Order Determination of Large Dimensional Dynamic Factor Model

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Joint work with
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Outline

1. Introduction
2. Limiting Spectral Distribution
3. Strong Limit of Extreme Eigenvalues
4. Application
Background

Consider the framework of a large dimensional dynamic $k$-factor model with lag $q$

$$R_t = \sum_{i=0}^{q} \Lambda_i F_{t-i} + e_t, \quad t = 1, \ldots, T$$

- $\Lambda_i$: $n \times k$ non-random matrices with full rank
- $F_t$: $k \times 1$ iid standard complex random vector
- $e_t$: $n \times 1$ iid complex, mean zero, variance $\sigma^2$, independent of $F_t$
- a **information-plus-noise** type model
  (Dozier & Silverstein, 2007a, b; Bai & Silverstein, 2012)
- $n, T \to \infty$, with $\frac{n}{T} \to c > 0$
- $k, q$ small and fixed but unknown
Under this high dimensional setting, an important statistical problem is to estimate $k$ and $q$ (Bai & Ng, 2002; Harding, 2012).
For fixed $\tau$, define

$$\Phi_n(\tau) = \frac{1}{2T} \sum_{j=1}^{T} (R_j R_{j+\tau}^* + R_{j+\tau} R_j^*)$$

$$= \frac{1}{2T} \left\{ \Lambda \left( F_0 F'_{\tau} + F_{\tau} F_0' \right) \Lambda' \right\} +$$

$$\frac{1}{2T} \left\{ \left( E_0 F'_{\tau} \Lambda' + \Lambda F_{\tau} E_0' \right) + \left( E_{\tau} F_0' \Lambda' + \Lambda F_0 E_{\tau}' \right) \right\} +$$

$$\frac{1}{2T} \left( E_0 E'_{\tau} + E_{\tau} E_0' \right),$$
For fixed $\tau$, define

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$$= \frac{1}{2T} \left\{ \Lambda \left( F_0 F'_\tau + F_\tau F'_0 \right) \Lambda' \right\} + \frac{1}{2T} \left\{ \left( E_0 F'_\tau \Lambda' + \Lambda F_\tau E'_0 \right) + \left( E_\tau F'_0 \Lambda' + \Lambda F_0 E'_\tau \right) \right\} + \frac{1}{2T} \left( E_0 E'_\tau + E_\tau E'_0 \right),$$

$$M_n(\tau) = \frac{1}{2T} \sum_{j=1}^{T} (e_j e^*_j + e_{j+\tau} e^*_j)$$

$$= \frac{1}{2T} \left( E_0 E'_\tau + E_\tau E'_0 \right).$$
Notations

Here,

\[ \Lambda = (\Lambda_0, \Lambda_1, \cdots, \Lambda_q)_{n \times k(q+1)}, \]

\[ F_\tau = \begin{pmatrix}
F_{T+\tau} & F_{T+\tau-1} & \cdots & F_{\tau+1} \\
F_{T+\tau-1} & F_{T+\tau-2} & \cdots & F_{\tau} \\
\vdots & \vdots & \vdots & \vdots \\
F_{T+\tau-q} & F_{T+\tau-1-q} & \cdots & F_{\tau+1-q}
\end{pmatrix}_{k(q+1) \times T}, \]

\[ E_\tau = (e_{T+\tau}, e_{T+\tau-1}, \cdots, e_{\tau+1})_{n \times T}. \]
Case $\tau = 0$

Fact 1:

$$M_n(0) = \frac{1}{T} \sum_{j=1}^{T} e_j e_j^*$$
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- the LSD is MP law (Marčenko and Pastur, 1967) with density

$$f_c(x) = \frac{1}{2\pi cx} \sqrt{(b_c - x)(x - a_c)}, x \in [a_c, b_c]$$

and a point mass $1 - 1/c$ at the origin if $c > 1$.

