$

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Define $\mathbf{B}_1 = \mathbf{\Lambda Q}$ and $\mathbf{B} = (\mathbf{B}_1 \vdots \mathbf{B}_2)$ is an $n \times n$ orthogonal matrix, where $\mathbf{Q} = (\mathbf{\Lambda}'\mathbf{\Lambda})^{-1/2}$.

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Then, $\mathbf{B}'\mathbf{\Phi}_n(\tau)\mathbf{B} = \begin{pmatrix} \mathbf{B}_1'\mathbf{\Phi}_n(\tau)\mathbf{B}_1 & \mathbf{B}_1'\mathbf{\Phi}_n(\tau)\mathbf{B}_2 \\ \mathbf{B}_2'\mathbf{\Phi}_n(\tau)\mathbf{B}_1 & \mathbf{B}_2'\mathbf{\Phi}_n(\tau)\mathbf{B}_2 \end{pmatrix}$.

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Note that $\mathbf{B}_2\boldsymbol{\Lambda} = 0$, we have

$$\begin{aligned}
\mathbf{B}_1'\boldsymbol{\Phi}_n(\tau)\mathbf{B}_1 &\sim \mathbf{Q}\mathbf{A}_\tau\mathbf{Q} + \mathbf{B}_1'\mathbf{M}_n\mathbf{B}_1 \\
\mathbf{B}_1'\boldsymbol{\Phi}_n(\tau)\mathbf{B}_2 &= \mathbf{B}_1'\mathbf{M}_n\mathbf{B}_2 + \frac{1}{2T}\mathbf{Q}(\mathbf{F}_0\mathbf{E}_\tau^* + \mathbf{F}_\tau\mathbf{E}_0^*)\mathbf{B}_2 \\
\mathbf{B}_2'\boldsymbol{\Phi}_n(\tau)\mathbf{B}_1 &= \mathbf{B}_2'\mathbf{M}_n\mathbf{B}_1 + \frac{1}{2T}\mathbf{B}_2'(\mathbf{E}_0\mathbf{F}_\tau^* + \mathbf{E}_\tau\mathbf{F}_0^*)\mathbf{Q} \\
\mathbf{B}_2'\boldsymbol{\Phi}_n(\tau)\mathbf{B}_2 &= \mathbf{B}_2'\mathbf{M}_n\mathbf{B}_2.
\end{aligned}$$

where $(\mathbf{A}_\tau)_{k(q+1)\times k(q+1)}$ is the matrix with 1's on upper and lower $k\tau$ diagonals and 0's elsewhere.

Estimation of k and q

If ℓ is a root of $\mathbf{\Phi}_n(\tau)$ but not a root of $\mathbf{B}_2'\mathbf{M}_n\mathbf{B}_2$, then

$$0 = \begin{vmatrix} \mathbf{B}_1'\mathbf{\Phi}_n(\tau)\mathbf{B}_1 - \ell\mathbf{I} & \mathbf{B}_1'\mathbf{\Phi}_n(\tau)\mathbf{B}_2 \\ \mathbf{B}_2'\mathbf{\Phi}_n(\tau)\mathbf{B}_1 & \mathbf{B}_2'\mathbf{\Phi}_n(\tau)\mathbf{B}_2 - \ell\mathbf{I} \end{vmatrix}$$

Since $|\mathbf{B}_2'\mathbf{\Phi}_n(\tau)\mathbf{B}_2 - \ell\mathbf{I}| = |\mathbf{B}_2'\mathbf{M}_n\mathbf{B}_2 - \ell\mathbf{I}| \neq 0$, we have

$$\left|\mathbf{B}_1'\mathbf{\Phi}_n(\tau)\mathbf{B}_1 - \ell\mathbf{I} - \mathbf{B}_1'\mathbf{\Phi}_n(\tau)\mathbf{B}_2(\mathbf{B}_2'\mathbf{M}_n\mathbf{B}_2 - \ell\mathbf{I})^{-1}\mathbf{B}_2'\mathbf{\Phi}_n(\tau)\mathbf{B}_1\right| = 0$$

After certain simplification, the equation above can be shown equivalent to

$$\left|\mathbf{A}_\tau - \left(\ell + \frac{cm(\ell)}{1 - c^2m^2(\ell) + \sqrt{1 - c^2m^2(\ell))}}\right)\mathbf{Q}^{-2} - \frac{cm(\ell)}{2\sqrt{1 - c^2m^2(\ell)}}\mathbf{I}_{k\times(q+1)}\right|$$

Estimation of k and q

The above equation is the key relation between signals and the observed spikes.

