Constrained Hawkes processes for modeling limit order books.

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joint work with Ban Zheng (Natixis) and Frédéric Abergel (ECP)

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March 19, 2014
Outline

1. Limit order books
2. Constrained Hawkes processes
3. Some applications

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1 Limit order books

2 Constrained Hawkes processes

3 Some applications
High frequency price data

Figure: Price of an asset over one day.
Signature plots

How do the usual models behave at small scales?

Figure: Red: Normalized realized volatility as a function of the sampling period.
High frequency price data: another asset

Figure: Price of an asset over one day.
Figure: Red: Normalized realized volatility as a function of the increment period.
The smallest time scale: Limit order book

Figure: A limit order book (LOB) at a given fixed time. Bid prices (red) and Ask prices (blue) available for market orders.
Limit order book events: limit order arrival

Figure: LOB before and after a limit order. Light blue: new ask limit.
Limit order book events: limit order cancellation

Figure: LOB before and after a limit order cancellation. Gray: canceled ask limit.
Time evolution of a LOB and mid-price.

Figure: Time evolution of a LOB and mid-price.
A simplified LOB: Best Bid (BB) and Best Ask (BA) prices.

BestBid/BestAsk dynamics, FTE.PA

Figure: Successive BB and BA events.

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Constrained Hawkes processes

March 19, 2014 12 / 40
Figure: Successive BB and BA events in physical time (mn).
Point process of a simplified LOB

We consider the marked point process describing the dynamics of the BB and BA prices,

\[ N = \sum_{k} \delta_{T_k, I_k} \quad \text{with} \quad 0 < T_1 < T_2 < \ldots \quad \text{and} \quad I_1, I_2, \ldots \in \{1, \ldots, p\}, \]

where each mark \( i \) in \( \{1, \ldots, p\} \) corresponds to a quantified shift of either the BB or the BA price, e.g.

- \( i = 1 \) Best Ask price moves upward one tick,
- \( i = 2 \) Best Ask price moves downward one tick,
- \( i = 3 \) Best Bid price moves upward one tick,
- \( i = 4 \) Best Bid price moves downward one tick.
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- $i = 3$ Best Bid price moves upward one tick,
- $i = 4$ Best Bid price moves downward one tick.

Extensions

By increasing the set of marks, one can consider one marked process describing the LOBs of several assets.
Point process and prices dynamics

One can recover the dynamics of BB, BA and mid prices from the point process $N$ through formulas of the form

$$P_t - P_0 = N \left( \mathbb{1}_{(0,t]} \otimes J \right) = \sum_{0 < T_k \leq t} J(I_k), \quad t > 0.$$ 

For instance, in the previous example,

$$J(1) = 1, J(2) = -1, J(3) = J(4) = 0$$

corresponds to $P_t = \text{BA price}$. 
Point process and BB-BA spread
The gap between the BB price and the BA price is called the spread, from now on denoted by

\[ S_t = \text{BA price}_t - \text{BB price}_t \in \{1, 2, 3, \ldots\} . \]

We will denote by \( J \) the corresponding function on \( \{1, \ldots, p\} \) such that, for all \( t > 0 \),

\[ S_t = S_0 + N \left( \mathbb{1}_{(0,t]} \times J \right) = S_0 + \sum_{0<T_k\leq t} J(I_k) . \]
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Important remarks

- \( J \) takes positive and negative values while \( S \) only takes positive ones.
- \( S_t \) behaves as a stationary random process.
- BB and BA prices typically behave as integrated (and thus co-integrated) stationary processes.

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Limit order books

Constrained Hawkes processes

Some applications
Hawkes processes

Consider a marked point process \( N = \sum_k \delta_{T_k,I_k} \) with

\[
\cdots < T_{-1} < T_0 \leq 0 < T_1 < T_2 < \cdots \quad \text{and} \quad \cdots, I_{-1}, I_0, I_1, I_2, \cdots \in \mathcal{I}.
\]

It is an Hawkes process if its conditional density is of the form

\[
\mu(t,i) = \mu_0(i) + \int_{(-\infty,t)} \phi(t - s, j; i) \, N(ds, dj),
\]

where \( \mu_0 : \mathcal{I} \to \mathbb{R}_+ \) is called the immigrant intensity and \( \phi : [0, \infty) \times \mathcal{I}^2 \to \mathbb{R}_+ \) is called the fertility function.
Multivariate Hawkes processes

