About the use of weak transport costs for concentration and functionals inequalities in discrete spaces.

Based on:
Joint work with N. Gozlan, C. Roberto et P. Tetali.

Institut d’Études Scientifiques de Cargèse

Concentration of measure and its applications

May 2018

P-M. Samson
Université Paris-Est Marne-la-Vallée
Notations:

- $\text{The Schrödinger}$
- $\text{the symmetric group}$
- $\text{Transport inequality on}$
- $\text{inequalities}$
- $\text{Universal transport}$
- $\text{Weak transport}$
- $\text{Kantorovich duality}$
- $\text{Introduction}$
- $\text{Curvature in discrete spaces}$

- $\text{Marton's type of cost}$
- $\text{for classical costs}$
- $\text{for weak costs}$

- $\text{Examples of weak cost}$
- $\text{Marton's inequality}$
- $\text{Strassen's result}$
- $\text{Martingale costs}$

- $\text{Weak transport inequalities}$
- $\text{Dual characterization}$
- $\text{to concentration}$

- $\text{Universal transport inequalities}$

- $\text{Barycentric transport inequalities}$
- $\text{examples}$
- $\text{characterisation on } \mathbb{R}$

- $\text{Transport inequality on the symmetric group}$
- $\text{introduction}$
- $\text{Ewens distribution}$
- $\text{deviation inequalities}$

- $\text{The Schrödinger minimization problem}$
- $\text{definition}$
- $\text{curvature in discrete spaces}$
- $\text{functional inequalities}$
- $\text{Examples in discrete}$
- $\text{Weak transport costs.2}$
Notations:

- $(\mathcal{X}, d)$: a metric space (complete separable)
Notations:

- \((X, d)\): a metric space (complete separable) \(\rightarrow X = \mathbb{Z}, S_n, \ldots\)
Notations:

- \((\mathcal{X}, d)\) : a metric space (complete separable) \(\rightarrow \mathcal{X} = \mathbb{Z}, S_n, \ldots\)
- \(\mathcal{M}(\mathcal{X})\) : the set of non-negative measure on \(\mathcal{X}\).
Notations:

- \((\mathcal{X}, d)\) : a metric space (complete separable) \(\to \mathcal{X} = \mathbb{Z}, S_n, \ldots\)
- \(\mathcal{M}(\mathcal{X})\) : the set of non-negative measure on \(\mathcal{X}\).
- \(\mathcal{P}(A)\) : the set of all probability measures on \(A \subset \mathcal{X}\).
Notations:

- \((\mathcal{X}, d)\): a metric space (complete separable) \(\to \mathcal{X} = \mathbb{Z}, S_n, \ldots\)
- \(\mathcal{M}(\mathcal{X})\): the set of non-negative measure on \(\mathcal{X}\).
- \(\mathcal{P}(A)\): the set of all probability measures on \(A \subset \mathcal{X}\).

More generally, given \(\gamma: \mathbb{R}^+ \to \mathbb{R}^+\), a lower semi-continuous function with

\[
\gamma(0) = 0 \quad \text{and} \quad \gamma(u + v) \leq C(\gamma(u) + \gamma(v)), \quad \forall u, v \in \mathbb{R}^+.
\]
Notations:
- \((\mathcal{X}, d)\) : a metric space (complete separable) \(\rightarrow \mathcal{X} = \mathbb{Z}, S_n, \ldots\)
- \(\mathcal{M}(\mathcal{X})\) : the set of non-negative measure on \(\mathcal{X}\).
- \(\mathcal{P}(A)\) : the set of all probability measures on \(A \subset \mathcal{X}\).
  More generally, given \(\gamma : \mathbb{R}^+ \rightarrow \mathbb{R}^+\), a lower semi-continuous function with
  \[
  \gamma(0) = 0 \quad \text{and} \quad \gamma(u + v) \leq C(\gamma(u) + \gamma(v)), \quad \forall u, v \in \mathbb{R}^+.
  \]
  \[\mathcal{P}_\gamma(\mathcal{X}) := \left\{ p \in \mathcal{P}(\mathcal{X}), \int \gamma(d(x_0, y))dp(y) < +\infty \right\}, \quad x_0 \in \mathcal{X}.\]
Notations:

- \((\mathcal{X}, d)\) : a metric space (complete separable) \(\rightarrow \mathcal{X} = \mathbb{Z}, S_n, \ldots\)
- \(\mathcal{M}(\mathcal{X})\) : the set of non-negative measure on \(\mathcal{X}\).
- \(\mathcal{P}(A)\) : the set of all probability measures on \(A \subset \mathcal{X}\).

More generally, given \(\gamma : \mathbb{R}^+ \rightarrow \mathbb{R}^+\), a lower semi-continuous function with

\[
\gamma(0) = 0 \quad \text{and} \quad \gamma(u + v) \leq C(\gamma(u) + \gamma(v)), \quad \forall u, v \in \mathbb{R}^+.
\]

\[\mathcal{P}_\gamma(\mathcal{X}) := \left\{ p \in \mathcal{P}(\mathcal{X}), \int \gamma(d(x_0, y)) dp(y) < +\infty \right\}, \quad x_0 \in \mathcal{X}.\]

Specific examples:
Notations:

- \((\mathcal{X}, d)\) : a metric space (complete separable) \(\rightarrow \mathcal{X} = \mathbb{Z}, S_n, \ldots\)
- \(\mathcal{M}(\mathcal{X})\) : the set of non-negative measure on \(\mathcal{X}\).
- \(\mathcal{P}(A)\) : the set of all probability measures on \(A \subset \mathcal{X}\).