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Generally, it is not easy to identify the point-wise transaction rule between the signals (eigenvalues of $\mathbf{Q}^2$) and spikes (solutions of the equation in ℓ).

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However, if the matrices $\mathbf{A}_\tau$ and $\mathbf{Q}^2$ are commutative, the transition phenomenon becomes clear, that is, there is a common orthogonal matrix $\mathbf{O}$ to simultaneously diagonalize the two matrices, i.e., we have $\mathbf{A}_\tau = \mathbf{O}\mathbf{D}_\tau\mathbf{O}'$ and $\mathbf{Q}^2 = \mathbf{O}\mathbf{D}_\lambda\mathbf{O}'$, where $\mathbf{D}_\tau = \mathrm{diag}[a_1, \cdots, a_{k(q+1)}]$ and $\mathbf{D}_\lambda = \mathrm{diag}[\lambda_1, \cdots, \lambda_{k(q+1)}]$.

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Then, the equation becomes

$$a_j - \left(\ell + \frac{cm(\ell)}{1-c^2m^2(\ell)+\sqrt{1-c^2m^2(\ell)})}\right)\lambda_j^{-1} - \frac{cm(\ell)}{2\sqrt{1-c^2m^2(\ell)}} = 0,$$
$$j = 1, 2\cdots, k(q+1).$$

Estimation of k and q

Case 1. If $a_j \geq 0$ and $g(d(c)) > a_j$, then the equation $a_j = g(\ell)$ doesn't have a solution in the interval $(d(c), \infty)$ because $g(\ell)$ is increasing and continuous, where

$$g(\ell) = \left(\ell + \frac{cm(\ell)}{1 - c^2m^2(\ell) + \sqrt{1 - c^2m^2(\ell))}}\right)\lambda_j^{-1} + \frac{cm(\ell)}{2\sqrt{1 - c^2m^2(\ell)}}.$$

On the interval $(-\infty, -d(c))$ it does not have solution either because $g(\ell) < g(-d(c)) = -g(d(c)) < 0$. Thus, the equation $a_j = g(\ell)$ does not have any solution.

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$$g(\ell) = \Big(\ell + \frac{cm(\ell)}{1 - c^2m^2(\ell) + \sqrt{1 - c^2m^2(\ell))}}\Big)\lambda_j^{-1} + \frac{cm(\ell)}{2\sqrt{1 - c^2m^2(\ell)}}.$$

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Case 2. If $a_j \geq 0$ and $a_j \geq g(d(c)) > 0$, then the equation $a_j = g(\ell)$ has a solution in the interval $(d(c), \infty)$, and on the interval $(-\infty, -d(c))$ it does not have solution either because $a_j \geq 0$ and $g(\ell) \leq g(-d(c)) < 0$. Thus, the equation $a_j = g(\ell)$ has only one solution.

Estimation of k and q

Case 3. If $a_j \geq 0$ and $a_j > -g(d(c)) \geq 0$, then the equation $a_j = g(\ell)$ has a solution in the interval $(d(c), \infty)$, and on the interval $(-\infty, -d(c))$ it does not have any solution because $a_j > g(-d(c)) \geq g(\ell)$ when $\ell < -d(c)$.

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Case 4. If $-g(d(c)) \geq a_j \geq g(d(c))$, then the equation $a_j = g(\ell)$ has a solution in the interval $(d(c), \infty)$ and another solution on the interval $(-\infty, -d(c))$. Especially when $a_j = 0$, the case is true.

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Similarly, we may discuss the cases when $a_j \leq 0$.

Since $m(d(c)) < 0$, we have $g(d(c)) < 0$ provided that λ_j is large enough. Thus, case 1 doesn't happen in general.

Estimation of k and q

- Therefore, the number of spiked eigenvalues of $\mathbf{\Phi}_n(\tau)$ satisfies

$$\hat{p}(\tau) \to \begin{cases} k(q+1), & \tau = 0 \\ 2k(q+1) - \mathrm{h}(\tau), & 1 \leq \tau \leq q \\ 2k(q+1), & \tau > q. \end{cases}$$

where $h(\tau) = 2 \cdot^{\#} \{j, g(d(c)) > |a_j|\} +^{\#} \{j, |a_j| > |g(d(c))| > 0\}$.

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- Generally, the first case doesn't happen, unless λ_j is very small.

- Transition threshold:

$$\lambda_0(c) = -\frac{2\sqrt{1 - c^2 m^2(d(c))}\left(d(c) + \frac{cm(d(c))}{(1-c^2 m^2(d(c))) + \sqrt{1-c^2 m^2(d(c))}}\right)}{cm(d(c))}$$

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- That is, when $\lambda_j > \lambda_0(c)$, then $g_j(d(c)) < 0$.