If $\mathcal{I} = \{1, \ldots, p\}$, The marked Hawkes process can be seen as a multivariate Hawkes process

$$N_i = N(\cdot \times \{i\}), \quad 1 \leq i \leq p,$$

the fertility is written as a $p \times p$ matrix $\phi(t) = [\phi_{i,j}(t)]_{i,j}$,

$$\mu(t, i) = \mu_0(i) + \int_{(-\infty,t)} \sum_{j=1}^{p} \phi_{i,j}(t - s) N_j(ds), \quad 1 \leq i \leq p,$$

or in a more compact form

$$\mu(t) = \mu_0 + \int_{(-\infty,t)} \phi(t - s) N(ds).$$
Basic stability condition

It can be shown that such a point process is well defined and admit a stationary version if

(BC) the spectral radius of the $p \times p$ matrix

$$\mathbb{N} = [\alpha_{i,j}]_{i,j} = \int_0^\infty \phi(t) \, dt$$

is in $[0, 1)$. 

Remarks

- A Hawkes process can be represented as a cluster process.
- It can be generalized to spatial point processes.
- It can be easily simulated using such representation or Ogata's algorithm (for temporal Hawkes processes).
- Second order properties (intensity, covariance measure, Bartlett's spectrum) are well known.
- The delay and mark of the next event can be forecasted using the Hazard rate.
- Hawkes processes can be made locally stationary (ongoing work).

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Constrained Hawkes processes

March 19, 2014 20 / 40
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Hawkes processes for modeling a simple LOB

Suppose that a stationary Hawkes process $N$ is used to model the dynamics of a simple LOB as defined previously yielding to

$$P_t - P_0 = N \left( \mathbb{1}_{(0,t]} \otimes J \right) = \sum_{0 < T_k \leq t} J(I_k), \quad t > 0.$$ 

for a BB, BA or mid-price $P$ (with an adequate $J$) and

$$S_t = S_0 + N \left( \mathbb{1}_{(0,t]} \times J \right) = S_0 + \sum_{0 < T_k \leq t} J(I_k).$$
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S_t = S_0 + N (\mathbb{1}_{(0,t]} \times J) = S_0 + \sum_{0 < T_k \leq t} J(I_k).
\]

However, for a Hawkes process, provided that \( \mu_0(i) > 0 \) for all \( i \), we have, for any \( k \),

\[
\min_{1 \leq i \leq p} \mathbb{P}(I_{k+1} = i \mid \mathcal{F}_{T_k}) = \frac{\mu(T_k, i)}{\mu(T_k, 1) + \cdots + \mu(T_k, p)} > 0
\]
What’s wrong with Hawkes processes?

As a consequence,

the spread has **positive probability** to eventually reach a **negative value**.
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In fact, $S_t$ behaves similarly to a random walk with drift $\mathbb{E}_0[J(I_k)]$, so cannot model a stationary and stable behavior.
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In other words, the BB and BA prices do behave as integrated processes but not as cointegrated ones.
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In other words, the BB and BA prices do behave as integrated processes but not as cointegrated ones.

Idea:

modify the conditional density by adding constraints depending on $S_t$. 
Constrained Hawkes processes

We consider a point process $N$ with marks in $\{1, \ldots, p\}$ with conditional intensity given by, for all $i = 1, \ldots, p$,

$$
\mu(t, i) = \begin{cases} 
0 & \text{if } S(t- \in A_i \\
\mu_0(i) + \int_{(0,t)} \sum_{j=1}^{p} \phi_{i,j}(t-s) N(ds \times \{j\}) & \text{otherwise},
\end{cases}
$$

where $S$ is a $q$-dimensional process valued in $\mathbb{N}^q$ and defined by

$$
S_t = S_0 + N(1_{(0,t]} \times J),
$$

for some $J : \{1, \ldots, p\} \rightarrow \mathbb{Z}^q$.

Here

- $p$ denotes the number of marks
- $q$ denotes the number of constraints.
- $A_1, \ldots, A_p$ are constraints subsets of $\mathbb{Z}^q$. 

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Simple facts and questions

Under the basic condition (BC), $N$ is always well defined.

It can be simulated by using the non-constrained Hawkes process $N'$ (Cluster method) by removing events (and its descendants) that do not meet the constraints.

It can also be simulated by Ogata's algorithm.

Forecasting the next event delay and mark is easy.

