Constrained Hawkes processes for modeling limit order books.

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Outline

High frequency price data

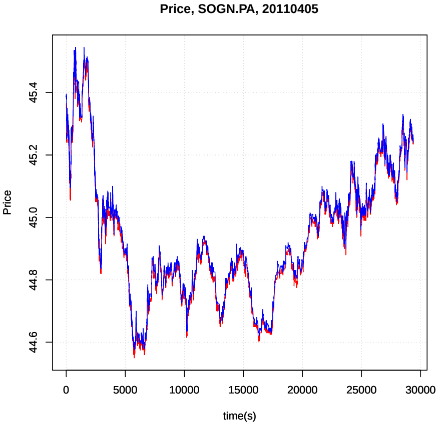


Figure: Price of an asset over one day.

Signature plots

How do the usual models behave at small scales ?

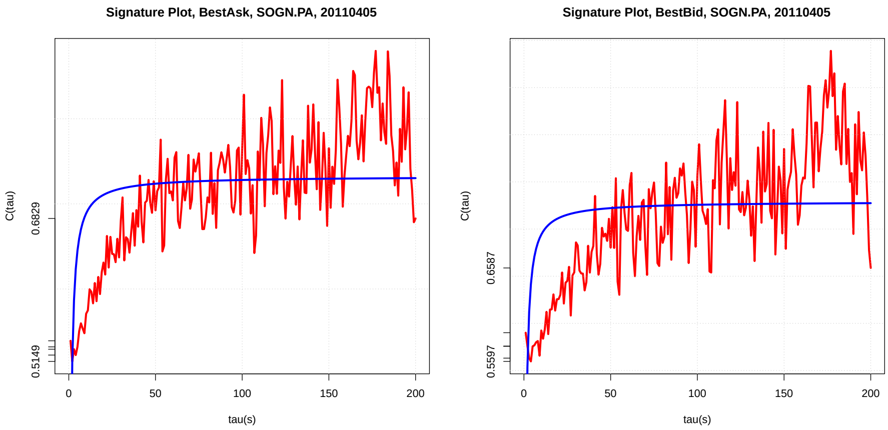
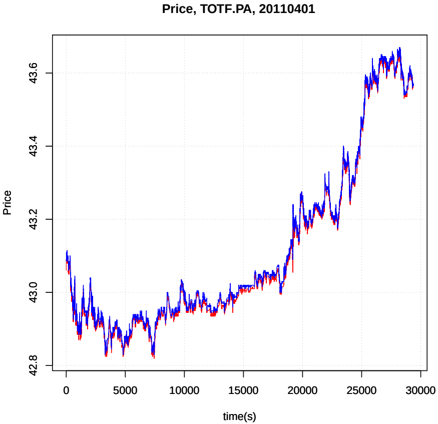


Figure: Red: Normalized realized volatility as a function of the sampling period.

High frequency price data: another asset



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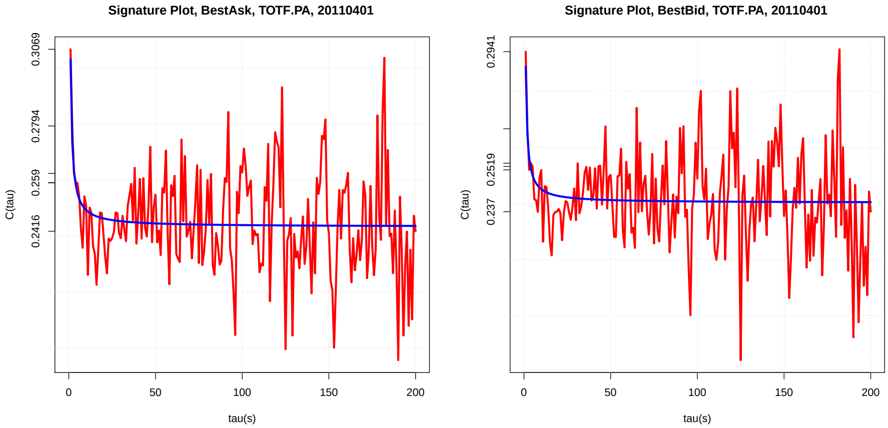


Figure: Red: Normalized realized volatility as a function of the increment period.

