Spectral theory for the q-Boson particle system

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A physicist's guide to solving the Kardar-Parisi-Zhang equation

\[
\frac{\partial U}{\partial t} = \frac{1}{2} \frac{\partial^2 U}{\partial x^2} + (\frac{\partial U}{\partial x})^2 + \dot{W}
\]

space-time white noise

1. Think of the Cole-Hopf transform instead: \(Z = e^U\) solves the SHE

\[
\frac{\partial Z}{\partial t} = \frac{1}{2} \frac{\partial^2 Z}{\partial x^2} + \dot{W} \cdot Z
\]

2. Look at the moments \(\langle Z(t,x_1)\cdots Z(t,x_n)\rangle\). They are solutions of the quantum delta Bose gas evolution [Kardar ‘87], [Molchanov ‘87].

\[
\frac{\partial}{\partial t} \langle Z(t,x_1)\cdots Z(t,x_n)\rangle = \frac{1}{2} \left( \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} + \sum_{i\neq j} \delta(x_i-x_j) \right) \langle Z(t,x_1)\cdots Z(t,x_n)\rangle
\]

3. Use Bethe ansatz to solve it [Lieb-Liniger ‘63], [McGuire ‘64], [Yang ‘67-68].

4. Reconstruct the solution using the known moments:

The replica trick.
Possible mathematician's interpretation. Be wise – discretize!

1. Start with a good **discrete system** that formally converges to KPZ. This should give a solution that we ought to care about.

2. Find `moments' that would solve an **integrable** autonomous system of equations.

3. Reduce it to a direct sum of 1d eq's + boundary cond's and use Bethe ansatz to solve it, for arbitrary initial conditions.

4. Reconstruct the solution using the known `moments' and take the limit to KPZ/SHE.

We can do 1-3 for two systems, q-TASEP and ASEP.

So far we can do 4 only for very special initial conditions.
**q-TASEP [B-Corwin '11]**

Particles jump by one to the right. Each particle has an independent exponential clock of rate \(1 - q^{\text{gap}}\), where `gap' is the number of empty spots ahead.

**Theorem [B-Corwin '11], [B-C-Sasamoto '12], [B-C-Gorin-Shakirov '13]**

For the q-TASEP with step initial data \(\{X_n(0) = -n\}_{n \geq 1}\)

\[
\mathbb{E} \left[ q^{\sum_{i=1}^{k} (x_{N_i}(t)+N_i)} \right] = \frac{(-1)^k}{(2\pi i)^k} \oint \cdots \oint \prod_{A < B} \frac{z_A - z_B}{z_A - q z_B} \prod_{j=1}^{k} \frac{e^{(q-1)t z_j}}{(1 - z_j)^{N_j}} \frac{d z_j}{z_j} \left( z_1 \cdots z_k \right)
\]

The original proof involved Macdonald processes. A simpler one?
q-Boson stochastic particle system [Sasamoto-Wadati '98]

Top particles at each location jump to the left by one indep. with rates $1-q^{\# \text{ of particles at the site}}$.

The generator is $(\vec{n}_j^\t) = \sum_{\text{clusters } i} (1-q^{c_i}) (f(\vec{n}_{c_i+\ldots+c_c}) - f(\vec{n}))$

Proposition [B-Corwin-Sasamoto '12] For a q-TASEP with finitely many particles on the right, $f(t, \vec{n}) = E \left[ \prod_{j=1}^{k} q^{n_j(t)+n_j} \right]$ is the unique solution of

$$\frac{d}{dt} f(t, \vec{n}) = (H f)(t, \vec{n}), \quad f(0, \vec{n}) = E \left[ \prod_{j=1}^{k} q^{n_j(o)+n_j} \right].$$

q-TASEP and q-Boson particle system are dual with respect to $f$.
q-TASEP gaps also evolve as a q-Boson particle system.