Application to a simple LOB: $q = 1$, $S_t$ is the spread at time $t$, the sets $A_i$ are chosen so that $S_t$ remains positive.

However, what about the stability of $S$?

Ergodicity of $(N, S)$. Application to LOB modeling.

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March 19, 2014 24 / 40
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- Ergodicity of $(N, S)$.
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- What about the stability of $S$?
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- Application to LOB modeling.
A very special case

Consider the simple LOB process, so that

- $p = 4$, $q = 1$,
- $S_t$ is the spread at time $t$ and, at each event, moves a tick upward or downward,
- $A_i = \{1\}$ for the events $i$ making the spread move downward, so that $S_t$ remains positive.

Take moreover the simple case $\phi = 0$ (no memory case: the conditional density does not depend on $N$).
A very special case

Consider the simple LOB process, so that

- $p = 4$, $q = 1$,
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Take moreover the simple case $\phi = 0$ (no memory case : the conditional density does not depend on $N$).

Then $S_t$ alone is a birth and death process on $\mathbb{N}$ and the ergodicity is equivalent to

$$J^T \mu_0 < 0.$$ (negative drift)
Markov assumption

Let us investigate the case where

\[ \phi_{i,i}(t) = \alpha_{i,j} \beta e^{-\beta t}, \quad t \geq 0, \]

so that the unknown parameters are reduced to \( \mathbf{\Lambda} = [\alpha_{i,j}], \beta > 0 \) and \( \mathbf{\mu}_0 \in (0, \infty)^p \). Then, defining the \( \mathbb{R}^p \) valued process

\[ \lambda(t) = \int_{-\infty}^{t} \phi(t - s) N(ds), \]

we have that \( \mathbf{X}(t) = (\mathbf{S}(t), \lambda(t)) \) is a Markov process (due to the exponential form of the fertility function).
Markov assumption

Let us investigate the case where

$$\phi_{i,i}(t) = \alpha_{i,j} \beta e^{-\beta t}, \quad t \geq 0,$$

so that the unknown parameters are reduced to $\mathcal{N} = [\alpha_{i,j}], \beta > 0$ and $\mu_0 \in (0, \infty)^p$. Then, defining the $\mathbb{R}^p$ valued process

$$\lambda(t) = \int_{-\infty}^{t} \phi(t-s) \mathcal{N}(ds),$$

we have that $X(t) = (S(t), \lambda(t))$ is a Markov process (due to the exponential form of the fertility function).

Moreover, the following discrete time processes are Markov chains:

- $X_k = X(T_k)$, with Markov kernel $Q$ on $X = \mathbb{N}^q \times (0, \infty)^p$,
- $Y_k = (I_k, X_k)$, with Markov kernel $\check{Q}$ on $Y = \{0, \ldots, p\} \times X$,
- $Z_k = (\Delta_k, I_k, X_k)$, where $\Delta_k = T_k - T_{k-1}$, with Markov kernel $\bar{Q}$ on $Z = \mathbb{R}_+ \times Y$. 
Irreducibility, aperiodicity, partial drift

Some conditions on $\mathbb{N}$ and $J$ are required to get that

- the above chains are $\psi$-irreducible and aperiodic (by adding an artificial mark $i = 0$ such that $J(0) = 0$).
Irreducibility, aperiodicity, partial drift

Some conditions on $\mathcal{N}$ and $\mathbf{J}$ are required to get that

- the above chains are $\psi$-irreducible and aperiodic (by adding an artificial mark $i = 0$ such that $\mathbf{J}(0) = 0$).
- For all $K = 1, 2, 3, \ldots$ and $M > 0$, all sets $\{1, \ldots, K\}^q \times (0, M]^p$ are petite-sets for $Q$. 

Irreducibility, aperiodicity, partial drift

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- the above chains are $\psi$-irreducible and aperiodic (by adding an artificial mark $i = 0$ such that $J(0) = 0$).
- For all $K = 1, 2, 3, \ldots$ and $M > 0$, all sets $\{1, \ldots, K\}^q \times (0, M]^p$ are petite-sets for $Q$.
- We have the partial drift condition,

$$[Q(1_{\mathbb{Z}_+^q} \otimes V_{1,\gamma})](s, \ell) \leq \theta V_{1,\gamma}(\ell) + b1_{(0,M]^p}(\ell),$$

where $M, b > 0$, $\theta \in (0, 1)$ and

$$V_{1,\gamma}(\ell) = e^{\gamma u^T \ell},$$

for some $\gamma > 0$ and $u$ some vector with positive entries.
Consider the case $q = 1$ and suppose that

$$J^T(I - \mathcal{N})^{-1} \mu_0 < 0.$$ 

This actually means that, would the constraints be removed, the process $J$ would have a negative drift under the stationary distribution (and thus be eventually negative with probability 1).
Geometric ergodicity: case $q = 1$