The smallest time scale: Limit order book

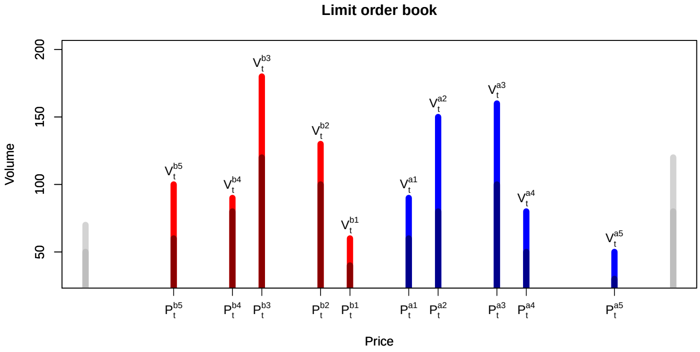


Figure: A limit order book (LOB) at a given fixed time. Bid prices (red) and Ask prices (blue) available for market orders.

Limit order book events: limit order arrival

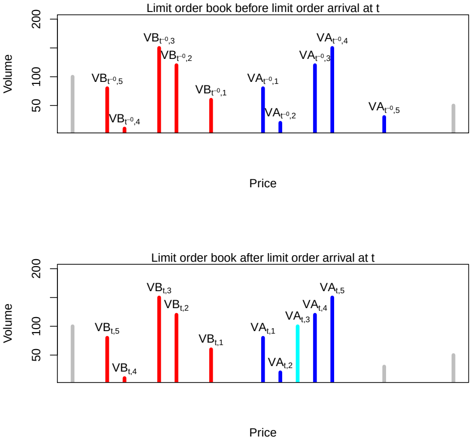


Figure: LOB before and after a limit order. Light blue: new ask limit.

Limit order book events: limit order cancellation

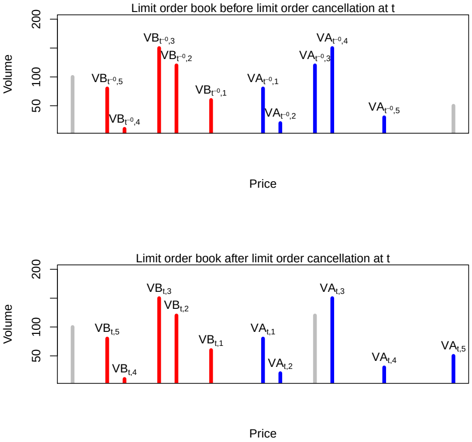
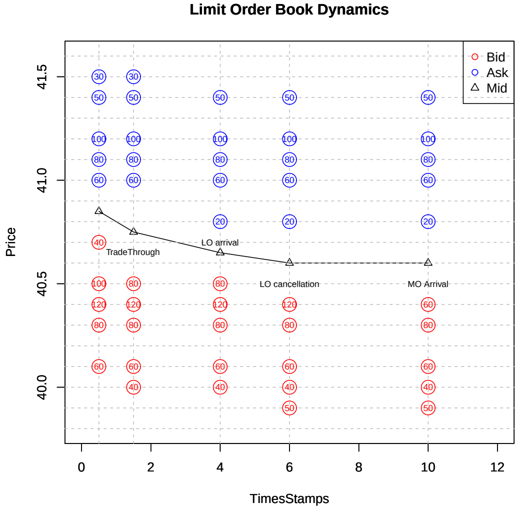


Figure: LOB before and after a limit order cancellation. Gray: canceled ask limit.

Time evolution of a LOB and mid-price.