More generally, given \(\gamma : \mathbb{R}^+ \rightarrow \mathbb{R}^+\), a lower semi-continuous function with

\[
\gamma(0) = 0 \quad \text{and} \quad \gamma(u + v) \leq C(\gamma(u) + \gamma(v)), \quad \forall u, v \in \mathbb{R}^+.
\]

\[\mathcal{P}_{\gamma}(\mathcal{X}) := \left\{ p \in \mathcal{P}(\mathcal{X}), \int \gamma(d(x_0, y)) dp(y) < +\infty \right\}, \quad x_0 \in \mathcal{X}.\]

Specific examples:

- \(\gamma_q(u) = u^q, q > 0\), \(\mathcal{P}_{\gamma_q}(\mathcal{X}) = \mathcal{P}_q(\mathcal{X})\).
Notations:

- \((X, d)\): a metric space (complete separable) \(\rightarrow X = \mathbb{Z}, S_n, \ldots\)
- \(\mathcal{M}(X)\): the set of non-negative measure on \(X\).
- \(\mathcal{P}(A)\): the set of all probability measures on \(A \subset X\).

More generally, given \(\gamma: \mathbb{R}^+ \rightarrow \mathbb{R}^+\), a lower semi-continuous function with

\[
\gamma(0) = 0 \quad \text{and} \quad \gamma(u + v) \leq C(\gamma(u) + \gamma(v)), \quad \forall u, v \in \mathbb{R}^+.
\]

\[
\mathcal{P}_\gamma(X) := \left\{ p \in \mathcal{P}(X), \int \gamma(d(x_0, y)) \, dp(y) < +\infty \right\}, \quad x_0 \in X.
\]

Specific examples:

- \(\gamma_q(u) = u^q, q > 0\), \(\mathcal{P}_{\gamma_q}(X) = \mathcal{P}_q(X)\).
- \(\gamma_0(u) = 1_{u \neq 0}\),
Notations:

- \((\mathcal{X}, d)\) : a metric space (complete separable) \(\rightarrow \mathcal{X} = \mathbb{Z}, S_n, \ldots\)
- \(\mathcal{M}(\mathcal{X})\) : the set of non-negative measure on \(\mathcal{X}\).
- \(\mathcal{P}(A)\) : the set of all probability measures on \(A \subset \mathcal{X}\).

More generally, given \(\gamma : \mathbb{R}^+ \rightarrow \mathbb{R}^+\), a lower semi-continuous function with

\[\gamma(0) = 0 \quad \text{and} \quad \gamma(u + v) \leq C(\gamma(u) + \gamma(v)), \quad \forall u, v \in \mathbb{R}^+\.

\(\mathcal{P}_\gamma(\mathcal{X}) := \left\{ \pi \in \mathcal{P}(\mathcal{X}), \int \gamma(d(x_0, y)) \, dp(y) < +\infty \right\}, \quad x_0 \in \mathcal{X}\).

Specific examples:

- \(\gamma_q(u) = u^q, q > 0\), \(\mathcal{P}_{\gamma_q}(\mathcal{X}) = \mathcal{P}_q(\mathcal{X})\).
- \(\gamma_0(u) = 1_{u \neq 0}\), \(\gamma_0(d(x, y)) = 1_{x \neq y}\).
**Notations:**

- \((\mathcal{X}, d)\) : a metric space (complete separable) \(\rightarrow \mathcal{X} = \mathbb{Z}, S_n, \ldots\)
- \(\mathcal{M}(\mathcal{X})\) : the set of non-negative measure on \(\mathcal{X}\).
- \(\mathcal{P}(A)\) : the set of all probability measures on \(A \subset \mathcal{X}\).

More generally, given \(\gamma : \mathbb{R}^+ \rightarrow \mathbb{R}^+\), a lower semi-continuous function with

\[
\gamma(0) = 0 \quad \text{and} \quad \gamma(u + v) \leq C(\gamma(u) + \gamma(v)), \quad \forall u, v \in \mathbb{R}^+.
\]

\(\mathcal{P}_\gamma(\mathcal{X}) := \left\{ p \in \mathcal{P}(\mathcal{X}), \int \gamma(d(x_0, y)) dp(y) < +\infty \right\}, \quad x_0 \in \mathcal{X}.
\)

**Specific examples:**

- \(\gamma_q(u) = u^q, q > 0\), \(\mathcal{P}_{\gamma_q}(\mathcal{X}) = \mathcal{P}_q(\mathcal{X}).\)
- \(\gamma_0(u) = 1_{u \neq 0}\), \(\gamma_0(d(x, y)) = 1_{x \neq y}\), \(\mathcal{P}_{\gamma_0}(\mathcal{X}) = \mathcal{P}(\mathcal{X}).\)
Notations:

- \((\mathcal{X}, d)\) : a metric space (complete separable) \(\to \mathcal{X} = \mathbb{Z}, S_n, \ldots\)
- \(\mathcal{M}(\mathcal{X})\) : the set of non-negative measure on \(\mathcal{X}\).
- \(\mathcal{P}(A)\) : the set of all probability measures on \(A \subset \mathcal{X}\).