Here $c = \lim_{n \to \infty} \frac{n}{T}$, $a_c = (1 - \sqrt{c})^2$ and $b_c = (1 + \sqrt{c})^2$. 
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![Graph of a standard sample covariance matrix](image)
Case $\tau = 0$

Fact 2:

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$$\Rightarrow \text{Cov}R_t = \sigma^2 I + \Lambda\Lambda^* \sim \begin{pmatrix} \sigma^2 I + \Lambda^*\Lambda & 0 \\ 0 & \sigma^2 I \end{pmatrix}$$
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- a spiked population model (Johnstone, 2001; Baik & Silverstein, 2006; Bai & Yao, 2008) with population eigenvalue $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{k(q+1)} > \sigma^2 = \cdots = \sigma^2$. 

when $\Lambda^* \Lambda$ is "not small", sample eigenvalue $\hat{\lambda}_1 \geq \hat{\lambda}_2 \geq \cdots \geq \hat{\lambda}_{k(q+1)}$ $> \sigma^2$ $> \hat{\lambda}_k(q+1) + 1 \cdots > \hat{\lambda}_n$. 

$\Rightarrow$ Can estimate $k(q+1)$ by counting the number of eigenvalues $> \sigma^2$. 

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Case \( \tau = 0 \)

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- a **spiked population model** (Johnstone, 2001; Baik & Silverstein, 2006; Bai & Yao, 2008) with population eigenvalue \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{k(q+1)} > \sigma^2 = \cdots = \sigma^2 \).

- when \( \Lambda^* \Lambda \) is “not small”, sample eigenvalue \( \hat{\lambda}_1 \geq \hat{\lambda}_2 \geq \cdots \geq \hat{\lambda}_{k(q+1)} > (\sigma^2 b_c) \geq \hat{\lambda}_{k(q+1)+1} \cdots \geq \hat{\lambda}_n \).
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Recall $\Lambda = (\Lambda_0, \Lambda_1, \cdots, \Lambda_q)_{n \times k(q+1)}$

$$\Rightarrow \text{Cov} \mathbf{R}_t = \sigma^2 \mathbf{I} + \Lambda \Lambda^* \sim \begin{pmatrix} \sigma^2 \mathbf{I} + \Lambda^* \Lambda & 0 \\ 0 & \sigma^2 \mathbf{I} \end{pmatrix}$$

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- when $\Lambda^* \Lambda$ is "not small", sample eigenvalue $\hat{\lambda}_1 \geq \hat{\lambda}_2 \geq \cdots \geq \hat{\lambda}_{k(q+1)} > (\sigma^2 b_c) \geq \hat{\lambda}_{k(q+1)+1} \cdots \geq \hat{\lambda}_n$.

$$\Rightarrow \text{Can estimate } k(q + 1) \text{ by counting the number of eigenvalues } > \sigma^2 b_c.$$
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To do so, we need to investigate the case for at least one $\tau \geq 1$. 
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Main Result

**Theorem 1 (Jin et al. (2014))**

Assume:

- (a) $\tau \geq 1$ is a fixed integer.
- (b) $e_k = (\varepsilon_{1k}, \cdots, \varepsilon_{nk})'$, $k = 1, 2, \ldots, T + \tau$, are $n$ dimensional vectors of independent standard complex components with $\sup_{1 \leq i \leq n, 1 \leq t \leq T+\tau} E|\varepsilon_{it}|^{2+\delta} \leq M < \infty$ for some $\delta \in (0, 2]$, and for any $\eta > 0$,

$$\frac{1}{\eta^{2+\delta}nT} \sum_{i=1}^{n} \sum_{t=1}^{T+\tau} E(|\varepsilon_{it}|^{2+\delta} I(|\varepsilon_{it}| \geq \eta T^{1/(2+\delta)})) = o(1). \quad (1)$$

- (c) $n/(T + \tau) \to c > 0$ as $n, T \to \infty$.
- (d) $M_n = \sum_{k=1}^{T} (\gamma_k \gamma_{k+\tau} + \gamma_{k+\tau} \gamma_k^*)$, where $\gamma_k = \frac{1}{\sqrt{2T}}e_k$. 