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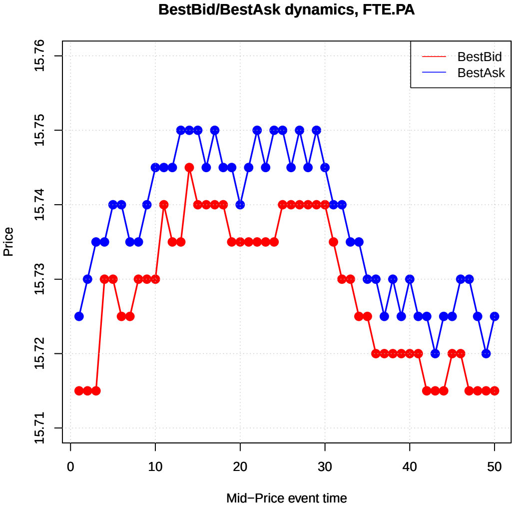
A simplified LOB: Best Bid (BB) and Best Ask (BA) prices.



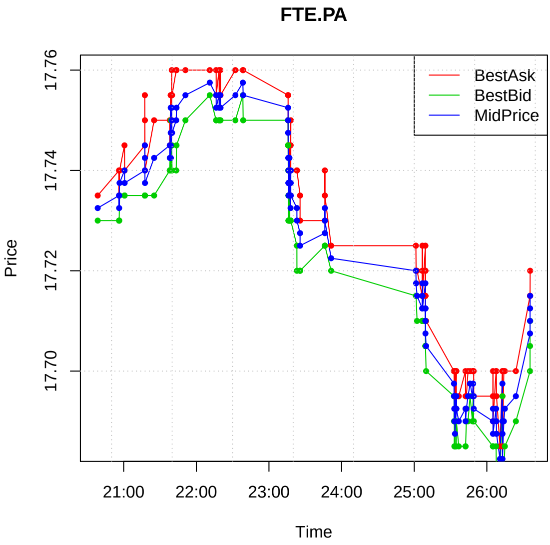


Figure: Successive BB and BA events in physical time (mn).

Point process of a simplified LOB

We consider the marked point process describing the dynamics of the BB and BA prices,

$$N = \sum_k \delta_{T_k, I_k} \quad \text{with} \quad 0 < T_1 < T_2 < \dots \quad \text{and} \quad I_1, I_2, \dots \in \{1, \dots, p\} ,$$

where each mark i in $\{1, \dots, p\}$ corresponds to a quantified shift of either the BB or the BA price, e.g.

- $i = 1$ Best Ask price moves upward one tick,
- $i = 2$ Best Ask price moves downward one tick,
- $i = 3$ Best Bid price moves upward one tick,
- $i = 4$ Best Bid price moves downward one tick.

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- $i = 4$ Best Bid price moves downward one tick.

Extensions

By increasing the set of marks, one can consider one marked process describing the LOBs of several assets.

Point process and prices dynamics

One can recover the dynamics of BB, BA and mid prices from the point process N through formulas of the form

$$P_t - P_0 = N\left(\mathbb{1}_{(0,t]} \otimes J\right) = \sum_{0<T_k\le t} J(I_k), \quad t > 0\,.$$

For instance, in the previous example,

$$J(1) = 1, J(2) = -1, J(3) = J(4) = 0$$

corresponds to $P_t =$ BA price.

Point process and BB-BA spread

The gap between the BB price and the BA price is called the spread, from now on denoted by

$$S_t = \text{BA price}_t - \text{BB price}_t \in \{1, 2, 3, \dots\} \; .$$

We will denote by J the corresponding function on $\{1, \dots, p\}$ such that, for all $t > 0$,

$$S_t = S_0 + N\left(\mathbb{1}_{(0,t]} \times J\right) = S_0 + \sum_{0 < T_k \leq t} J(I_k) \; .$$

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Important remarks

- J takes positive and negative values while S only takes positive ones.
- S_t behaves as a stationary random process.
- BB and BA prices typically behave as integrated (and thus co-integrated) stationary processes.