More generally, given \(\gamma : \mathbb{R}^+ \to \mathbb{R}^+\), a lower semi-continuous function with

\[
\gamma(0) = 0 \quad \text{and} \quad \gamma(u + v) \leq C(\gamma(u) + \gamma(v)), \quad \forall u, v \in \mathbb{R}^+.
\]

\(\mathcal{P}_{\gamma}(\mathcal{X}) := \left\{ p \in \mathcal{P}(\mathcal{X}), \int \gamma(d(x_0, y))dp(y) < +\infty \right\}, \quad x_0 \in \mathcal{X}.
\]

Specific examples:
- \(\gamma_q(u) = u^q, q > 0, \quad \mathcal{P}_{\gamma_q}(\mathcal{X}) = \mathcal{P}_q(\mathcal{X}).\)
- \(\gamma_0(u) = 1_{u \neq 0}, \quad \gamma_0(d(x, y)) = 1_{x \neq y}, \quad \mathcal{P}_{\gamma_0}(\mathcal{X}) = \mathcal{P}(\mathcal{X}).\)

- \(\Pi(\mu, \nu)\) : the set of probability measures in \(\mathcal{P}_{\gamma}(\mathcal{X} \times \mathcal{X})\) with marginals \(\mu\) and \(\nu\).
Notations:

- \((\mathcal{X}, d)\) : a metric space (complete separable) \(\rightarrow \mathcal{X} = \mathbb{Z}, S_n, \ldots\)
- \(\mathcal{P}(\mathcal{X})\) : the set of non-negative measure on \(\mathcal{X}\).
- \(\mathcal{P}(A)\) : the set of all probability measures on \(A \subset \mathcal{X}\).

More generally, given \(\gamma : \mathbb{R}^+ \rightarrow \mathbb{R}^+\), a lower semi-continuous function with

\[
\gamma(0) = 0 \quad \text{and} \quad \gamma(u + v) \leq C(\gamma(u) + \gamma(v)), \quad \forall u, v \in \mathbb{R}^+.
\]

\(\mathcal{P}_\gamma(\mathcal{X}) := \left\{ p \in \mathcal{P}(\mathcal{X}), \int \gamma(d(x_0, y))\,dp(y) < +\infty \right\}, \quad x_0 \in \mathcal{X}.
\)

Specific examples:
- \(\gamma_q(u) = u^q, \quad q > 0, \quad \mathcal{P}_{\gamma_q}(\mathcal{X}) = \mathcal{P}_q(\mathcal{X})\).
- \(\gamma_0(u) = 1_{u \neq 0}, \quad \gamma_0(d(x, y)) = 1_{x \neq y}, \quad \mathcal{P}_{\gamma_0}(\mathcal{X}) = \mathcal{P}(\mathcal{X}).\)

- \(\Pi(\mu, \nu)\) : the set of probability measures in \(\mathcal{P}_\gamma(\mathcal{X} \times \mathcal{X})\) with marginals \(\mu\) and \(\nu\).

Any measure \(\pi \in \Pi(\mu, \nu)\) admits a decomposition

\[
d\pi(x, y) = dp_x(y)\,d\mu(x), \quad \pi = \mu \otimes p
\]
Notations:

- \((\mathcal{X}, d)\) : a metric space (complete separable) \(\rightarrow \mathcal{X} = \mathbb{Z}, S_n, \ldots\)
- \(\mathcal{M}(\mathcal{X})\) : the set of non-negative measure on \(\mathcal{X}\).
- \(\mathcal{P}(A)\) : the set of all probability measures on \(A \subset \mathcal{X}\).

More generally, given \(\gamma : \mathbb{R}^+ \rightarrow \mathbb{R}^+\), a lower semi-continuous function with

\[\gamma(0) = 0 \quad \text{and} \quad \gamma(u + v) \leq C(\gamma(u) + \gamma(v)), \quad \forall u, v \in \mathbb{R}^+.\]

\[\mathcal{P}_{\gamma}(\mathcal{X}) := \left\{ p \in \mathcal{P}(\mathcal{X}), \int \gamma(d(x_0, y)) d\mu(y) < +\infty \right\}, \quad x_0 \in \mathcal{X}.\]

Specific examples:

- \(\gamma_q(u) = u^q, q > 0, \quad \mathcal{P}_{\gamma_q}(\mathcal{X}) = \mathcal{P}_q(\mathcal{X}).\)
- \(\gamma_0(u) = 1_{u \neq 0}, \quad \gamma_0(d(x, y)) = 1_{x \neq y}, \quad \mathcal{P}_{\gamma_0}(\mathcal{X}) = \mathcal{P}(\mathcal{X}).\)

- \(\Pi(\mu, \nu)\) : the set of probability measures in \(\mathcal{P}_{\gamma}(\mathcal{X} \times \mathcal{X})\) with marginals \(\mu\) and \(\nu\).

Any measure \(\pi \in \Pi(\mu, \nu)\) admits a decomposition

\[d\pi(x, y) = dp_x(y)d\mu(x), \quad \pi = \mu \otimes p\]

\[\pi(A \times B) = \int_A p_x(B)d\mu(x), \quad \text{for all Borel set } A, B \text{ of } \mathcal{X}.\]
Notations:

- \((\mathcal{X}, d)\) : a metric space (complete separable) \(\rightarrow \mathcal{X} = \mathbb{Z}, S_n, \ldots\)
- \(\mathcal{M}(\mathcal{X})\) : the set of non-negative measure on \(\mathcal{X}\).
- \(\mathcal{P}(A)\) : the set of all probability measures on \(A \subset \mathcal{X}\).

More generally, given \(\gamma : \mathbb{R}^+ \rightarrow \mathbb{R}^+\), a lower semi-continuous function with

\[\gamma(0) = 0 \quad \text{and} \quad \gamma(u + v) \leq C(\gamma(u) + \gamma(v)), \quad \forall u, v \in \mathbb{R}^+.\]

\(\mathcal{P}_\gamma(\mathcal{X}) := \left\{ p \in \mathcal{P}(\mathcal{X}), \int \gamma(d(x_0, y)) dp(y) < +\infty \right\}, \quad x_0 \in \mathcal{X}.

