Norms of submatrices and entropic uncertainty relations for high dimensional random unitaries

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Joint work with R. Latała, Z. Puchała, K. Życzkowski

Random matrices and their applications
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Quantum states and von Neumann measurements

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- in this talk: N-level systems – $\mathcal{H} = \mathcal{H}_N \simeq \mathbb{C}^N$

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- In principle all unitary matrices can be realised in experiments (Reck, Zeilinger, Bernstein, Bertani, 1994)
- If $a_1, \ldots, a_N \in \mathbb{R}$ then one associates with the measurement a Hermitian operator (observable) $A = \sum_{i=1}^{N} a_i |e_i\rangle\langle e_i|$. We then have
  - mean output: $\langle A\rangle_\psi = \langle\psi|A|\psi\rangle$
  - standard deviation: $(\Delta_\psi(A))^2 = \langle\psi|A^2|\psi\rangle - \langle\psi|A|\psi\rangle^2$.

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Theorem (Heisenberg UP)

For any two Hermitian operators A, B on $\mathcal{H}$ and any state $|\psi\rangle$

$$\Delta_\psi^2(A)\Delta_\psi^2(B) \geq \frac{1}{4}\Big|\langle \psi | [A, B] | \psi \rangle\Big|^2,$$

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- Depends on A,B rather then just on the measurement basis, the outputs have to be numerical

Entropic uncertainty principle

A natural way to quantify uncertainty corresponding to a random variable is Shannon's entropy

Definition

Shannon's entropy of a probability vector $p = (p_1, \ldots, p_N)$ *is defined as*

$$H(p) = \sum_{i=1}^{N} -p_i \ln p_i.$$

- $H(p) \geq 0$ ($H(p) = 0$ only if $p = \delta_i$ - no uncertainty),
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- Jensen's ineq. $\implies$ $H(p) \leq \ln N$ (equality only for the uniform distr. – greatest uncertainty),
- **Question:** For two basis $|e_1\rangle, \dots, |e_N\rangle$ and $|v_1\rangle, \dots, |v_N\rangle$ can we find conditions guaranteeing that

$$\min_{\psi} \Big(H(p^{\psi}) + H(q^{\psi}) \Big)$$

is large where $p^{\psi} = (|\langle \psi | e_i \rangle|^2)_{i=1}^{N}$, $q^{\psi} = (|\langle \psi | v_i \rangle|^2)_{i=1}^{N}$?

Some history – continuous case

- The use of Shannon's entropy (of probability densities) was postulated first independently by Hirschmann and Everett (1957) who conjectured an uncertainty principle for the position and momentum operators.
- The proof was provided by Białynicki-Birula and Mycielski and by Beckner in 1975 (both based on Beckner's results for the Fourier transform)
- The entropic version of Heisenberg's principle for position and momentum is known to imply the version with standard deviations.

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Question: For two bases $|e_1\rangle, \ldots, |e_N\rangle$ and $|v_1\rangle, \ldots, |v_N\rangle$. Find lower bounds on

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$$U = [U_{ij}]_{i,j=1}^N := [\langle e_i|v_j\rangle]_{i,j=1}^N, \quad c := \max_{i,j} |U_{ij}|.$$

- **Deutsch (1983):** $\min_{\psi} \Big(H(p^{\psi}) + H(q^{\psi}) \Big) \geq -2 \ln \frac{1+c}{2}$

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- This is optimal for *mutually unbiased bases* ($|\langle e_i | v_j \rangle|^2 = \frac{1}{N}$ for all i, j, e.g. standard and Fourier bases):

$$\min_{\psi} \Big(H(p^{\psi}) + H(q^{\psi}) \Big) \geq \ln N.$$

More than two measurements

- One can also consider a larger number of measurements, given by unitaries $U^{(1)}, \ldots, U^{(L)}$. If for $i = 1, \ldots, L$, the probability vectors $p^{(\psi,i)} = (p_1^{(i)}, \ldots, p_N^{(i)})$ are given by $p_j^{(i)} = |\langle\psi|U^{(i)}|e_j\rangle|^2$, what can be said about