Estimation of k and q

Algorithm

- Count the number of spiked eigenvalues of $\mathbf{\Phi}_n(0)$, $k\widehat{(q+1)}$.
- For $\tau = 1, 2, \cdots$, count the number of spiked eigenvalues of $\mathbf{\Phi}_n(\tau)$ and stop at the smallest lag $\widehat{q+1}$, at which the number jumps to $2k\widehat{(q+1)}$.
- Set $\hat{k} = \frac{k\widehat{(q+1)}}{\widehat{q+1}}$ and $\hat{q} = \widehat{q+1} - 1$.

Simulation

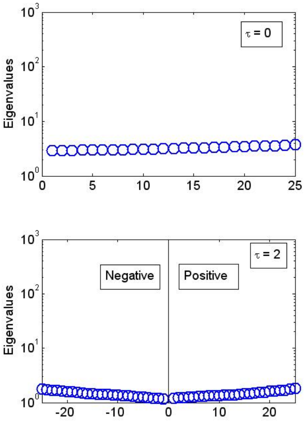

Figure 5 : Sample eigenvalues plots for a factor model with no factors with $n = 450,\ T = 500,\ k = 0,\ q = 0$ and $\sigma_\varepsilon^2 = 1$.

Simulation

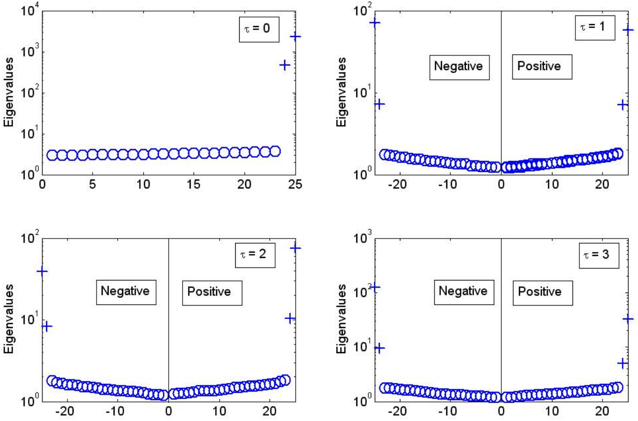

Figure 6 : Sample eigenvalues plots for a factor model with $n = 450$, $T = 500$, $k = 2$, $q = 0$ and $\sigma_\varepsilon^2 = 1$.

Simulation

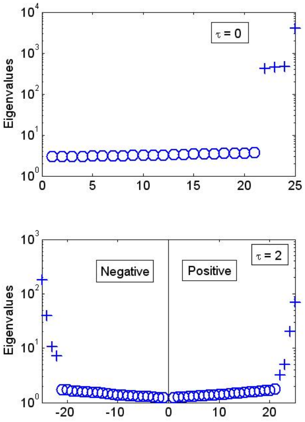

Figure 7 : Sample eigenvalues plots for a factor model with $n = 450$, $T = 500$, $k = 2$, $q = 1$ and $\sigma_\varepsilon^2 = 1$.

Simulation

$\tau = 0$	$\tau = 1$	$\tau = 2$	$\tau = 3$	$\tau = 4$	$\tau = 5$
10.4834	**13.2847**	**9.6398**	**4.5707**	**4.1978**	**4.3652**
10.1067	**12.8983**	**9.3054**	**4.5100**	**3.7982**	**4.2543**
9.5428	**10.5731**	**8.9893**	**3.7854**	**3.5023**	**3.6568**
8.1918	**9.7048**	**8.8115**	**3.4964**	**3.2956**	**3.2796**
7.8733	**2.7132**	**3.0196**	**3.4424**	**2.9948**	**3.2131**
7.6733	**2.2934**	**2.8472**	**3.2752**	**2.8658**	**3.0014**
1.8057	**2.0844**	**2.7571**	**3.1088**	**2.8206**	**2.9009**
1.7851	**1.9410**	**2.7238**	**2.4418**	**2.6166**	**2.7364**
1.7475	1.7971	1.8099	**2.4222**	**2.6032**	**2.5338**
1.7273	1.7096	1.7313	**2.3283**	**2.4414**	**2.1618**
1.7090	1.7068	1.7232	**2.1798**	**2.3751**	**2.1310**
1.6787	1.6803	1.6998	**2.0149**	**2.1294**	**1.9938**
1.6619	1.6418	1.6874	1.8028	1.7561	1.7109

Table 1 : Absolute values of the largest eigenvalues of $\mathbf{\Phi}_n$ at various lags, for $c = 0.9,\ b_c = (1+\sqrt{c})^2 = 3.7974,\ d_c = 1.8573$.

Thank you!