Solving q-Boson system means finding q-TASEP q-moments.
Coordinate integrability of the $q$-Boson system

The generator of $k$ free (distant) particles is

$$(L u)(\vec{n}) = (1-q) \sum_{i=1}^{k} (\nabla_i u)(\vec{n}), \quad \nabla_i \text{ is } (\nabla f)(x) = f(x-1)-f(x)$$

Define the boundary conditions as

$$\left(\nabla_i - q \nabla_{i+1}\right) u \bigg|_{n_i=n_{i+1}} = 0 \quad \text{for all } 1 \leq i \leq k-1$$

Proposition [B-Corwin-Sasamoto '12] If $u: \mathbb{Z}^k \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{C}$ satisfies the free evolution equation $\frac{d}{dt} u = Lu$ and boundary conditions, then its restriction to $\{n_1 \geq \ldots \geq n_k\}$ satisfies the $q$-Boson system evolution equation $\frac{d}{dt} u = Hu$.

This suffices to re-prove the nested integral formula

$$\int \prod q^{(x_1(t)+N_1) + \ldots + (x_k(t)+N_k)} = \frac{(-1)^k q^{k(k-1)/2}}{(2\pi i)^k} \oint \ldots \oint \prod_{A<B} \frac{z_A-z_B}{z_A-qz_B} \prod_{j=1}^{k} \frac{e^{(q-1)t \frac{z_{j}}{z_{j}}}}{(1-z_j)^{N_j} z_j}$$

**free evolution**

**boundary conditions**
Algebraic integrability of the q-Boson system

[Sasamoto-Wadati '98] showed that periodic $H$ is the image of a q-Boson Hamiltonian

$$\hat{H} = - \sum_{j=1}^{M} (B_{j-1}^+ - B_j^+) B_j,$$

under ( $\nu_j$ is the number of particles at site $j$)

$$(B_j f)(\vec{\nu}) = \frac{1 - q^{\nu_j}}{1 - q} f(\ldots, \nu_{j-1}, \ldots), \quad (B_j^+ f)(\vec{\nu}) = f(\ldots, \nu_{j+1}, \ldots), \quad (N_j f)(\vec{\nu}) = \nu_j f(\vec{\nu}),$$

and that $\hat{H}$ arises from the monodromy matrix of a quantum integrable system with trigometric $R$-matrix, same as in XXZ/ASEP.

Actually, ASEP has a parallel story.
The ASEP story (briefly)

Set $\tau = \frac{p}{q} < 1$, $n_y(t) = \# \{m \geq 1 : x_m(t) \geq y\}$, $Q_y = \frac{\tau^{n_y} - \tau^{n_y-1}}{\tau - 1}$.

Theorem [B-C-Sasamoto, 2012] For ASEP with step initial data $\{x_n(0) = -n\}_{n \geq 1}$

$$E\left[ Q_{y_1}(t) \cdots Q_{y_k}(t) \right] = \frac{\tau^{k(k-1)/2}}{(2\pi i)^k} \oint \cdots \oint_{A < B} \frac{z_A - z_B}{z_A - \tau z_B} \prod_{j=1}^{k} \left( \frac{z_j - (p-q)^2 t}{(1+z_j)(p+q z_j)} \right)^{y_j} \frac{dz_j}{\tau + z_j}$$

The dual of ASEP is another ASEP [Schutz '97], which is also integrable in both coordinate and algebraic sense. [Tracy-Widom '08+] used Bethe ansatz approach to study ASEP's transition probabilities.
To be able to solve q-Boson system (thus q-TASEP) for general initial conditions, we want to diagonalize $H$. It is not self-adjoint, but PT-invariance (under joint space reflection and time inversion) effectively replaces self-adjointness:

Let $\mu$ be an invariant product measure

$$\mu(d\nu) = \bigotimes_{n \in \mathbb{Z}} \mu_0(d\nu_n), \quad \mu_0(k) = \text{const.} \{\alpha^k, k \geq 0; 0^-, k < 0\}.$$

Then $H = PH^*P^{-1}$ in $L^2(\{\nu_n\}_{n \in \mathbb{Z}}, \mu)$ with $(Pf)(\{\nu_n\}_{n \in \mathbb{Z}}) = f(\{-\nu_n\}_{n \in \mathbb{Z}})$ (parity transformation).
Coordinate Bethe ansatz [Bethe ’31]