Consider the case $q = 1$ and suppose that

$$J^T (I - \mathbb{N})^{-1} \mu_0 < 0.$$ 

This actually means that, would the constraints be removed, the process $J$ would have a negative drift under the stationary distribution (and thus be eventually negative with probability 1).

Then we obtain a complete drift condition which implies that $Q$ is $(V_{0,\gamma_0} \otimes V_{1,\gamma_1})$-geometrically ergodic for some $\gamma_0, \gamma_1 > 0$, with

$$V_{0,\gamma_0}(s) = e^{\gamma_0 s}$$

and $V_{1,\gamma_1}$ defined as above.
Consequences

All the usual good properties of geometrically ergodic Markov chains follow:

Scaling limit (Donsker Theorem)

\[
T^{-1/2} (P_t - P_0 - \mathbb{E} \left[ J \right]) \xrightarrow{t \to \infty} \sigma(J) \left( B_t \right),
\]

where \( B \) is the standard Brownian motion.

We expect \( \mathbb{E} \left[ J \right] = 0 \) for the BB or BA price, and \( P_t \) behaves as a random walk at large scales.

It is of course not the case for \( S \) for which \( \sigma(J) = 0 \).

The result can be extended to all \( q \geq 1 \) by recursively checking negative drifts on the chains obtained by removing an arbitrary set of constraints.
Consequences

All the usual good properties of geometrically ergodic Markov chains follow:

- Ergodicity, central limit theorems for the chains $Q$, $\tilde{Q}$, $\bar{Q}$,
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- Ergodicity, central limit theorems for the chains $Q$, $\tilde{Q}$, $\bar{Q}$,
- Ergodicity, central limit theorems for any continuous time processes

$$P_t = P_0 + \mathcal{N}((0, t] \times J) = P_0 + \sum_{0<T_k<t} J(I_k).$$

We expect $E_0[J] = 0$ for the BB or BA price, and $P_t$ behaves as a random walk at large scales.

It is of course not the case for $S$ for which $\sigma(J) = 0$.

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$$T^{-1/2} \left( P_{tT} - P_0 - tT\mathbb{E}^0[J] \right)_{t \in [0, 1]} \Rightarrow \sigma(J) (B_t)_{t \in [0, 1]} \text{ in } D,$$ (1)

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$$P_t = P_0 + N((0, t] \times J) = P_0 + \sum_{0<T_k<t} J(I_k).$$

- Scaling limit (Donsker Theorem)

$$T^{-1/2} \left( P_{tT} - P_0 - tT\mathbb{E}^0[J] \right)_{t\in[0,1]} \Rightarrow \sigma(J) (B_t)_{t\in[0,1]} \text{ in } D,$$  (1)

where $B$ is the standard Brownian motion.

- we expect $\mathbb{E}^0[J] = 0$ for the BB or BA price, and $P_t$ behaves as a random walk at large scales.

- It is of course not the case for $S$ for which $\sigma(J) = 0$. 
Consequences

All the usual good properties of geometrically ergodic Markov chains follow:

- Ergodicity, central limit theorems for the chains $Q$, $\tilde{Q}$, $\bar{Q}$,
- Ergodicity, central limit theorems for any continuous time processes $P_t = P_0 + N((0, t] \times J) = P_0 + \sum_{0<T_k<t} J(I_k)$.

- Scaling limit (Donsker Theorem)

\[ T^{-1/2} \left( P_{tT} - P_0 - tT\mathbb{E}^0[J] \right)_{t\in[0,1]} \Rightarrow \sigma(J) (B_t)_{t\in[0,1]} \text{ in } D, \quad (1) \]

where $B$ is the standard Brownian motion.