Hawkes processes

Consider a marked point process $N = \sum_k \delta_{T_k, I_k}$ with

$$\cdots < T_{-1} < T_0 \leq 0 < T_1 < T_2 < \dots \quad \text{and} \quad \dots, I_{-1}, I_0, I_1, I_2, \dots \in \mathcal{I} \;.$$

It is an Hawkes process if its conditional density is of the form

$$\mu(t,i) = \mu_0(i) + \int_{(-\infty,t)} \phi(t-s, j; i)\; N(\mathrm{d}s, \mathrm{d}j)\;,$$

where $\mu_0 : \mathcal{I} \to \mathbb{R}_+$ is called the immigrant intensity and $\phi : [0,\infty) \times \mathcal{I}^2 \to \mathbb{R}_+$ is called the fertility function.

Multivariate Hawkes processes

If $\mathcal{I} = \{1, \dots, p\}$, The marked Hawkes process can be seen as a multivariate Hawkes process

$$N_i = N(\cdot \times \{i\}), \quad 1 \leq i \leq p \,,$$

the fertility is written as a $p \times p$ matrix $\boldsymbol{\phi}(t) = [\phi_{i,j}(t)]_{i,j}$,

$$\mu(t,i) = \mu_0(i) + \int_{(-\infty,t)} \sum_{j=1}^{p} \phi_{i,j}(t-s)\, N_j(\mathrm{d}s), \quad 1 \leq i \leq p \,,$$

or in a more compact form

$$\boldsymbol{\mu}(t) = \boldsymbol{\mu}_0 + \int_{(-\infty,t)} \boldsymbol{\phi}(t-s)\, \mathbf{N}(\mathrm{d}s)\,.$$

Basic stability condition

It can be shown that such a point process is well defined and admit a stationary version if

(BC) the spectral radius of the $p \times p$ matrix

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Hawkes processes for modeling a simple LOB

Suppose that a stationary Hawkes process N is used to model the dynamics of a simple LOB as defined previously yielding to

$$P_t - P_0 = N\left(\mathbb{1}_{(0,t]} \otimes J\right) = \sum_{0<T_k\leq t} J(I_k), \quad t > 0 \ .$$

for a BB, BA or mid-price P (with an adequate J) and

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However, for a Hawkes process, provided that $\mu_0(i) > 0$ for all i, we have, for any k,

$$\min_{1\leq i\leq p} \mathbb{P}(I_{k+1} = i \,|\, \mathcal{F}_{T_k}) = \frac{\mu(T_k, i)}{\mu(T_k, 1) + \cdots + \mu(T_k, p)} > 0$$

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Idea :

modify the conditional density by adding constraints depending on S_t.

Constrained Hawkes processes

We consider a point process N with marks in $\{1, \ldots, p\}$ with conditional intensity given by, for all $i = 1, \ldots, p$,

$$\mu(t, i) = \begin{cases} 0 & \text{if } \mathbf{S}(t-) \in \mathbf{A}_i \\ \mu_0(i) + \int_{(-\infty,t)} \sum_{j=1}^{p} \phi_{i,j}(t-s)\, N(\mathrm{d}s \times \{j\}) & \text{otherwise} , \end{cases}$$

where $\mathbf{S}$ is a q-dimensional process valued in $\mathbb{N}^q$ and defined by

$$\mathbf{S}_t = \mathbf{S}_0 + N\left(\mathbb{1}_{(0,t]} \times \mathbf{J}\right) ,$$

for some $\mathbf{J} : \{1, \ldots, p\} \to \mathbb{Z}^q$.

Here

- p denotes the number of marks
- q denotes the number of constraints.
- $\mathbf{A}_1, \ldots, \mathbf{A}_p$ are constraints subsets of $\mathbb{Z}^q$.

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- Application to LOB modeling.

A very special case

Consider the simple LOB process, so that

- $p = 4$, $q = 1$,
- S_t is the spread at time t and, at each event, moves a tick upward or downward,
- $A_i = \{1\}$ for the events i making the spread move downward, so that S_t remains positive.

Take moreover the simple case $\phi = 0$ (no memory case : the conditional density does not depend on N).