(Algebraic) eigenfunctions for a sum of 1d operators

\[(L \Psi)(\vec{x}) = \sum_{i=1}^{k} (L_{x_i} \Psi)(\vec{x}), \quad \vec{x} = (x_1, \ldots, x_k) \in \mathcal{X},\]

that satisfy boundary conditions

\[B_{x_i, x_{i+1}} \Psi|_{x_i = x_{i+1}} = 0, \quad 1 \leq i \leq k-1, \quad B : \{ \text{functions on } \mathcal{X}^2 \} \to \mathbb{C},\]

can be found via

1. Diagonalizing 1d operator \[L \psi_z = \lambda_z \psi_z, \quad \psi_z : \mathcal{X} \to \mathbb{C},\]
2. Taking linear combinations \[\psi_{\vec{z}}(\vec{x}) = \sum_{\sigma \in S(k)} A_{\sigma}(\vec{z}) \psi_{z_{\sigma(1)}}(x_1) \cdots \psi_{z_{\sigma(k)}}(x_k),\]
3. Choosing \[A_{\sigma}(\vec{z}) = \text{sgn}(\sigma) \prod_{a > \ell} \frac{S(z_{\sigma(a)}, z_{\sigma(\ell)})}{S(z_a, z_\ell)}, \quad S(z_1, z_2) = \frac{B(\psi_{z_1}(x) \psi_{z_2}(y))|_{y=x}}{\psi_{z_1}(x) \psi_{z_2}(x)}\]

No quantization of spectrum (Bethe equations) in infinite volume.
Left and right eigenfunctions

For q-Boson gen.  $(Hf)(\vec{n}) = \sum_{\text{clusters } i} (1-q^{c_i})(f(\vec{n}_{c_i}^{-}) - f(\vec{n}))$ that reduces to

$$(Lu)(\vec{n}) = (1-q) \sum_{i=1}^{k} (\nabla_i u)(\vec{n}), \quad (\nabla_i - q \nabla_{i+1})u|_{n_i = n_{i+1} = 0},$$

Bethe ansatz yields ($z_1, \ldots, z_k \in \mathbb{C} \setminus \{1\}$)

$$\Psi^l_{\vec{z}}(\vec{n}) = \sum_{\sigma \in S(k)} \prod_{a > b} \frac{Z_{\sigma(a)}^{q-1}Z_{\sigma(b)}^{q-1}}{Z_{\sigma(a)}Z_{\sigma(b)}^{q-1}} \prod_{j=1}^{k} \frac{1}{(1-Z_{\sigma(j)})^{n_j}}$$

$$\Psi^r_{\vec{z}}(\vec{n}) = \frac{1}{C_q(\vec{n})} \sum_{\sigma \in S(k)} \prod_{a > b} \frac{Z_{\sigma(a)}^{q-1}Z_{\sigma(b)}^{q-1}}{Z_{\sigma(a)}Z_{\sigma(b)}^{q-1}} \prod_{j=1}^{k} \frac{Z_{\sigma(a)}^{q-1}Z_{\sigma(b)}^{q-1}}{(1-Z_{\sigma(j)})^{n_j}}$$

with $C_q(\vec{n}) = (-1)^{k} q^{\frac{k(k-1)}{2}}(c_1^k)(c_2)^{k_2} \ldots$ and

$$H \Psi^l_{\vec{z}} = (1-q)(z_1 + \ldots + z_k)\Psi^l_{\vec{z}}, \quad H^{\text{transpose}} \Psi^r_{\vec{z}} = (1-q)(z_1 + \ldots + z_k)\Psi^r_{\vec{z}}.$$
**Direct and inverse Fourier type transforms**

Let
\[ W^k = \{ f: \{ n, \ldots, n_k \mid n_j \in \mathbb{Z} \} \rightarrow C \text{ of compact support} \} \]
\[ C^k = C \left[ (z_1 - 1)^{\pm 1}, \ldots, (z_k - 1)^{\pm 1} \right] S(k) = \text{symmetric Laurent poly's in } (z_j - 1), 1 \leq j \leq k. \]

**Direct transform:** \( F: W^k \rightarrow C^k \)
\[
F: f \mapsto \sum_{n_1, \ldots, n_k} f(\vec{n}) \cdot \psi^r_{\vec{z}}(\vec{n}) = \langle f, \psi^r_{\vec{z}} \rangle_W
\]