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Theorem 1 (Jin et al. (2014)) (cont’d)

Then as \( n, T \to \infty \), \( F_{Mn}^D \Rightarrow F_\tau \) a.s. and \( F_\tau \) has a density function

\[
\phi_c(x) = \frac{1}{2c\pi} \sqrt{\frac{y_0^2}{1+y_0} - \left(\frac{1-c}{x} + \frac{1}{\sqrt{1+y_0}}\right)^2}, \quad |x| \leq d_c,
\]

where

\[
d_c = \begin{cases} 
\frac{(1-c)\sqrt{1+y_1}}{y_1-1}, & c \neq 1, \\
2, & c = 1,
\end{cases}
\]

\( y_0 \) is the largest real root of the equation:

\[
y^3 - \frac{(1-c)^2-x^2}{x^2} y^2 - \frac{4}{x^2} y - \frac{4}{x^2} = 0;
\]

and \( y_1 \) is the only real root of the equation:

\[
((1-c)^2 - 1)y^3 + y^2 + y - 1 = 0
\]

such that \( y_1 > 1 \) if \( c < 1 \) and \( y_1 \in (0,1) \) if \( c > 1 \).

Further, if \( c > 1 \), then \( F_\tau \) has a point mass \( 1 - 1/c \) at the origin.
Main Result

Figure 1: $\phi_c(x)$ with $c = 0.2$ (black), 0.5 (blue) and 0.7 (red).

Figure 2: $\phi_c(x)$ with $c = 1.5$ (black), 2 (blue) and 2.5 (red). The area under each curve is $1/c$. 
1. Introduction

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Motivation

Once the LSD of $M_n(\tau)$ is derived, it is observed that the number of eigenvalues of $\Phi_n(\tau)$ that lie outside the support of the LSD of $M_n(\tau)$ at lags $1 \leq \tau \leq q$ is different from that at lags $\tau > q$. Thus, the estimates of $k$ and $q$ can be separated by counting the number of eigenvalues of $\Phi_n(\tau)$ that lie outside the support of the LSD of $M_n(\tau)$ from $\tau = 0, 1, 2, \cdots, q, q + 1, \cdots$. 
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It is worth noting that for this method to work, we require that with probability 1, there is no eigenvalues outside the the support of the LSD of $M_n(\tau)$ so that if an eigenvalue of $\Phi_n(\tau)$ goes out of the support of the LSD of $M_n(\tau)$, it must come from the information part.
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It is worth noting that for this method to work, we require that with probability 1, there is no eigenvalues outside the the support of the LSD of $M_n(\tau)$ so that if an eigenvalue of $\Phi_n(\tau)$ goes out of the support of the LSD of $M_n(\tau)$, it must come from the information part.

This motivates us to establish the limits of the largest and smallest eigenvalues of $M_n(\tau)$, after showing that with probability 1 no eigenvalues exist outside the support of the LSD of $M_n(\tau)$. 
Theorem 2
Assume:

(a) $\tau \geq 1$ is a fixed integer.

(b) $e_k = (\varepsilon_{1k}, \cdots, \varepsilon_{nk})'$, $k = 1, 2, \ldots, T + \tau$, are $n$-vectors of independent standard complex components with $\sup_{i,t} E|\varepsilon_{it}|^4 \leq M$ for some $M > 0$.

(c) There exist $K > 0$ and a random variable $X$ with finite fourth order moment such that, for any $x > 0$, for all $n, T$
\[
\frac{1}{nT} \sum_{i=1}^{n} \sum_{t=1}^{T+\tau} P(|\varepsilon_{it}| > x) \leq KP(|X| > x).
\]

(d) $c_n \equiv n/T \to c > 0$ as $n \to \infty$.

(e) $M_n = \sum_{k=1}^{T} (\gamma_k \gamma_{k+\tau}^* + \gamma_{k+\tau} \gamma_k^*)$, where $\gamma_k = \frac{1}{\sqrt{2T}} e_k$.

(f) The interval $[a, b]$ lies outside the support of $F_{\tau}$.

Then $P(\text{no eigenvalues of } M_n \text{ appear in } [a, b] \text{ for all large } n) = 1$. 
Main Results

**Theorem 3**

Assuming conditions (a)–(e) in Theorem 2 hold, we have

$$\lim_{n \to \infty} \lambda_{\min}(M_n) = -d_c \text{ a.s. and } \lim_{n \to \infty} \lambda_{\max}(M_n) = d_c \text{ a.s.}$$

Here, $-d_c$ and $d_c$ are the left and right boundary points of the support of the LSD of $M_n$, as defined in Theorem 1.
Figure 3: $\phi_c(x)$ and plot of sample eigenvalues with $\tau = 1, c = 0.2$ ($n = 200, T = 1000$).