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Take moreover the simple case $\phi = 0$ (no memory case : the conditional density does not depend on N).
Then S_t alone is a birth and death process on $\mathbb{N}$ and the ergodicity is equivalent to

$$\mathbf{J}^T \boldsymbol{\mu}_0 < 0 \; .$$

(negative drift)

Markov assumption

Let us investigate the case where

$$\phi_{i,i}(t) = \alpha_{i,j}\beta \mathrm{e}^{-\beta t}, \quad t \geq 0 ,$$

so that the unknown parameters are reduced to $\aleph = [\alpha_{i,j}]$, $\beta > 0$ and $\boldsymbol{\mu}_0 \in (0, \infty)^p$. Then, defining the $\mathbb{R}^p$ valued process

$$\boldsymbol{\lambda}(t) = \int_{-\infty}^{t} \boldsymbol{\phi}(t-s)\, \boldsymbol{N}(\mathrm{d}s) ,$$

we have that $\mathbf{X}(t) = (\mathbf{S}(t), \boldsymbol{\lambda}(t))$ is a Markov process (due to the exponential form of the fertility function).

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Moreover, the following discrete time processes are Markov chains :

- $\mathbf{X}_k = \mathbf{X}(T_k)$, with Markov kernel Q on $\mathrm{X} = \mathbb{N}^q \times (0, \infty)^p$,
- $\mathbf{Y}_k = (I_k, \mathbf{X}_k)$, with Markov kernel $\check{Q}$ on $\mathrm{Y} = \{0, \dots, p\} \times \mathrm{X}$,
- $\mathbf{Z}_k = (\Delta_k, I_k, \mathbf{X}_k)$, where $\Delta_k = T_k - T_{k-1}$, , with Markov kernel $\bar{Q}$ on $\mathrm{Z} = \mathbb{R}_+ \times \mathrm{Y}$,.

Irreducibility, aperiodicity, partial drift

Some conditions on $\aleph$ and $\mathbf{J}$ are required to get that

- the above chains are ψ-irreducible and aperiodic (by adding an artificial mark $i = 0$ such that $J(0) = 0$).

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- For all $K = 1, 2, 3, \ldots$ and $M > 0$, all sets $\{1, \ldots, K\}^q \times (0, M]^p$ are petite-sets for Q.
- we have the *partial* drift condition,

$$[Q(\mathbb{1}_{\mathbb{Z}_+^q} \otimes V_{1,\gamma})](\mathbf{s}, \boldsymbol{\ell}) \leq \theta\, V_{1,\gamma}(\boldsymbol{\ell}) + b \mathbb{1}_{(0,M]^p}(\boldsymbol{\ell}) ,$$

where $M, b > 0$, $\theta \in (0, 1)$ and

$$V_{1,\gamma}(\boldsymbol{\ell}) = \mathrm{e}^{\gamma \mathbf{u}^T \boldsymbol{\ell}} ,$$

for some $\gamma > 0$ and $\mathbf{u}$ some vector with positive entries.

Geometric ergodicity: case $q = 1$

Consider the case $q = 1$ and suppose that

$$\mathbf{J}^T (I - \aleph)^{-1} \boldsymbol{\mu}_0 < 0 \ .$$

This actually means that, would the constraints be removed, the process J would have a negative drift under the stationary distribution (and thus be eventually negative with probability 1).

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Then we obtain a complete drift condition which implies that Q is $(V_{0,\gamma_0} \otimes V_{1,\gamma_1})$-geometrically ergodic for some $\gamma_0, \gamma_1 > 0$, with

$$V_{0,\gamma_0}(s) = \mathrm{e}^{\gamma_0 s}$$

and V_{1,γ_1} defined as above.

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- ▷ Scaling limit (Donsker Theorem)

$$T^{-1/2} \left(P_{tT} - P_0 - tT\mathbb{E}^0[J] \right)_{t\in[0,1]} \Rightarrow \sigma(J) \; (B_t)_{t\in[0,1]} \quad \text{in } D \; , \qquad (1)$$

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  where B is the standard Brownian motion.
- we expect $\mathbb{E}^0[J] = 0$ for the BB or BA price, and P_t behaves as a random walk at large scales.
- It is of course not the case for S for which $\sigma(J) = 0$.