**Inverse transform:** \( G: C^k \rightarrow W^k \)
\[
G: g \mapsto (q - 1)^k q^{-\frac{k(k+1)}{2}} (2\pi i)^k k! \int \cdots \int \det \left[ \frac{1}{q \omega_i \omega_j - w_j} \right]_{i,j=1}^k \prod_{j=1}^k \frac{W_j}{1 - \omega_j} \psi^l_w(\vec{n}) G(\vec{w}) d\vec{w}
= \langle G, \psi^l_{\vec{w}}(\vec{n}) \rangle_C
\]
Contour deformations

Inverse transform: \( \mathcal{Y}: \mathbb{C}^k \rightarrow \mathcal{W}^k \)

\[
\mathcal{Y}: G \rightarrow (q^{-1})^k q^{-\frac{1}{2} \frac{k(k-1)}{2}} \frac{1}{(2\pi i)^k} \oint \cdots \oint \det \left[ \frac{1}{qw_i - w_j} \right] w_j^{i-1} \frac{\psi^l(\tilde{n})}{\tilde{w}} G(\tilde{w}) \, dw
\]

\[
= \frac{1}{(2\pi i)^k} \oint \cdots \oint \frac{Z_a - Z_b}{Z_a - qZ_b} \prod_{j=1}^k \frac{1}{(1 - Z_j)^{r_j+1}} G(Z) \, dZ =
\]

\[
= \sum \frac{(q^{-1})^k q^{-\frac{k^2}{2}}}{m_1! m_2! \cdots} \frac{1}{(2\pi i)^l} \oint \cdots \oint \det \left[ \frac{1}{q^{\lambda_i} w_i - w_j} \right] w_j^{\lambda_i-1} \frac{\psi^l(\tilde{n})}{\tilde{w}} G(\tilde{w}; \lambda) \, dw_1 \cdots dw_l
\]

\( \lambda_i + \cdots + \lambda_k = k \quad \lambda_i = m_1, m_2, \ldots \)

different sets of Bethe states

large

nested

small

\( \tilde{w}; \lambda = (w_1, qw_1, \ldots, q^{\lambda_1-1} w_1, w_2, qw_2, \ldots, q^{\lambda_2-1} w_2, \ldots, w_l, qw_l, \ldots, q^{\lambda_l-1} w_l) \)

\[\tilde{w} \rightarrow \tilde{w}; \lambda \rightarrow \tilde{w}; \lambda^{\prime} \]

\[\tilde{w}; \lambda \rightarrow \tilde{w}; \lambda^{\prime} \]

symmetrization

residue calculus

\[0 \rightarrow Z_1 \rightarrow Z_2 \rightarrow \cdots \rightarrow Z_k \]

\[Z_1 \rightarrow Z_2 \rightarrow \cdots \rightarrow Z_k \]
Theorem [B-Corwin-Petrov-Sasamoto '13] On spaces $\mathcal{W}^k$ and $\mathcal{C}^k$, operators $\mathcal{F}$ and $\mathcal{G}$ are mutual inverses of each other.

**Isometry:**

\[
\langle f, g \rangle_{\mathcal{W}} = \langle \mathcal{F}f, \mathcal{F}g \rangle_{\mathcal{C}} \quad \text{for} \quad f, g \in \mathcal{W}^k
\]
\[
\langle F, G \rangle_{\mathcal{C}} = \langle \mathcal{G}F, \mathcal{G}G \rangle_{\mathcal{W}} \quad \text{for} \quad F, G \in \mathcal{C}^k
\]

**Biorthogonality:**

\[
\langle \psi^l_{-\hat{m}}(\vec{m}), \psi^r_{-\hat{n}}(\vec{n}) \rangle_{\mathcal{C}} = \delta_{\vec{m}, \vec{n}}
\]

in a certain weak sense.

\[
\langle \psi^l_{-\vec{z}}(\cdot), \psi^r_{-\vec{w}}(\cdot) \rangle_{\mathcal{W}} = \frac{1}{k!} \prod_{a+b} \frac{z_a - q z_b}{z_a - z_b} \prod_{j=1}^k \frac{1}{1 - z_j} \det \left[ \delta(z_i - \omega_j) \right]_{i,j=1}^k
\]

This diagonalizes the generator of the $q$-Boson stochastic system and proves completeness of the Bethe ansatz for it.
Back to the $q$-Boson particle system