Figure 4: $\phi_c(x)$ and plot of sample eigenvalues with $\tau = 1, c = 2.5$ ($n = 2500, T = 1000$).
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Recall that $k(\tau + 1)$ can be estimated by counting the number of spiked eigenvalues of $\Phi_n(0)$.

For $\tau \geq 1$, we have

$$\Phi_n(\tau) = \frac{1}{2} T \left\{ \Lambda(F_0 F_\tau + F_\tau F_0)^\Lambda \right\} + \frac{1}{2} T \left\{ (E_0 F_\tau \Lambda' + \Lambda F_\tau E_\tau') + (E_\tau F_0 \Lambda' + \Lambda F_0 E_\tau') \right\} + M_n,$$
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$$
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\frac{1}{2T} \left\{ \left( E_0 F_{\tau}' \Lambda' + \Lambda F_{\tau} E_0' \right) + \left( E_{\tau} F_0' \Lambda' + \Lambda F_0 E_{\tau}' \right) \right\} + M_n,
$$
Estimation of $k$ and $q$

Define $B_1 = \Lambda Q$ and $B = (B_1 \ldots B_2)$ is an $n \times n$ orthogonal matrix, where $Q = (\Lambda' \Lambda)^{-1/2}$.

Then, $B_1' \Phi_n(\tau) B_1 = (B_1' \Phi_n(\tau) B_1 B_1' \Phi_n(\tau) B_1 B_1' \Phi_n(\tau) B_2 B_2' \Phi_n(\tau) B_2 B_2' \Phi_n(\tau) B_1 B_1' \Phi_n(\tau) B_1 B_1' \Phi_n(\tau) B_2 B_2' \Phi_n(\tau) B_2 B_2')$.

Note that $B_2 \Lambda = 0$, we have

$B_1' \Phi_n(\tau) B_1 \sim Q \tau Q + B_1' M_n B_1 B_1' \Phi_n(\tau) B_2 B_2' \Phi_n(\tau) B_1 B_1' \Phi_n(\tau) B_1 B_1' \Phi_n(\tau) B_2 B_2' \Phi_n(\tau) B_2 B_2'$. Where $(A \tau)_{k(q+1) \times k(q+1)}$ is the matrix with 1's on upper and lower $k\tau$ diagonals and 0's elsewhere.
Estimation of $k$ and $q$

Define $B_1 = \Lambda Q$ and $B = (B_1 : B_2)$ is an $n \times n$ orthogonal matrix, where $Q = (\Lambda' \Lambda)^{-1/2}$. 
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Then, \( B' \Phi_n(\tau) B = \begin{pmatrix} B'_1 \Phi_n(\tau) B_1 & B'_1 \Phi_n(\tau) B_2 \\ B'_2 \Phi_n(\tau) B_1 & B'_2 \Phi_n(\tau) B_2 \end{pmatrix} \).
Estimation of $k$ and $q$

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Note that $B_2 \Lambda = 0$, we have

$$B_1' \Phi_n(\tau) B_1 \sim QA_\tau Q + B_1' M_n B_1$$

$$B_1' \Phi_n(\tau) B_2 = B_1' M_n B_2 + \frac{1}{2T} \frac{Q(F_0 E_\tau^* + F_\tau E_0^*) B_2}{2T}$$

$$B_2' \Phi_n(\tau) B_1 = B_2' M_n B_1 + \frac{1}{2T} B_2' (E_0 F_\tau^* + E_\tau F_0^*) Q$$

$$B_2' \Phi_n(\tau) B_2 = B_2' M_n B_2.$$
Estimation of \( k \) and \( q \)

If \( \ell \) is a root of \( \Phi_n(\tau) \) but not a root of \( B'_2 M_n B_2 \), then

\[
0 = \begin{vmatrix}
B'_1 \Phi_n(\tau) B_1 - \ell I & B'_1 \Phi_n(\tau) B_2 \\
B'_2 \Phi_n(\tau) B_1 & B'_2 \Phi_n(\tau) B_2 - \ell I
\end{vmatrix}
\]