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$$P_t = P_0 + N((0,t] \times J) = P_0 + \sum_{0<T_k<t} J(I_k) .$$

- Scaling limit (Donsker Theorem)

$$T^{-1/2} \left(P_{tT} - P_0 - tT\mathbb{E}^0[J]\right)_{t\in[0,1]} \Rightarrow \sigma(J) \, (B_t)_{t\in[0,1]} \quad \text{in } D \; , \qquad (1)$$

  where B is the standard Brownian motion.
- we expect $\mathbb{E}^0[J] = 0$ for the BB or BA price, and P_t behaves as a random walk at large scales.
- It is of course not the case for S for which $\sigma(J) = 0$.
- The result can be extended to all $q \geq 1$ by recursively checking negative drifts on the chains obtained by removing an arbitrary set of constraints.

Simple LOB

We use the Constrained Hawkes process to describe the dynamics of a simple LOB using the marks

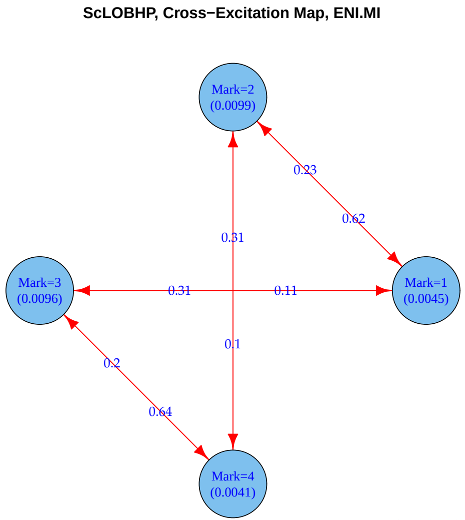
- $i = 1$ Best Ask price moves upward one tick,
- $i = 2$ Best Ask price moves downward one tick,
- $i = 3$ Best Bid price moves upward one tick,
- $i = 4$ Best Bid price moves downward one tick.

In this case we have

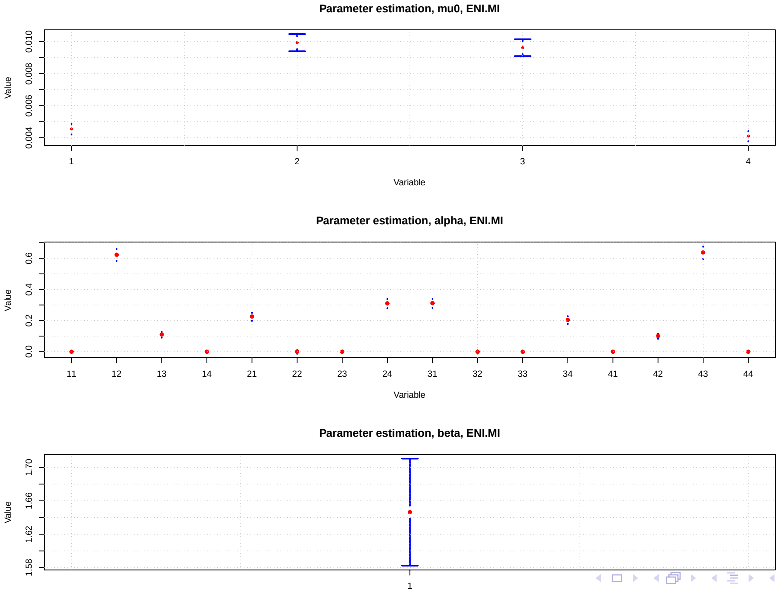
- $p = 4$, $q = 1$, S_t is the spread at time t.

All the parameters are estimated by numerically maximizing the likelihood.