Corollary  The (unique) solution of the $q$-Boson evolution equation

$$\frac{d}{dt} f(t, \vec{n}) = (H f)(t, \vec{n}), \quad f(0, \vec{n}) = f_0,$$

has the form

$$\hat{f}(t, \vec{n}) = \mathcal{Y} \left( e^{t (q-1)(z_1 + \ldots + z_k)} \hat{F} f_0 \right) = \frac{1}{(2\pi i)^k} \int \ldots \int \prod_{\text{nested}} \frac{Z_a}{Z_a - q Z_b} \prod_{j=1}^k \frac{e^{t(q-1)z_j}}{(1-z_j)^{n_{j+1}}} \langle f_0, \psi_r^\ast \rangle W \, d\vec{z}$$

The computation of $\hat{F} f_0$ can still be difficult. It is, however, automatic if $f_0 = \mathcal{Y} G \Rightarrow \hat{F} f_0 = \hat{F} \mathcal{Y} G = G$.

In the case of $q$-TASEP's step initial condition

$$f_0(\vec{n}) = \prod_{\{n_i \geq 1, 1 \leq i \leq k\}}, \quad G(\vec{z}) = q^{\frac{k(k-1)}{2}} \prod_{j=1}^k \frac{z_i - 1}{z_j}.$$
For q-TASEP, we define

\[ x_1(o) = -1 - X_1, \quad x_2(o) = -1 - x_1(o) - X_2, \ldots, \quad x_n(o) = -1 - x_{n-1}(o) - X_n, \ldots \]

where \( X_1, X_2, \ldots \) are i.i.d. with

\[
\text{Prob} \{ X = k \} = \text{const} \cdot \left\{ \frac{\alpha^k}{(1-q)^k} \right\}, \quad k \geq 0,
\]

\[ 0, \quad k < 0. \]

Then

\[ f_0(\vec{n}) = \prod_{j=1}^{k} \left( 1 - \frac{\alpha}{q} \right)^{n_j} = \mathcal{J} \left( q^{\frac{k(k-1)}{2}} \prod_{j=1}^{k} \frac{z_j-1}{z_j-\alpha/q} \right). \]

Hence

\[ \mathbb{E} q^{(X_{N_1}(t)+N_1)+\ldots+(X_{N_k}(t)+N_k)} = \frac{(-1)^k q^{\frac{k(k-1)}{2}}}{(2\pi i)^k} \prod_{\text{nested}} \int_A \int_B \frac{z_A - z_B}{z_A - qz_B} \prod_{j=1}^{k} \frac{e^{(q-1)t}z_j^{N_j}}{(1-z_j)^{N_j}} \frac{dz_j}{z_j - \alpha/q}. \]

Large time asymptotics of q-TASEP and KPZ in [B-Corwin-Ferrari '12].
Extension to equilibrium:[Imamura-Sasamoto '12] via replica, [BCF-Veto '14]
Other systems

1. A very similar story takes place for ASEP/XXZ in infinite volume [B-Corwin-Petrov-Sasamoto ‘14]. Analogous results are contained in [Babbitt-Thomas ‘77] for SSEP/XXX, [Babbitt-Gutkin ‘90], yet complete proofs seem to be inaccessible.

2. Our Plancherel theorem nontrivially degenerates to two different discrete versions of the delta Bose gas (one of them was treated by [Van Diejen ‘04], [Macdonald ‘71]), and further down to the standard continuous delta Bose gas (where we recover results of [Yang ‘68], [Oxford ‘79], [Heckman-Opdam ‘97]).

Different degenerations require different form of $\gamma$. 
Degenerations of wave functions

\[ \sum_{\sigma \in S(k)} \sigma \left( \prod_{a > b} \frac{Z_a - q Z_b}{Z_a - Z_b} \prod_{j=1}^{k} \frac{1}{(1 - Z_j)^{n_j}} \right) \]

- \( q_j = e^{-\varepsilon} \to 1 \)
- \( 1 - z_j = O(\varepsilon) \)

- \( z_j \gg 1 \)

\[ \sum_{\sigma \in S(k)} \sigma \left( \prod_{a > b} \frac{Z_a - q Z_b}{Z_a - Z_b} \prod_{j=1}^{k} z_j^{-n_j} \right) \]