Specific examples:
- \(\gamma_q(u) = u^q, \quad q > 0, \quad \mathcal{P}_{\gamma_q}(\mathcal{X}) = \mathcal{P}_q(\mathcal{X}).\)
- \(\gamma_0(u) = \mathbb{1}_{u \neq 0}, \quad \gamma_0(d(x, y)) = \mathbb{1}_{x \neq y}, \quad \mathcal{P}_{\gamma_0}(\mathcal{X}) = \mathcal{P}(\mathcal{X}).\)

- \(\Pi(\mu, \nu)\) : the set of probability measures in \(\mathcal{P}_\gamma(\mathcal{X} \times \mathcal{X})\) with marginals \(\mu\) and \(\nu\).
Any measure \(\pi \in \Pi(\mu, \nu)\) admits a decomposition

\[d\pi(x, y) = dp_x(y)d\mu(x), \quad \pi = \mu \otimes p\]

\[\pi(A \times B) = \int_A p_x(B)d\mu(x), \quad \text{for all Borel set} \ A, \ B \text{ of } \mathcal{X}.\]

- \(H(\nu|m)\) : the relative entropy of \(\nu \in \mathcal{P}(\mathcal{X})\) with respect to a measure \(m\),
Notations:

- \((\mathcal{X}, d)\) : a metric space (complete separable) \(\to \mathcal{X} = \mathbb{Z}, S_n, \ldots\)
- \(\mathcal{M}(\mathcal{X})\) : the set of non-negative measure on \(\mathcal{X}\).
- \(\mathcal{P}(A)\) : the set of all probability measures on \(A \subset \mathcal{X}\).

More generally, given \(\gamma : \mathbb{R}^+ \to \mathbb{R}^+\), a lower semi-continuous function with

\[
\gamma(0) = 0 \quad \text{and} \quad \gamma(u + v) \leq C(\gamma(u) + \gamma(v)), \quad \forall u, v \in \mathbb{R}^+.
\]

\(\mathcal{P}_\gamma(\mathcal{X}) := \left\{ p \in \mathcal{P}(\mathcal{X}), \int \gamma(d(x_0, y))dp(y) < +\infty \right\}, \quad x_0 \in \mathcal{X}.
\]

Specific examples:
- \(\gamma_q(u) = u^q, \quad q > 0, \quad \mathcal{P}_{\gamma_q}(\mathcal{X}) = \mathcal{P}_q(\mathcal{X}).\)
- \(\gamma_0(u) = 1_{u \neq 0}, \quad \gamma_0(d(x, y)) = 1_{x \neq y}, \quad \mathcal{P}_{\gamma_0}(\mathcal{X}) = \mathcal{P}(\mathcal{X}).\)

- \(\Pi(\mu, \nu)\) : the set of probability measures in \(\mathcal{P}_\gamma(\mathcal{X} \times \mathcal{X})\) with marginals \(\mu\) and \(\nu\).

Any measure \(\pi \in \Pi(\mu, \nu)\) admits a decomposition

\[
d\pi(x, y) = dp_x(y)d\mu(x), \quad \pi = \mu \otimes p
\]

\[
\pi(A \times B) = \int_A p_x(B)d\mu(x), \quad \text{for all Borel set} \ A, B \text{ of } \mathcal{X}.
\]

- \(H(\nu|\mu)\) : the relative entropy of \(\nu \in \mathcal{P}(\mathcal{X})\) with respect to a measure \(\mu\),

\[
H(\nu|\mu) := \int \log \left( \frac{d\nu}{dm} \right) d\nu, \quad \text{if} \ \nu \ll m,
\]

and \(H(\nu|\mu) := +\infty\) otherwise.
Introduction : Marton’s transport inequality

In the 1990s, K. Marton introduced a weak transport cost \( r_{T^2}^p \nu \mid \mu \). He proved a variant of the Csizàr-Kullback-Pinsker inequality to recover a Talagrand's concentration inequality on product spaces, related to the so-called convex-hull method.

The Csizár-Kullback-Pinsker inequality:

\[
\text{for any } \mu, \nu, \quad \text{TV}_2 \leq 2 \inf_{\pi} \mathbb{E} \left[ |x - y|^2 \right]^{1/2},
\]

where \( \text{TV}_2 \):

\[
\text{TV}_2 = \sup_{A} |\mu - \nu|_1 - \inf_{\pi} \mathbb{E} \left[ |x - y|^2 \right]^{1/2},
\]

The Marton's transport inequalities:

\[
\text{for any } \mu, \nu, \quad r_{T^2}^p \nu \mid \mu \leq 2 \inf_{\pi} \mathbb{E} \left[ |x - y|^2 \right]^{1/2},
\]

with \( r_{T^2}^p \nu \mid \mu \) :

\[
\inf_{\pi} \mathbb{E} \left[ |x - y|^2 \right]^{1/2}.
\]

By Jensen's inequality,

\[
\frac{1}{4} \text{TV}_2 \leq \frac{1}{2} \text{TV}_2.
\]
Introduction: Marton’s transport inequality

In the 1990s, K. Marton introduced a weak transport cost $\tilde{T}_2(\nu|\mu)$. 