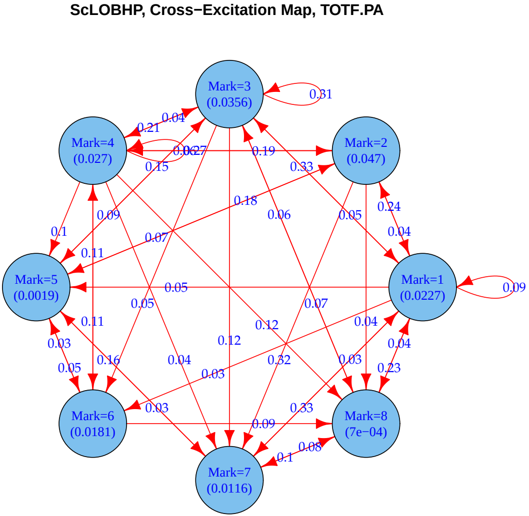
Excitation and immigrant intensities for ENI.MI, over ten days, time unit = seconds

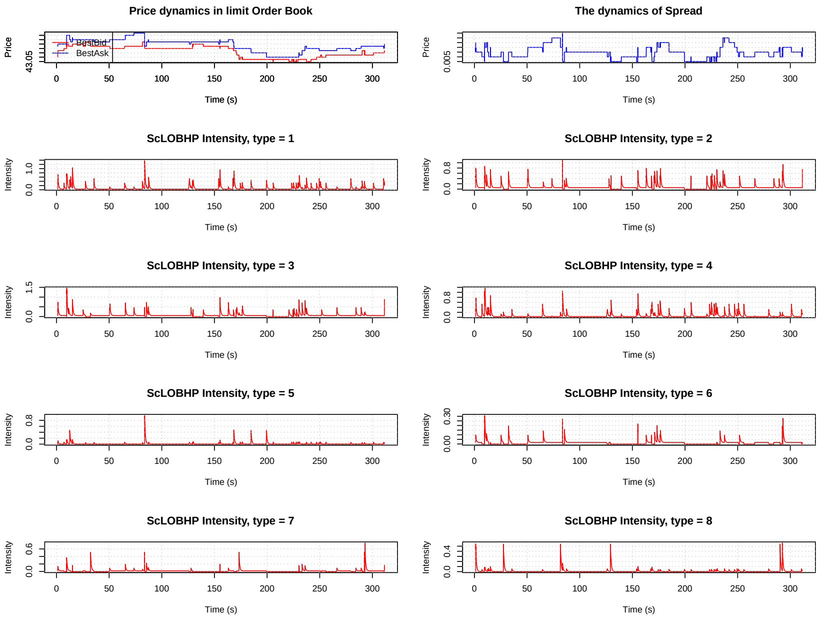


All parameters, same data, same unit



One/two ticks events, TOTF.PA




Price dynamics in limit Order Book
Price
43.05
BestBid
BestAsk
0
50
100
150
200
250
300
Time (s)
The dynamics of Spread
Price
0.005
Time (s)
ScLOBHP Intensity, type = 1
Intensity
0.0
1.0
Time (s)
ScLOBHP Intensity, type = 2
Intensity
0.0
0.8
Time (s)
ScLOBHP Intensity, type = 3
Intensity
0.0
1.5
Time (s)
ScLOBHP Intensity, type = 4
Intensity
0.0
0.8
Time (s)
ScLOBHP Intensity, type = 5
Intensity
0.0
0.8
Time (s)
ScLOBHP Intensity, type = 6
Intensity
0.00
0.30
Time (s)
ScLOBHP Intensity, type = 7
Intensity
0.0
0.6
Time (s)
ScLOBHP Intensity, type = 8
Intensity
0.0
0.4
Time (s)

Cross excitation, LOB with two assets

We use the Constrained Hawkes process to describe the dynamics of a simple LOB using the marks