Hall-Littlewood polynomials

\[ \sum_{\sigma \in S(k)} \sigma \left( \prod_{a > b} \frac{Z_a - Z_b - 1}{Z_a - Z_b} \prod_{j=1}^{k} z_j^{-n_j} \right) \]

semi-discrete Brownian polymer

\[ \sum_{\sigma \in S(k)} \sigma \left( \prod_{a > b} \frac{Z_a - Z_b - 1}{Z_a - Z_b} \prod_{j=1}^{k} e^{x_j z_j} \right) \]

continuous delta Bose gas / KPZ

rescale near crit. pt. of \( e^{t z}/z^n \) with \( t \sim \sqrt{n} \to \infty \) (time is involved)
Noncommutative harmonic analysis

Plancherel theorems often ride on top of noncommutative harmonic analysis statements via imposing additional symmetry.

A classical example [Frobenius, 1896] Let $K$ be a finite group. Its double $G=K \times K$ acts on $L^2(K)$ by left and right argument shifts:

$((g, h) \cdot f)(x) = f(g^{-1}x h)$.

Decomposition on irreducibles has the form

$L^2(K) = \bigoplus_{\pi \in \text{Irr}(K)} \pi \otimes \pi^*.$

For functions that are inv. wrt conjugation this gives Plancherel:

$F(x) = \sum_{\pi \in \text{Irr}(K)} \frac{\dim^2 \pi}{|K|} \frac{\chi_{\pi}(x)}{\dim \pi} \langle F, \frac{\chi_{\pi}}{\dim \pi} \rangle_{L^2(K)}$.

The above expression relates the Plancherel measure to the direct transform via characters (traces) of irreducible representations.
Harmonic analysis on symmetric spaces

For a Lie group $G$ and its subgroup $K$, $G$ acts in $L^2(G/K)$. Plancherel theorem for $K$-inv functions captures the decomposition on irreps of $G$ and diagonalizes $K$-invariant part of the Laplacian on $G/K$.

$G=SO(3), \ K=SO(2), \ G/K=S^2$

$$\Delta_{inv} = \frac{\partial^2}{\partial \theta^2} + \frac{\cos \theta}{\sin \theta} \frac{\partial}{\partial \theta}$$

Eigenfunctions $P_\ell(\cos \theta)$, with Legendre poly's $\{P_\ell\}_{\ell \geq 0}$

$$\int_0^\pi P_m(\cos \theta) P_n(\cos \theta) \sin \theta \, d\theta = \int_{-1}^1 P_m(x) P_n(x) \, dx = c_n \delta_{mn}$$

For real semi-simple $G$ and maximal compact $K$, this is the celebrated theory of [Gelfand-Naimark, 1946+] and [Harish-Chandra, 1947+].
In the case of the continuous delta Bose gas, $H = \frac{1}{2}(\Delta + \sum_{i \neq j} \delta(x_i - x_j))$, the Plancherel theorem rides on top of harmonic analysis for the degenerate (or graded) Hecke algebra of type A, that is generated by permutations and $\{\frac{\partial}{\partial x_i}\}_{i=1}^n$ subject to relations

$$r_j \cdot \frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_{r_j(i)}} \cdot r_j = \begin{cases} \frac{1}{i=j}, & \text{if } i=j+1, \\ 0, & \text{otherwise} \end{cases} \quad r_j = (j \, j+1) \in S(n).$$

Its representation in $C^\infty(\mathbb{R}^n)$ is given by [Yang '67], [Gutkin '82]

$$(Q(r_j)f)(x_1, \ldots, x_n) = f(\ldots, x_{j+1}, x_j, \ldots) - \int_0^1 f(\ldots, x_{j-t}, x_{j+1+t}, \ldots) dt, \quad Q\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial x_i}.$$

