Norms of submatrices and entropic uncertainty relations for high dimensional random unitaries

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Random matrices and their applications
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Quantum states and von Neumann measurements

- a state of a quantum system – a unit element $|\psi\rangle$ of a Hilbert space $\mathcal{H}$
- in this talk: $N$-level systems – $\mathcal{H} = \mathcal{H}_N \simeq \mathbb{C}^N$
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  - an orthonormal basis $|e_1\rangle, \ldots, |e_N\rangle \in \mathcal{H}_N$.
  - the probability of getting answer $a_i$ for a system in state $|\psi\rangle$ is $p_i = |\langle \psi | e_i \rangle|^2$.
  - sometimes we will identify $|e_i\rangle$’s with columns of a unitary matrix.
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If $a_1, \ldots, a_N \in \mathbb{R}$ then one associates with the measurement a Hermitian operator (observable) $A = \sum_{i=1}^{N} a_i |e_i\rangle\langle e_i|$. We then have
  - mean output: $\langle A \rangle_\psi = \langle \psi | A |\psi \rangle$
  - standard deviation: $(\Delta_\psi(A))^2 = \langle \psi | A^2 |\psi \rangle - \langle \psi | A |\psi \rangle^2$. 
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**Theorem (Heisenberg UP)**

*For any two Hermitian operators $A, B$ on $\mathcal{H}$ and any state $|\psi\rangle$*

$$\Delta^2_\psi(A)\Delta^2_\psi(B) \geq \frac{1}{4} \left| \langle \psi | [A, B] | \psi \rangle \right|^2,$$

*where $[A, B] = AB - BA$.*
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- First for position and momentum operators on $L^2(\mathbb{R})$ (Heisenberg)
- Depends on $A,B$ rather then just on the measurement basis, the outputs have to be numerical
Entropic uncertainty principle

A natural way to quantify uncertainty corresponding to a random variable is Shannon’s entropy

**Definition**

Shannon’s entropy of a probability vector \( p = (p_1, \ldots, p_N) \) is defined as

\[
H(p) = \sum_{i=1}^{N} -p_i \ln p_i.
\]

- \( H(p) \geq 0 \) (\( H(p) = 0 \) only if \( p = \delta_i \) - no uncertainty),
- Jensen’s ineq. \( \implies H(p) \leq \ln N \) (equality only for the uniform distr. – greatest uncertainty),
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- Jensen’s ineq. \( \Rightarrow H(p) \leq \ln N \) (equality only for the uniform distr. – greatest uncertainty),
- **Question**: For two basis \( |e_1\rangle, \ldots, |e_N\rangle \) and \( |v_1\rangle, \ldots, |v_N\rangle \) can we find conditions guaranteeing that

\[
\min_{\psi} \left( H(p^\psi) + H(q^\psi) \right)
\]

is large where \( p^\psi = (|\langle \psi | e_i \rangle|^2)_{i=1}^{N} \), \( q^\psi = (|\langle \psi | v_i \rangle|^2)_{i=1}^{N} \)?
Some history – continuous case

- The use of Shannon’s entropy (of probability densities) was postulated first independently by Hirschmann and Everett (1957) who conjectured an uncertainty principle for the position and momentum operators.

- The proof was provided by Białynicki-Birula and Mycielski and by Beckner in 1975 (both based on Beckner’s results for the Fourier transform)

- The entropic version of Heisenberg’s principle for position and momentum is known to imply the version with standard deviations.
**Question:** For two bases $|e_1\rangle, \ldots, |e_N\rangle$ and $|v_1\rangle, \ldots, |v_N\rangle$. Find lower bounds on

$$\min_{\psi} \left( H(p^{\psi}) + H(q^{\psi}) \right),$$

where $p = (\langle \psi | e_i \rangle^2)^{N}_{i=1}$, $q = (\langle \psi | v_i \rangle^2)^{N}_{i=1}$?
Back to finite dim.: Deutsch, Maasen-Uffink & Coles-Piani ineq.

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$$U = [U_{ij}]_{i,j=1}^N := [\langle e_i | v_j \rangle]_{i,j=1}^N, \quad c := \max_{i,j} |U_{ij}|.$$

- **Deutsch (1983):** $\min_{\psi} \left( H(p^\psi) + H(q^\psi) \right) \geq -2 \ln \frac{1+c}{2}$
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- **Coles-Piani (2014):**
  \[ \min_{\psi} \left( H(p^\psi) + H(q^\psi) \right) \geq - \ln c^2 + (1 - c) \ln(c/c_2), \]
  where $c_2$ – second largest element of $U$. 
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- This is optimal for *mutually unbiased bases* ($|\langle e_i | v_j \rangle|^2 = \frac{1}{N}$ for all $i, j$, e.g. standard and Fourier bases):
  $$\min_{\psi} \left( H(p^\psi) + H(q^\psi) \right) \geq \ln N.$$
More than two measurements