Marton's inequality

Talagrand's concentration

Kantorovich duality

for classical costs

for weak costs

Examples of weak cost

Marton's type of cost

Barycentric cost

Strassen's result

Martingale costs

Weak transport inequalities

Dual characterization
to concentration

Universal transport inequalities

Barycentric transport inequalities

examples

characterisation on $\mathbb{R}$

Transport inequality on the symmetric group

introduction

Ewens distribution
delegation inequalities

The Schrödinger minimization problem

definition

curvature in discrete spaces

functional inequalities

Examples in discrete

Weak transport costs.3
Introduction : Marton’s transport inequality

In the 1990s, K. Marton introduced a weak transport cost $\tilde{T}_2(\nu|\mu)$. She proved a variant of the Csizàr-Kullback-Pinsker inequality to recover a Talagrand’s concentration inequality on product spaces, related to the so-called convex-hull method.
Introduction : Marton’s transport inequality

In the 1990s, K. Marton introduced a weak transport cost $\tilde{T}_2(\nu \| \mu)$. She proved a variant of the Csizàr-Kullback-Pinsker inequality to recover a Talagrand’s concentration inequality on product spaces, related to the so-called convex-hull method.

The Csizár-Kullback-Pinsker inequality :
Introduction: Marton’s transport inequality

In the 1990s, K. Marton introduced a weak transport cost $\tilde{r}_2(\nu|\mu)$. She proved a variant of the Csizàr-Kullback-Pinsker inequality to recover a Talagrand’s concentration inequality on product spaces, related to the so-called convex-hull method.

The Csizár-Kullback-Pinsker inequality: for any $\mu, \nu \in \mathcal{P}(\mathcal{X})$

$$\|\mu - \nu\|_{TV}^2 \leq 2 \, H(\nu|\mu),$$
Introduction: Marton’s transport inequality

In the 1990s, K. Marton introduced a weak transport cost \( \tilde{T}_2(\nu|\mu) \).
She proved a variant of the Csizàr-Kullback-Pinsker inequality to recover a
Talagrand’s concentration inequality on product spaces, related to the so-called
convex-hull method.

The Csizár-Kullback-Pinsker inequality: for any \( \mu, \nu \in \mathcal{P}(X) \)
\[ \|\mu - \nu\|_{TV}^2 \leq 2 H(\nu|\mu), \]
where \( \|\mu - \nu\|_{TV} := 2 \sup_A |\mu(A) - \nu(A)| \)
\[ = 2 \inf_{\pi \in \Pi(\mu, \nu)} \int \int 1_{x \neq y} d\pi(x, y). \]
Introduction : Marton’s transport inequality

In the 1990s, K. Marton introduced a weak transport cost $\tilde{T}_2(\nu|\mu)$. She proved a variant of the Csizàr-Kullback-Pinsker inequality to recover a Talagrand’s concentration inequality on product spaces, related to the so-called convex-hull method.

The Csizár-Kullback-Pinsker inequality : for any $\mu, \nu \in \mathcal{P}(X)$

$$\|\mu - \nu\|_{TV}^2 \leq 2 H(\nu|\mu),$$

where

$$\|\mu - \nu\|_{TV} := 2 \sup_A |\mu(A) - \nu(A)|$$

$$= 2 \inf_{\pi \in \Pi(\mu, \nu)} \iint 1_{x \neq y} d\pi(x, y).$$

The Marton’s transport inequalities :
**Introduction : Marton’s transport inequality**

In the 1990s, K. Marton introduced a weak transport cost $\tilde{T}_2(\nu|\mu)$. She proved a variant of the Csizàr-Kullback-Pinsker inequality to recover a Talagrand’s concentration inequality on product spaces, related to the so-called convex-hull method.

**The Csizár-Kullback-Pinsker inequality :** for any $\mu, \nu \in \mathcal{P}(X)$

$$\|\mu - \nu\|_{T^2}^2 \leq 2 H(\nu|\mu),$$

where

$$\|\mu - \nu\|_{T^2} := 2 \sup_A |\mu(A) - \nu(A)|
= 2 \inf_{\pi \in \Pi(\mu, \nu)} \iint 1_{x \neq y} d\pi(x, y).$$

**The Marton’s transport inequalities :** for any $\mu, \nu \in \mathcal{P}(X)$

$$\tilde{T}_2(\nu|\mu) \leq 2 H(\nu|\mu), \quad \tilde{T}_2(\mu|\nu) \leq 2 H(\nu|\mu),$$
**Introduction : Marton’s transport inequality**

In the 1990s, K. Marton introduced a weak transport cost $\tilde{T}_2(\nu|\mu)$. She proved a variant of the Csizràr-Kullback-Pinsker inequality to recover a Talagrand’s concentration inequality on product spaces, related to the so-called convex-hull method.

**The Csizár-Kullback-Pinsker inequality** : for any $\mu, \nu \in \mathcal{P}(\mathcal{X})$

$$\|\mu - \nu\|_{TV}^2 \leq 2H(\nu|\mu),$$

where

$$\|\mu - \nu\|_{TV} := 2 \sup_A |\mu(A) - \nu(A)|$$

$$= 2 \inf_{\pi \in \Pi(\mu, \nu)} \int \int 1_{x \neq y} d\pi(x, y).$$

**The Marton’s transport inequalities** : for any $\mu, \nu \in \mathcal{P}(\mathcal{X})$

$$\tilde{T}_2(\nu|\mu) \leq 2H(\nu|\mu), \quad \tilde{T}_2(\mu|\nu) \leq 2H(\nu|\mu),$$

with $\tilde{T}_2(\nu|\mu) := \inf_{\pi \in \Pi(\mu, \nu)} \int c(x, px) d\mu(x), \quad \pi = \mu \otimes p$.
Introduction: Marton’s transport inequality

In the 1990s, K. Marton introduced a weak transport cost \( \tilde{T}_2(\nu|\mu) \). She proved a variant of the Csizàr-Kullback-Pinsker inequality to recover a Talagrand’s concentration inequality on product spaces, related to the so-called convex-hull method.