Since \( |B'_2 \Phi_n(\tau) B_2 - \ell I| = |B'_2 M_n B_2 - \ell I| \neq 0 \), we have

\[
\begin{vmatrix}
B'_1 \Phi_n(\tau) B_1 - \ell I - B'_1 \Phi_n(\tau) B_2 (B'_2 M_n B_2 - \ell I)^{-1} B'_2 \Phi_n(\tau) B_1
\end{vmatrix} = 0
\]

After certain simplification, the equation above can be shown equivalent to

\[
|A_\tau - \left( \ell + \frac{cm(\ell)}{1 - c^2 m^2(\ell) + \sqrt{1 - c^2 m^2(\ell)}} \right) Q^{-2} - \frac{cm(\ell)}{2\sqrt{1 - c^2 m^2(\ell)}} I_{k \times (q+1)} |
\]
Estimation of $k$ and $q$

The above equation is the key relation between signals and the observed spikes.
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Generally, it is not easy to identify the point-wise transaction rule between the signals (eigenvalues of $Q^2$) and spikes (solutions of the equation in $\ell$).

However, if the matrices $A_\tau$ and $Q^2$ are commutative, the transition phenomenon becomes clear, that is, there is a common orthogonal matrix $O$ to simultaneously diagonalize the two matrices, i.e., we have $A_\tau = OD_\tau O'$ and $Q^2 = OD_\lambda O'$, where $D_\tau = \text{diag}[a_1, \cdots, a_{k(q+1)}]$ and $D_\lambda = \text{diag}[\lambda_1, \cdots, \lambda_{k(q+1)}]$. 
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Then, the equation becomes

$$a_j - \left( \ell + \frac{cm(\ell)}{1-c^2m^2(\ell)+\sqrt{1-c^2m^2(\ell)}} \right) \lambda_j^{-1} - \frac{cm(\ell)}{2\sqrt{1-c^2m^2(\ell)}} = 0,$$

$$j = 1, 2 \cdots, k(q+1).$$
Estimation of $k$ and $q$

Case 1. If $a_j \geq 0$ and $g(d(c)) > a_j$, then the equation $a_j = g(\ell)$ doesn’t have a solution in the interval $(d(c), \infty)$ because $g(\ell)$ is increasing and continuous, where

$$
g(\ell) = \left(\ell + \frac{cm(\ell)}{1 - c^2 m^2(\ell) + \sqrt{1 - c^2 m^2(\ell)}}\right)\lambda_j^{-1} + \frac{cm(\ell)}{2 \sqrt{1 - c^2 m^2(\ell)}}.
$$

On the interval $(-\infty, -d(c))$ it does not have solution either because $g(\ell) < g(-d(c)) = -g(d(c)) < 0$. Thus, the equation $a_j = g(\ell)$ does not have any solution.
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Case 2. If $a_j \geq 0$ and $a_j \geq g(d(c)) > 0$, then the equation $a_j = g(\ell)$ has a solution in the interval $(d(c), \infty)$, and on the interval $(-\infty, -d(c))$ it does not have solution either because $a_j \geq 0$ and $g(\ell) \leq g(-d(c)) < 0$. Thus, the equation $a_j = g(\ell)$ has only one solution.
Case 3. If $a_j \geq 0$ and $a_j > -g(d(c)) \geq 0$, then the equation $a_j = g(\ell)$ has a solution in the interval $(d(c), \infty)$, and on the interval $(-\infty, -d(c))$ it does not have any solution because $a_j > g(-d(c)) \geq g(\ell)$ when $\ell < -d(c)$. 

Similarly, we may discuss the cases when $a_j \leq 0$. Since $m(d(c)) < 0$, we have $g(d(c)) < 0$ provided that $\lambda_j$ is large enough. Thus, case 1 doesn't happen in general.
Estimation of $k$ and $q$

Case 3. If $a_j \geq 0$ and $a_j > -g(d(c)) \geq 0$, then the equation $a_j = g(\ell)$ has a solution in the interval $(d(c), \infty)$, and on the interval $(-\infty, -d(c))$ it does not have any solution because $a_j > g(-d(c)) \geq g(\ell)$ when $\ell < -d(c)$.