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- Taking pairwise mutually unbiased bases we get via Maasen-Uffink's bound:

$$\min_{\psi} \frac{1}{L} \sum_{i=1}^{L} H(p^{(\psi,i)}) \geq \frac{1}{2} \ln N.$$

This turns out to be optimal for MUB's if $L \leq \sqrt{N} + 1$, $N = P^{2l}$, P - prime (Ballester-Wehner 2007).

- Pairwise MUB's:

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- For $L > 2$ but small wrt. N, random constructions only: Hayden et al. (2004). If $U_1, \ldots, U_L$ are random unitary matrices and $L \geq (\ln N)^4$, then with high probability as $N \to \infty$

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Question

Can you do it for smaller L? Motivation:

- $L = 2$ – optimal deterministic constructions known, but what's the behaviour for generic bases?
- $2 < L \ll N$ – no deterministic constructions. Proof of existence by probabilistic methods.
- $L = 2$ – check optimality of known uncertainty relations on generic data.

Theorem (Latała, Puchała, Życzkowski, A. (2014))

Let U be an $N \times N$ random unitary matrix. With probability converging to one as $N \to \infty$ for any two basis $(|e_i\rangle)_{i=1}^N$, $(|v_i\rangle)_{i=1}^N$, such that $U = [\langle e_i | v_j \rangle]_{i,j=1}^N$

$$\ln N - C_0 \geq \min_{\psi} \left(H(p^{\psi}) + H(q^{\psi}) \right) \geq \ln N - C_1,$$

for any $C_0 < 1 - \gamma \simeq 0.42$ and $C_1 \simeq 3.49$.

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Theorem (Latała, Puchała, Życzkowski, A. (2014))

In the setting with L measurements, if the bases are given by i.i.d. random unitary matrices, then with probability converging to one (uniformly in $L \geq 2$) as $N \to \infty$,

$$\min_{\psi} \frac{1}{L} \sum_{i=1}^{L} H(p^{(\psi,i)}) \geq \frac{L-1}{L} \ln N - C_2,$$

where C_2 is a universal constant.

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- One can also show that the Coles-Piani ineq. gives on generic matrices a bound not better than

$$\min_{\psi} \Big(H(p^{\psi}) + H(q^{\psi}) \Big) \geq \ln N - \ln\ln N - \frac{1}{2}\ln 2.$$

Main tool. Majorization and Schur concavity

Definition

If $p = (p_1, \ldots, p_n)$, $q = (q_1, \ldots, q_n)$ are two non-negative vectors than we say that p is majorized by q ($p \prec q$) if

$$\sum_{i=1}^{k} p_i^{\downarrow} \leq \sum_{i=1}^{k} q_i^{\downarrow}, \quad k = 1, \ldots, n,$$

with equality for $k = n$, where $x_1^{\downarrow} \geq \ldots \geq x_n^{\downarrow}$ is the non-increasing rearrangement of the coordinates of x.
We say that a function $F\colon [0, \infty)^n \to \mathbb{R}$ is Schur concave if $f(p) \geq f(q)$, whenever $p \prec q$.

Theorem (Schur)

A differentiable function F is Schur concave iff it is permutation invariant and for all x, $(x_1 - x_2)(\frac{\partial F(x)}{\partial x_1} - \frac{\partial F(x)}{\partial x_2}) \leq 0$.

Corollary: $F(x) = -\sum_i x_i \ln x_i$ is Schur concave. In particular if $p \prec q$, then $H(p) \geq H(q)$.