The Csizár-Kullback-Pinsker inequality: for any \( \mu, \nu \in \mathcal{P}(X) \)

\[
\|\mu - \nu\|_{TV}^2 \leq 2 \mathcal{H}(\nu|\mu),
\]

where

\[
\|\mu - \nu\|_{TV} := 2 \sup_A |\mu(A) - \nu(A)|
\]

\[
= 2 \inf_{\pi \in \Pi(\mu, \nu)} \int \int 1_{x \neq y} d\pi(x, y).
\]

The Marton’s transport inequalities: for any \( \mu, \nu \in \mathcal{P}(X) \)

\[
\tilde{T}_2(\nu|\mu) \leq 2 \mathcal{H}(\nu|\mu), \quad \tilde{T}_2(\mu|\nu) \leq 2 \mathcal{H}(\nu|\mu),
\]

with \( \tilde{T}_2(\nu|\mu) := \inf_{\pi \in \Pi(\mu, \nu)} \int c(x, p_x) d\mu(x), \quad c(x, p_x) = \left( \int 1_{x \neq y} d\mu_x(y) \right)^2. \)
Introduction: Marton’s transport inequality

In the 1990s, K. Marton introduced a weak transport cost $\tilde{T}_2(\nu|\mu)$. She proved a variant of the Csizàr-Kullback-Pinsker inequality to recover a Talagrand’s concentration inequality on product spaces, related to the so-called convex-hull method.

The Csizár-Kullback-Pinsker inequality: for any $\mu, \nu \in \mathcal{P}(X)$

$$\|\mu - \nu\|_{TV}^2 \leq 2H(\nu|\mu),$$

where

$$\|\mu - \nu\|_{TV} := 2\sup_A |\mu(A) - \nu(A)|$$

$$= 2\inf_{\pi \in \Pi(\mu, \nu)} \int\int 1_{x \neq y} \, d\pi(x, y).$$

The Marton’s transport inequalities: for any $\mu, \nu \in \mathcal{P}(X)$

$$\tilde{T}_2(\nu|\mu) \leq 2H(\nu|\mu), \quad \tilde{T}_2(\mu|\nu) \leq 2H(\nu|\mu),$$

with $\tilde{T}_2(\nu|\mu) := \inf_{\pi \in \Pi(\mu, \nu)} \int c(x, p_x) \, d\mu(x), \quad c(x, p_x) = \left(\int 1_{x \neq y} \, dp_x(y)\right)^2$. 

The Schrödinger minimization problem

Definition

Curvature in discrete spaces

Functional inequalities

Examples in discrete weak transport costs
Introduction: Marton’s transport inequality

In the 1990s, K. Marton introduced a weak transport cost \( \tilde{T}_2(\nu|\mu) \).
She proved a variant of the Csizàr-Kullback-Pinsker inequality to recover a Talagrand’s concentration inequality on product spaces, related to the so-called convex-hull method.

The Csizár-Kullback-Pinsker inequality: for any \( \mu, \nu \in \mathcal{P}(\mathcal{X}) \)
\[
\|\mu - \nu\|_{TV}^2 \leq 2 H(\nu|\mu),
\]
where \( \|\mu - \nu\|_{TV} := 2 \sup_A |\mu(A) - \nu(A)| \)
\[
= 2 \inf_{\pi \in \Pi(\mu, \nu)} \int \int 1_{x \neq y} d\pi(x, y).
\]

The Marton’s transport inequalities: for any \( \mu, \nu \in \mathcal{P}(\mathcal{X}) \)
\[
\tilde{T}_2(\nu|\mu) \leq 2 H(\nu|\mu), \quad \tilde{T}_2(\mu|\nu) \leq 2 H(\nu|\mu),
\]
with \( \tilde{T}_2(\nu|\mu) := \inf_{\pi \in \Pi(\mu, \nu)} \int c(x, p_x) d\mu(x), \quad c(x, p_x) = \left( \int 1_{x \neq y} dp_x(y) \right)^2 . \)

By Jensen’s inequality,
\[
\frac{1}{4} \|\mu - \nu\|_{TV}^2 \leq \tilde{T}_2(\nu|\mu) \leq \frac{1}{2} \|\mu - \nu\|_{TV}
\]
Symmetric version of Marton’s transport inequalities:

\[
\inf_{\pi \in \Pi \atop \pi_1 = \nu_1, \pi_2 = \nu_2} \int_X d\nu(x) \leq \inf_{\pi \in \Pi \atop \pi_1 = \nu_1, \pi_2 = \nu_2} \int_X d\nu(x)
\]

where \( r \) denotes the \( i \)-th marginal of \( p \).
Symmetric version of Marton’s transport inequalities:

\[
\frac{1}{2} \tilde{T}_2(\nu_2 | \nu_1) \leq \left( \sqrt{H(\nu_1 | \mu)} + \sqrt{H(\nu_2 | \mu)} \right)^2, \quad \forall \mu \in \mathcal{P}(\mathcal{X}), \nu_1, \nu_2 \in \mathcal{P}(\mathcal{X}),
\]
Symmetric version of Marton’s transport inequalities:

\[ \frac{1}{2} \tilde{T}_2(\nu_2 | \nu_1) \leq \left( \sqrt{H(\nu_1 | \mu)} + \sqrt{H(\nu_2 | \mu)} \right)^2, \quad \forall \mu \in \mathcal{P}(\mathcal{X}), \nu_1, \nu_2 \in \mathcal{P}(\mathcal{X}), \]

or equivalently, since \( \left( \sqrt{H_1} + \sqrt{H_2} \right)^2 = \inf_{s \in (0,1)} \{ H_1/s + H_2/(1 - s) \} \),
Symmetric version of Marton’s transport inequalities:

\[
\frac{1}{2} \tilde{T}_2(\nu_2 | \nu_1) \leq \left( \sqrt{H(\nu_1 | \mu)} + \sqrt{H(\nu_2 | \mu)} \right)^2, \quad \forall \mu \in \mathcal{P}(X), \nu_1, \nu_2 \in \mathcal{P}(X),
\]

or equivalently, since \( \left( \sqrt{H_1} + \sqrt{H_2} \right)^2 = \inf_{s \in (0,1)} \{ H_1/s + H_2/(1 - s) \} \),

\[
\frac{1}{2} \tilde{T}_2(\nu_2 | \nu_1) \leq \frac{1}{s} H(\nu_1 | \mu) + \frac{1}{1 - s} H(\nu_2 | \mu), \quad \forall s \in (0, 1).
\]
Symmetric version of Marton’s transport inequalities:

\[
\frac{1}{2} \tilde{T}_2(\nu_2|\nu_1) \leq \left( \sqrt{H(\nu_1|\mu)} + \sqrt{H(\nu_2|\mu)} \right)^2, \quad \forall \mu \in \mathcal{P}(X), \nu_1, \nu_2 \in \mathcal{P}(X),
\]

or equivalently, since \( \left( \sqrt{H_1} + \sqrt{H_2} \right)^2 = \inf_{s \in (0,1)} \{ H_1/s + H_2/(1 - s) \}, \)

\[
\frac{1}{2} \tilde{T}_2(\nu_2|\nu_1) \leq \frac{1}{s} H(\nu_1|\mu) + \frac{1}{1 - s} H(\nu_2|\mu), \quad \forall s \in (0, 1).
\]

Transport-entropy inequalities tensorize:
Symmetric version of Marton’s transport inequalities:

\[ \frac{1}{2} \tilde{H}_2(\nu_2|\nu_1) \leq \left( \sqrt{H(\nu_1|\mu)} + \sqrt{H(\nu_2|\mu)} \right)^2, \quad \forall \mu \in \mathcal{P}(\mathcal{X}), \nu_1, \nu_2 \in \mathcal{P}(\mathcal{X}), \]

or equivalently, since \( \left( \sqrt{H_1} + \sqrt{H_2} \right)^2 = \inf_{s \in (0,1)} \{ H_1/s + H_2/(1-s) \}, \)

\[ \frac{1}{2} \tilde{H}_2(\nu_2|\nu_1) \leq \frac{1}{s} H(\nu_1|\mu) + \frac{1}{1-s} H(\nu_2|\mu), \quad \forall s \in (0,1). \]

Transport-entropy inequalities tensorize: setting \( \mu^n = \mu \times \cdots \times \mu \in \mathcal{P}(\mathcal{X}^n), \)
Symmetric version of Marton’s transport inequalities:

\[ \frac{1}{2} \tilde{T}_2(\nu_2 | \nu_1) \leq \left( \sqrt{H(\nu_1 | \mu)} + \sqrt{H(\nu_2 | \mu)} \right)^2, \quad \forall \mu \in \mathcal{P}(\mathcal{X}), \nu_1, \nu_2 \in \mathcal{P}(\mathcal{X}), \]

or equivalently, since \( \left( \sqrt{H_1} + \sqrt{H_2} \right)^2 = \inf_{s \in (0,1)} \{ H_1/s + H_2/(1 - s) \}, \)

\[ \frac{1}{2} \tilde{T}_2(\nu_2 | \nu_1) \leq \frac{1}{s} H(\nu_1 | \mu) + \frac{1}{1 - s} H(\nu_2 | \mu), \quad \forall s \in (0,1). \]

Transport-entropy inequalities tensorize: setting \( \mu^n = \mu \times \cdots \times \mu \in \mathcal{P}(\mathcal{X}^n), \)

\[ \frac{1}{2} \tilde{T}_2(\nu_2 | \nu_1) \leq \frac{1}{s} H(\nu_1 | \mu^n) + \frac{1}{1 - s} H(\nu_2 | \mu^n), \quad \forall s \in (0,1), \quad \forall \nu_1, \nu_2 \in \mathcal{P}(\mathcal{X}^n). \]
Symmetric version of Marton’s transport inequalities:

\[
\frac{1}{2} \tilde{T}_2(\nu_2 | \nu_1) \leq \left( \sqrt{H(\nu_1 | \mu)} + \sqrt{H(\nu_2 | \mu)} \right)^2, \quad \forall \mu \in \mathcal{P}(\mathcal{X}), \nu_1, \nu_2 \in \mathcal{P}(\mathcal{X}),
\]

or equivalently, since \( \left( \sqrt{H_1} + \sqrt{H_2} \right)^2 = \inf_{s \in (0,1)} \{ H_1/s + H_2/(1 - s) \}, \)

\[
\frac{1}{2} \tilde{T}_2(\nu_2 | \nu_1) \leq \frac{1}{s} H(\nu_1 | \mu) + \frac{1}{1 - s} H(\nu_2 | \mu), \quad \forall s \in (0, 1).
\]