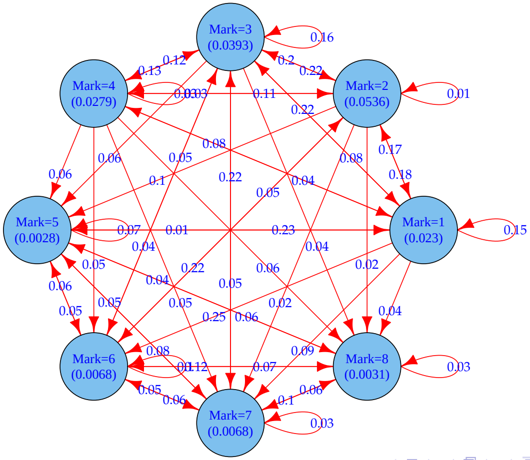
- $i = 1, 5$ Best Ask price moves upward one tick for asset 1,2, respectively,
- $i = 2, 6$ Best Ask price moves downward one tick for asset 1,2, respectively,
- $i = 3, 7$ Best Bid price moves upward one tick for asset 1,2, respectively,
- $i = 4, 8$ Best Bid price moves downward one tick for asset 1,2, respectively.

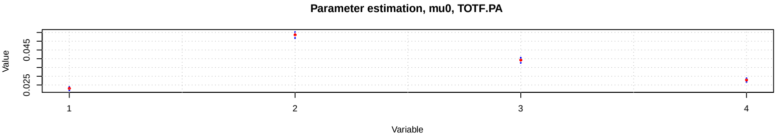
In this case we have

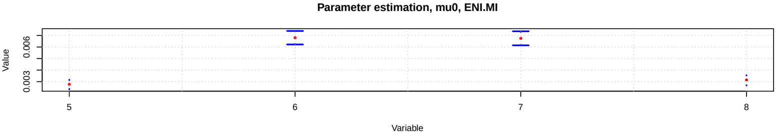
- $p = 8$, $q = 2$, $\mathbf{S}_t$ contains the spreads of the two assets at time t.

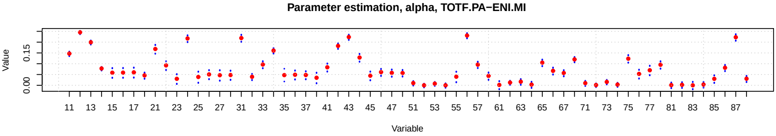
Excitation and immigrant intensities for ENI.MI and TOTF.PA

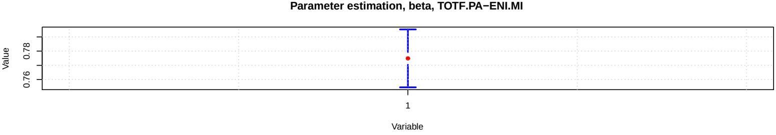
ScLOBHP, Cross−Excitation Map, TOTF.PA−ENI.MI




Parameter estimation, mu0, TOTF.PA
Value
0.025
0.045
1
2
3
4
Variable


Parameter estimation, mu0, ENI.MI
Value
0.003
0.006
5
6
7
8
Variable


Parameter estimation, alpha, TOTF.PA−ENI.MI
Value
0.00
0.15
11 13 15 17 21 23 25 27 31 33 35 37 41 43 45 47 51 53 55 57 61 63 65 67 71 73 75 77 81 83 85 87
Variable


Parameter estimation, beta, TOTF.PA−ENI.MI
Value
0.76
0.78
1
Variable

Back to the case $q = 1$, conclusion

Using the estimated parameters one can evaluate the drift appearing in the stability condition :

$$\mathbf{J}^T (I - \aleph)^{-1} \boldsymbol{\mu}_0 \ .$$

It seems to be a good indicator of the volatility.

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Some directions for future work

- Computation of the asymptotic deviation $\sigma(J)$ of the midprice appearing in (1), which seems a more sensible estimate of the volatility.

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- Possible extensions to other applications for describing the dynamics of an object driven by a Point process within some boundary conditions.

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- Possible extensions to other applications for describing the dynamics of an object driven by a Point process within some boundary conditions.
- Markov assumption should not be necessary.
- Locally stationary case (work in progress for standard Hawkes processes).

Further reading

Ban Zheng, François Roueff, and Frédéric Abergel. Modelling bid and ask prices using constrained Hawkes processes: Ergodicity and scaling limit. *SIAM J. Finan. Math.*, 5(1):99–136, February 2014. doi: 10.1137/130912980. Preprint available at [HAL] or [arXiv].