One can also consider a larger number of measurements, given by unitaries $U^{(1)}, \ldots, U^{(L)}$. If for $i = 1, \ldots, L$, the probability vectors $p^{(\psi,i)} = (p_1^{(i)}, \ldots, p_N^{(i)})$ are given by $p_j^{(i)} = |\langle \psi | U^{(i)} | e_j \rangle|^2$, what can be said about

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- Taking pairwise mutually unbiased bases we get via Maasen-Uffink’s bound:
  \[
  \min_{\psi} \frac{1}{L} \sum_{i=1}^{L} H(p^{(\psi,i)}) \geq \frac{1}{2} \ln N.
  \]

This turns out to be optimal for MUB’s if $L \leq \sqrt{N} + 1$, $N = P^{2l}$, $P$ - prime (Ballester-Wehner 2007).
Pairwise MUB’s:

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For \( L = N + 1 \) MUB’s (maximal possible), then

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• For \( L > 2 \) but small wrt. \( N \), random constructions only: Hayden et al. (2004). If \( U_1, \ldots, U_L \) are random unitary matrices and \( L \geq (\ln N)^4 \), then with high probability as \( N \to \infty \)

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**Question**

Can you do it for smaller $L$? Motivation:

- $L = 2$ – optimal deterministic constructions known, but what’s the behaviour for generic bases?
- $2 < L \ll N$ – no deterministic constructions. Proof of existence by probabilistic methods.
- $L = 2$ – check optimality of known uncertainty relations on generic data.
Theorem (Latała, Puchała, Życzkowski, A. (2014))

Let $U$ be an $N \times N$ random unitary matrix. With probability converging to one as $N \to \infty$ for any two basis $(|e_i\rangle)_{i=1}^N, (|v_i\rangle)_{i=1}^N$, such that $U = [\langle e_i|v_j \rangle]_{i,j=1}^N$

$$\ln N - C_0 \geq \min_{\psi} \left( H(p^\psi) + H(q^\psi) \right) \geq \ln N - C_1,$$

for any $C_0 < 1 - \gamma \simeq 0.42$ and $C_1 \simeq 3.49$. 
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Theorem (Latała, Puchała, Życzkowski, A. (2014))

In the setting with $L$ measurements, if the bases are given by i.i.d. random unitary matrices, then with probability converging to one (uniformly in $L \geq 2$) as $N \to \infty$,

$$\min_{\psi} \frac{1}{L} \sum_{i=1}^{L} H(p^{(\psi,i)}) \geq \frac{L - 1}{L} \ln N - C_2,$$

where $C_2$ is a universal constant.
Recall the Maasen-Uffink bound:

$$\min_{\psi} \left( H(p^\psi) + H(q^\psi) \right) \geq -\ln c^2,$$

where $c = \max |U_{ij}|$. 

For a random unitary matrix $c \simeq \sqrt{2 \ln N}$ (Jiang).

Therefore the Maasen-Uffink ineq. gives

$$\min_{\psi} \left( H(p^\psi) + H(q^\psi) \right) \geq \ln N - \ln \ln N - \ln 2,$$

and therefore is suboptimal for generic data.

One can also show that the Coles-Piani ineq. gives on generic matrices a bound not better than

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Main tool. Majorization and Schur concavity

**Definition**
If \( p = (p_1, \ldots, p_n) \), \( q = (q_1, \ldots, q_n) \) are two non-negative vectors than we say that \( p \) is majorized by \( q \) \((p \prec q)\) if

\[
\sum_{i=1}^{k} p_i \leq \sum_{i=1}^{k} q_i, \quad k = 1, \ldots, n,
\]

with equality for \( k = n \), where \( x_1 \geq \ldots \geq x_n \) is the non-increasing rearrangement of the coordinates of \( x \).

We say that a function \( F: [0, \infty)^n \to \mathbb{R} \) is Schur concave if \( f(p) \geq f(q) \), whenever \( p \prec q \).

**Theorem (Schur)**
A differentiable function \( F \) is Schur concave iff it is permutation invariant and for all \( x \), \((x_1 - x_2)(\frac{\partial F(x)}{\partial x_1} - \frac{\partial F(x)}{\partial x_2}) \leq 0\).
Corollary: \( F(x) = - \sum_i x_i \ln x_i \) is Schur concave. In particular if \( p \prec q \), then \( H(p) \geq H(q) \).