Majorization entropic uncertainty relations

For the unitary matrix $U = [\langle e_i | v_j \rangle]_{i,j=1}^N$ and set $s_0 = 0$ and for $k \geq 1$,

$$s_k = \max\{\|A\| : A \text{ is an } n \times m \text{ submatrix of } U, n + m = k + 1\}.$$

Theorem (Rudnicki, Puchała, Życzkowski (2014))

For any two bases $(|e_i\rangle)_{i=1}^N$ and $(|v_i\rangle)_{i=1}^N$ and any state $|\psi\rangle$, Let $x_1, \ldots, x_{2N}$ be the coordinates of $p^\psi \oplus q^\psi$. Then for all k,

$$x_1^\downarrow + \ldots + x_k^\downarrow \leq 1 + s_{k-1}.$$

As a consequence $p^\psi \oplus q^\psi \prec (1, s_1, s_2 - s_1, \ldots, s_{N-1} - s_{N-2})$ and $\min_\psi (H(p^\psi) + H(q^\psi)) \geq -\sum_i (s_i - s_{i-1}) \ln(s_i - s_{i-1})$.

Remark: This is not directly comparable with the Maasen-Uffink bound.

Random unitaries. Norms of submatrices

Lemma (Latała, Puchała, Życzkowski, A.)

Let U be an $N \times N$ random unitary matrix and

$$U(n,m) = \max\{\|A\| : A \textit{ is an } n \times m \textit{ submatrix of } U\}$$

Then for all m, n and all $\varepsilon \in [0, 1/3]$

$$\mathbb{E}\|U(n,m)\| \leq$$
$$\frac{1}{1 - 2\varepsilon - \varepsilon^2}\sqrt{\frac{2}{2N-1}}\left(m \ln\frac{eN}{m} + n \ln\frac{eN}{n} + 2(n+m)\ln(1 + \frac{2}{\varepsilon})\right)^{1/2}.$$

The method of proof is completely standard, just the union bound and concentration of measure on the sphere (however now we deal with 1-Lipschitz functions). Note that for fixed n, m (indep. of N) it gives

$$U(n,m) \leq (1 + o_P(1))\sqrt{\frac{n+m}{N}\ln N} \quad \text{as } N \to \infty.$$

Asymptotic uncertainty relation for two measurements

As a consequence with probability tending to one as $N \to \infty$, for all $1 \leq k \leq N-1$,

$$s_k \leq m_k := \sqrt{4.18 \frac{k+1}{N} \left(1 + \ln\left(\frac{2N}{k+1}\right)\right)}.$$

This bound is clearly suboptimal for large k (as the rhs exceeds one), but it suffices for proving the uncertainty principle for random unitaries by slightly tedious but straightforward calculations:

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- $m_k - m_{k-1} \le \frac{1}{N} \frac{\sqrt{4.18} \ln \frac{2N}{i}}{2\sqrt{\frac{i}{N} \ln \frac{2eN}{i}}} =: r_i$.

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As a consequence with probability tending to one as $N \to \infty$, for all $1 \leq k \leq N-1$,

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Many measurements: similar ideas

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Question: What is the precise behaviour of $U(n,m)$ for large N?

Lower bounds on norms of submatrices

Theorem (Latała, Puchała, Życzkowski, A. (2014))

If n, m are fixed (independent of N), then for every $\varepsilon > 0$ with pr. tending to one,

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A few words about the proof of the lower bounds

- For the case of fixed n, m we use a result by Jiang on coupling of U and a complex Ginibre matrix and then some simple combinatorics. It turns out that in this case the maximum spectral norm of a submatrix is roughly the same as the maximum Hilbert-Schmidt norm.
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    It is known that $(|X_i|^2)_{i=1}^N$ is distributed uniformly on the simplex, so expectation reduces to calculating barycenters.

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- How to construct explicit matrices satisfying almost optimal entropic uncertainty relations for $L > 2$?
- What is the precise behaviour of maximum norms of submatrices of an $N \times N$ random unitary matrix beyond the cases of fixed size or $n \times 1$ submatrices?

Thank you