- we expect $\mathbb{E}^0[J] = 0$ for the BB or BA price, and $P_t$ behaves as a random walk at large scales.
- It is of course not the case for $S$ for which $\sigma(J) = 0$.
- The result can be extended to all $q \geq 1$ by recursively checking negative drifts on the chains obtained by removing an arbitrary set of constraints.
1 Limit order books

2 Constrained Hawkes processes

3 Some applications
Simple LOB

We use the Constrained Hawkes process to describe the dynamics of a simple LOB using the marks

\( i = 1 \) Best Ask price moves upward one tick,
\( i = 2 \) Best Ask price moves downward one tick,
\( i = 3 \) Best Bid price moves upward one tick,
\( i = 4 \) Best Bid price moves downward one tick.

In this case we have

\( p = 4, \quad q = 1, \quad S_t \) is the spread at time \( t \).

All the parameters are estimated by numerically maximizing the likelihood.
Excitation and immigrant intensities for ENI.MI, over ten days, time unit = seconds

ScLOBHP, Cross-Excitation Map, ENI.MI
All parameters, same data, same unit

Parameter estimation, $\mu_0$, ENI.MI

Parameter estimation, $\alpha$, ENI.MI

Parameter estimation, $\beta$, ENI.MI

Francois Roueff

http://perso.telecom-paristech.fr/~roueff/

joint work with Ban Zheng (Natixis) and Frederic Abergel (Institut Mines-Telecom; Telecom ParisTech; CNRS LTCI)

Constrained Hawkes processes

March 19, 2014 33 / 40
One/two ticks events, TOTF.PA

ScLOBHP, Cross–Excitation Map, TOTF.PA
Price dynamics in limit Order Book

ScLOBHP Intensity, type = 1

ScLOBHP Intensity, type = 2

ScLOBHP Intensity, type = 3

ScLOBHP Intensity, type = 4

ScLOBHP Intensity, type = 5

ScLOBHP Intensity, type = 6

ScLOBHP Intensity, type = 7

ScLOBHP Intensity, type = 8

The dynamics of Spread

Constrained Hawkes processes
Cross excitation, LOB with two assets

We use the Constrained Hawkes process to describe the dynamics of a simple LOB using the marks

- $i = 1, 5$ Best Ask price moves upward one tick for asset 1,2, respectively,
- $i = 2, 6$ Best Ask price moves downward one tick for asset 1,2, respectively,
- $i = 3, 7$ Best Bid price moves upward one tick for asset 1,2, respectively,
- $i = 4, 8$ Best Bid price moves downward one tick for asset 1,2, respectively.

In this case we have

- $p = 8, q = 2$, $S_t$ contains the spreads of the two assets at time $t$. 
Excitation and immigrant intensities for ENI.MI and TOTF.PA

ScLOBHP, Cross−Excitation Map, TOTF.PA−ENI.MI

Mark=1 (0.0023)
Mark=2 (0.0536)
Mark=3 (0.0393)
Mark=4 (0.0279)
Mark=5 (0.0028)
Mark=6 (0.0068)
Mark=7 (0.0068)
Mark=8 (0.0031)
Parameter estimation, $\mu_0$, TOTF.PA

Parameter estimation, $\mu_0$, ENI.MI

Parameter estimation, $\alpha$, TOTF.PA−ENI.MI

Parameter estimation, $\beta$, TOTF.PA−ENI.MI
Back to the case $q = 1$, conclusion

Using the estimated parameters one can evaluate the drift appearing in the stability condition:

$$J^T(I - \mathcal{H})^{-1}\mu_0.$$  

It seems to be a good indicator of the volatility.
Back to the case $q = 1$, conclusion

Using the estimated parameters one can evaluate the drift appearing in the stability condition:

$$J^T(I - \mathcal{N})^{-1} \mu_0 .$$

It seems to be a good indicator of the volatility.

Some directions for future work

- Computation of the asymptotic deviation $\sigma(J)$ of the midprice appearing in (1), which seems a more sensible estimate of the volatility.
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Some directions for future work

- Computation of the asymptotic deviation $\sigma(J)$ of the midprice appearing in (1), which seems a more sensible estimate of the volatility.
- Possible extensions to other applications for describing the dynamics of an object driven by a Point process within some boundary conditions.
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- Markov assumption should not be necessary.
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Some directions for future work

- Computation of the asymptotic deviation $\sigma(J)$ of the midprice appearing in (1), which seems a more sensible estimate of the volatility.

- Possible extensions to other applications for describing the dynamics of an object driven by a Point process within some boundary conditions.

- Markov assumption should not be necessary.

- Locally stationary case (work in progress for standard Hawkes processes).
Further reading