**Majorization entropic uncertainty relations**

For the unitary matrix \( U = [\langle e_i | v_j \rangle]_{i,j=1}^N \) and set \( s_0 = 0 \) and for \( k \geq 1 \),

\[
s_k = \max \{ \| A \| : A \text{ is an } n \times m \text{ submatrix of } U, n + m = k + 1 \}\.
\]

**Theorem (Rudnicki, Puchała, Życzkowski (2014))**

For any two bases \( (|e_i\rangle)_{i=1}^N \) and \( (|v_i\rangle)_{i=1}^N \) and any state \( |\psi\rangle \), Let \( x_1, \ldots, x_{2N} \) be the coordinates of \( p^\psi \oplus q^\psi \). Then for all \( k \),

\[
x_1^{\downarrow} + \ldots + x_k^{\downarrow} \leq 1 + s_{k-1}.
\]

As a consequence \( p^\psi \oplus q^\psi \prec (1, s_1, s_2 - s_1, \ldots, s_{N-1} - s_{N-2}) \) and

\[
\min_{\psi}(H(p^\psi) + H(q^\psi)) \geq -\sum_i (s_i - s_{i-1}) \ln(s_i - s_{i-1}).
\]

**Remark:** This is not directly comparable with the Maasen-Uffink bound.
Lemma (Latała, Puchała, Życzkowski, A.)

Let $U$ be an $N \times N$ random unitary matrix and

$$U(n, m) = \max \{ \|A\| : A \text{ is an } n \times m \text{ submatrix of } U \}$$

Then for all $m, n$ and all $\varepsilon \in [0, 1/3]$

$$\mathbb{E}\|U(n, m)\| \leq \frac{1}{1 - 2\varepsilon - \varepsilon^2} \sqrt{\frac{2}{2N - 1}} \left( m \ln \frac{eN}{m} + n \ln \frac{eN}{n} + 2(n + m) \ln (1 + \frac{2}{\varepsilon}) \right)^{1/2}.$$ 

The method of proof is completely standard, just the union bound and concentration of measure on the sphere (however now we deal with 1-Lipschitz functions). Note that for fixed $n, m$ (indep. of $N$) it gives

$$U(n, m) \leq (1 + o_P(1)) \sqrt{\frac{n + m}{N}} \ln N \quad \text{as } N \to \infty.$$
Asymptotic uncertainty relation for two measurements

As a consequence with probability tending to one as $N \to \infty$, for all $1 \leq k \leq N - 1$,

$$s_k \leq m_k := \sqrt{4.18 \frac{k + 1}{N} \left(1 + \ln \left(\frac{2N}{k + 1}\right)\right)}.$$

This bound is clearly suboptimal for large $k$ (as the rhs exceeds one), but it suffices for proving the uncertainty principle for random unitaries by slightly tedious but straightforward calculations:
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$$m_k - m_{k-1} \leq \frac{1}{N} \sqrt{\frac{4.18 \ln \frac{2N}{i}}{2\sqrt{\frac{i}{N} \ln \frac{2eN}{i}}}} =: r_i.$$
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This bound is clearly suboptimal for large $k$ (as the rhs exceeds one), but it suffices for proving the uncertainty principle for random unitaries by slightly tedious but straightforward calculations:

- $m_k - m_{k-1} \leq \frac{1}{N} \frac{\sqrt{4.18 \ln \frac{2N}{i}}}{2 \sqrt{\frac{i}{N} \ln \frac{2eN}{i}}} =: r_i$.
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Asymptotic uncertainty relation for two measurements

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Random unitaries. Norms of submatrices

Recall that

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It is not difficult to obtain lower and upper bounds on \( U(n, m) \) which differ by an absolute constant:

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\frac{1}{C} \sqrt{\frac{n}{N} \ln \left( \frac{eN}{n} \right) + \frac{m}{N} \ln \left( \frac{eN}{m} \right)} \leq U(n, m) \leq C \sqrt{\frac{n}{N} \ln \left( \frac{eN}{n} \right) + \frac{m}{N} \ln \left( \frac{eN}{m} \right)}
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Question: What is the precise behaviour of \( U(n, m) \) for large \( N \)?
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Theorem (Latała, Puchała, Życzkowski, A. (2014))

If $n, m$ are fixed (independent of $N$), then for every $\varepsilon > 0$ with pr. tending to one,

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A few words about the proof of the lower bounds

- For the case of fixed $n, m$ we use a result by Jiang on coupling of $U$ and a complex Ginibre matrix and then some simple combinatorics. It turns out that in this case the maximum spectral norm of a submatrix is roughly the same as the maximum Hilbert-Schmidt norm.

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$$\sqrt{n} \sum_{i=1}^{n} (|X_i|^2)^\downarrow.$$

It is known that $(|X_i|^2)_{i=1}^{N}$ is distributed uniformly on the simplex, so expectation reduces to calculating barycenters.
Final comments

- **The main message:** Random unitaries satisfy with high probability almost optimal entropic uncertainty relations for an arbitrary number of measurements. The analysis of this phenomenon becomes quite easy if one uses majorization and Schur concavity.
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For random measurements

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- How to construct explicit matrices satisfying almost optimal entropic uncertainty relations for \( L > 2 \)?
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How to construct explicit matrices satisfying almost optimal entropic uncertainty relations for \( L > 2 \)?

What is the precise behaviour of maximum norms of submatrices of an \( N \times N \) random unitary matrix beyond the cases of fixed size or \( n \times 1 \) submatrices?
Thank you