$C^\infty(\mathbb{R}^n)$ is embedded into continuous functions satisfying correct boundary conditions via $f \mapsto f^+_\sigma$, $f^+_\sigma(\sigma^{-1}x) = (Q(\sigma)f)(x)$, $x_1 \geq \ldots \geq x_n$, $\sigma \in S(n)$. Restricting the harmonic analysis to symmetric functions gives the Plancherel theorem [Heckman-Opdam '97].
• One of the discretizations of the delta Bose gas that we obtain, for which the wave functions are the Hall–Littlewood polynomials, is connected to the harmonic analysis on $G/K$, where

$$G = \text{GL}(n, F), \quad K = \text{GL}(n, \mathcal{O})$$

$F$ is a non-archimedean local field (like $\mathbb{Q}_p$ or $\mathbb{F}_p((t))$), $\mathcal{O}$ is its ring of integers (like $\mathbb{Z}_p$ or $\mathbb{F}_p[[t]]$), $q = p^{-\frac{1}{m}}$. [Macdonald ’71]

• The other discretization corresponds to $H = \sum_{i=1}^{b_1} \nabla_{n_i} + \sum_{i<j} \delta_{n_i = n_j}$, that arises from moments of the semi-discrete Brownian polymer. First in this case, and later in the $q$-case, [Takeyama ’12, ’14] constructed a representation of a rational twist of the affine Hecke algebra, but so far there is no harmonic analysis.
Mysterious connection to Macdonald polynomials

Macdonald polynomials $P_\lambda(x_1, \ldots, x_N) \in \mathbb{Q}(q,t)[x_1, \ldots, x_N]^{S(N)}$ labelled by partitions $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_N \geq 0)$ form a basis in symmetric polynomials in $N$ variables over $\mathbb{Q}(q,t)$. They diagonalize

$$D_1^{(N)} = \sum_{i=1}^{N} \left( \prod_{a \neq b} (x_a - x_b)^4 \right) T_{t,x_i} \prod_{a \leq b} T_{q,x_i} = \sum_{i=1}^{N} \prod_{j \neq i} \frac{t x_i - x_j}{x_i - x_j} T_{q,x_i}$$

with (generically) pairwise different eigenvalues

$$(T_q f)(z) = f(qz)$$

$$D_1^{(N)} P_\lambda = (q^{\lambda_1} t^{N-1} + q^{\lambda_2} t^{N-2} + \ldots + q^{\lambda_N}) P_\lambda.$$

Proposition [B-Corwin '13] Assume $t=0$. Then

$$\left[ \left( D_1^{(N)} \right)^k, \text{ multiplication by } (x_1 + \ldots + x_N) \right] = (1 - q^k) x_N \left( D_1^{(N-1)} - D_1^{(N)} \right) \left( D_1^{(N)} \right)^{k-1}.$$
Mysterious connection to Macdonald polynomials

\[
\left[ (\tilde{D}_1^{(N)})^k, \text{ multiplication by } (x_1 + \ldots + x_N) \right] = (1 - q^k) x_N (\tilde{D}_1^{(N-1)} - \tilde{D}_1^{(N)}) (\tilde{D}_1^{(N)})^{k-1}
\]

**Corollary** For any symmetric analytic \( F(x_1, \ldots, x_N) \)

\[
f(t, \vec{\nu}) = \begin{cases} 
0, & \text{if at least one } \nu_m > 0, \ m > 0 \\
\frac{\lambda}{\lambda^{(1)}} \frac{(\tilde{D}_1^{(1)})^{\nu_1}(\tilde{D}_1^{(2)})^{\nu_2} \ldots (\tilde{D}_1^{(N)})^{\nu_N}}{(x_1 + \ldots + x_N)} e^{\nu t (x_1 + \ldots + x_N)} F(x_1, \ldots, x_N) \bigg|_{x_1 = \ldots = x_{N-1}} & \text{otherwise}
\end{cases}
\]

solves the evolution equation of the q-Boson system

\[
\frac{d}{dt} f(t, \vec{\nu}) = (H f)(t, \vec{\nu}),
\]

where \( H \) is the generator.

**q-TASEP's step initial condition** corresponds to \( F(x_1, \ldots, x_N) \equiv 1 \).

How does this relate to Plancherel theory?
Summary

• The wish to analyze $q$-TASEP for arbitrary initial conditions lead to new Plancherel theory of Bethe type.
• Its degenerations include that for quantum delta Bose gas, and $q$-TASEP moments do not suffer from intermittency, thus can be used for rigorous replica like computations.
• Similar Plancherel theory exists for ASEP.
• The connection to Macdonald processes is apparent but remains somewhat mysterious.
• More work needed to turn the algebraic advances into new analytic results.