Case 4. If $-g(d(c)) \geq a_j \geq g(d(c))$, then the equation $a_j = g(\ell)$ has a solution in the interval $(d(c), \infty)$ and another solution on the interval $(-\infty, -d(c))$. Especially when $a_j = 0$, the case is true.
Estimation of $k$ and $q$

Case 3. If $a_j \geq 0$ and $a_j > -g(d(c)) \geq 0$, then the equation $a_j = g(\ell)$ has a solution in the interval $(d(c), \infty)$, and on the interval $(-\infty, -d(c))$ it does not have any solution because $a_j > g(-d(c)) \geq g(\ell)$ when $\ell < -d(c)$.

Case 4. If $-g(d(c)) \geq a_j \geq g(d(c))$, then the equation $a_j = g(\ell)$ has a solution in the interval $(d(c), \infty)$ and another solution on the interval $(-\infty, -d(c))$. Especially when $a_j = 0$, the case is true.

Similarly, we may discuss the cases when $a_j \leq 0$. 
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Similarly, we may discuss the cases when $a_j \leq 0$.

Since $m(d(c)) < 0$, we have $g(d(c)) < 0$ provided that $\lambda_j$ is large enough. Thus, case 1 doesn’t happen in general.
Estimation of $k$ and $q$

Therefore, the number of spiked eigenvalues of $\Phi_n(\tau)$ satisfies

$$
\hat{p}(\tau) \rightarrow \begin{cases} 
k(q + 1), & \tau = 0 \\
2k(q + 1) - h(\tau), & 1 \leq \tau \leq q \\
2k(q + 1), & \tau > q.
\end{cases}
$$

where $h(\tau) = 2 \cdot \# \{ j, g(d(c)) > |a_j| \} + \# \{ j, |a_j| > |g(d(c))| > 0 \}$. 

Generally, the first case doesn't happen, unless $\lambda_j$ is very small.

Transition threshold: $\lambda_0(c) = -\frac{2}{\sqrt{1 - c^2}}m(d(c)) + \sqrt{1 - c^2}m(d(c)) (1 - c^2)$. That is, when $\lambda_j > \lambda_0(c)$, then $g_j(d(c)) < 0.$
Estimation of $k$ and $q$

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  k(q+1), & \tau = 0 \\
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  $$

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Generally, the first case doesn’t happen, unless $\lambda_j$ is very small.

Transition threshold:

$$
\lambda_0(c) = -\frac{2 \sqrt{1 - c^2 m^2(d(c))} \left( d(c) + \frac{cm(d(c))}{(1-c^2 m^2(d(c))) + \sqrt{1-c^2 m^2(d(c))}} \right)}{cm(d(c))}
$$
Estimation of $k$ and $q$

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- That is, when $\lambda_j > \lambda_0(c)$, then $g_j(d(c)) < 0$. 
Estimation of $k$ and $q$

**Algorithm**

- Count the number of spiked eigenvalues of $\Phi_n(0)$, $k(q+1)$.
- For $\tau = 1, 2, \cdots$, count the number of spiked eigenvalues of $\Phi_n(\tau)$ and stop at the smallest lag $q+1$, at which the number jumps to $2k(q+1)$.
- Set $\hat{k} = \frac{k(q+1)}{q+1}$ and $\hat{q} = q + 1 - 1$. 
Figure 5: Sample eigenvalues plots for a factor model with no factors with $n = 450$, $T = 500$, $k = 0$, $q = 0$ and $\sigma^2_{\varepsilon} = 1$. 
Simulation

Figure 6: Sample eigenvalues plots for a factor model with $n = 450$, $T = 500$, $k = 2$, $q = 0$ and $\sigma^2_\varepsilon = 1$. 

\[ \tau = 0 \]
\[ \tau = 1 \]
\[ \tau = 2 \]
\[ \tau = 3 \]
Figure 7: Sample eigenvalues plots for a factor model with $n = 450$, $T = 500$, $k = 2$, $q = 1$ and $\sigma^2_\varepsilon = 1$. 
Table 1: Absolute values of the largest eigenvalues of $\Phi_n$ at various lags, for $c = 0.9$, $b_c = (1 + \sqrt{c})^2 = 3.7974$, $d_c = 1.8573$. 

<table>
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<th>$\tau = 0$</th>
<th>$\tau = 1$</th>
<th>$\tau = 2$</th>
<th>$\tau = 3$</th>
<th>$\tau = 4$</th>
<th>$\tau = 5$</th>
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Thank you!