Transport-entropy inequalities tensorize: setting \( \mu^n = \mu \times \cdots \times \mu \in \mathcal{P}(\mathcal{X}^n), \)

\[
\frac{1}{2} \tilde{T}_2(\nu_2 | \nu_1) \leq \frac{1}{s} H(\nu_1 | \mu^n) + \frac{1}{1 - s} H(\nu_2 | \mu^n), \quad \forall s \in (0, 1), \quad \forall \nu_1, \nu_2 \in \mathcal{P}(\mathcal{X}^n).
\]

where

\[
\tilde{T}_2(\nu_2 | \nu_1) := \inf_{\pi \in \Pi(\nu_1, \nu_2)} \int c^n(x, p_x) \, d\nu_1(x),
\]

with for \( x = x_1, \ldots, x_n \in X^n, \)

\[
c^n(x, p_x) := \hat{c} \frac{d}{d} x_i \hat{x}^i \equiv \hat{c} \sum_{i=1}^n x_i \cdot \hat{x}^i.
\]

\( p_i \) denotes the \( i \)-th marginal of \( p. \)
Symmetric version of Marton’s transport inequalities:

\[
\frac{1}{2} \tilde{T}_2(\nu_2|\nu_1) \leq \left( \sqrt{H(\nu_1|\mu)} + \sqrt{H(\nu_2|\mu)} \right)^2, \quad \forall \mu \in \mathcal{P}(\mathcal{X}), \nu_1, \nu_2 \in \mathcal{P}(\mathcal{X}),
\]

or equivalently, since \( \left( \sqrt{H_1} + \sqrt{H_2} \right)^2 = \inf_{s \in (0,1)} \{ H_1/s + H_2/(1 - s) \}, \)

\[
\frac{1}{2} \tilde{T}_2(\nu_2|\nu_1) \leq \frac{1}{s} H(\nu_1|\mu) + \frac{1}{1 - s} H(\nu_2|\mu), \quad \forall s \in (0,1).
\]

Transport-entropy inequalities tensorize: setting \( \mu^n = \mu \times \cdots \times \mu \in \mathcal{P}(\mathcal{X}^n), \)

\[
\frac{1}{2} \tilde{T}_2(\nu_2|\nu_1) \leq \frac{1}{s} H(\nu_1|\mu^n) + \frac{1}{1 - s} H(\nu_2|\mu^n), \quad \forall s \in (0,1), \quad \forall \nu_1, \nu_2 \in \mathcal{P}(\mathcal{X}^n).
\]

where

\[
\tilde{T}_2(\nu_2|\nu_1) := \inf_{\pi \in \Pi(\nu_1, \nu_2)} \int c^n(x, \rho_x) d\nu_1(x),
\]

with for \( x = (x_1, \ldots, x_n) \in \mathcal{X}^n, \)

\[
c^n(x, \rho) := \sum_{i=1}^n c(x_i, \rho_i),
\]
Symmetric version of Marton’s transport inequalities:

\[
\frac{1}{2} \tilde{T}_2(\nu_2|\nu_1) \leq \left( \sqrt{H(\nu_1|\mu)} + \sqrt{H(\nu_2|\mu)} \right)^2, \quad \forall \mu \in \mathcal{P}(\mathcal{X}), \nu_1, \nu_2 \in \mathcal{P}(\mathcal{X}),
\]

or equivalently, since \( \left( \sqrt{H_1} + \sqrt{H_2} \right)^2 = \inf_{s \in (0,1)} \left\{ H_1/s + H_2/(1 - s) \right\}, \)

\[
\frac{1}{2} \tilde{T}_2(\nu_2|\nu_1) \leq \frac{1}{s} H(\nu_1|\mu) + \frac{1}{1 - s} H(\nu_2|\mu), \quad \forall s \in (0,1).
\]

Transport-entropy inequalities tensorize: setting \( \mu^n = \mu \times \cdots \times \mu \in \mathcal{P}(\mathcal{X}^n), \)

\[
\frac{1}{2} \tilde{T}_2(\nu_2|\nu_1) \leq \frac{1}{s} H(\nu_1|\mu^n) + \frac{1}{1 - s} H(\nu_2|\mu^n), \quad \forall s \in (0,1), \forall \nu_1, \nu_2 \in \mathcal{P}(\mathcal{X}^n).
\]

where

\[
\tilde{T}_2(\nu_2|\nu_1) := \inf_{\pi \in \Pi(\nu_1, \nu_2)} \int c^n(x, p_x) \, d\nu_1(x),
\]

with for \( x = (x_1, \ldots, x_n) \in \mathcal{X}^n, \)

\[
c^n(x, p) := \sum_{i=1}^{n} c(x_i, p_i), \quad c(x_i, p_i) = \left( \int 1_{x_i \neq y_i} \, dp_i(y_i) \right)^2.
\]
Symmetric version of Marton’s transport inequalities:

\[
\frac{1}{2} \tilde{T}_2(\nu_2 | \nu_1) \leq \left( \sqrt{H(\nu_1 | \mu)} + \sqrt{H(\nu_2 | \mu)} \right)^2, \quad \forall \mu \in \mathcal{P}(\mathcal{X}), \nu_1, \nu_2 \in \mathcal{P}(\mathcal{X}),
\]

or equivalently, since \( \left( \sqrt{H_1} + \sqrt{H_2} \right)^2 = \inf_{s \in (0,1)} \{ H_1/s + H_2/(1 - s) \}, \)

\[
\frac{1}{2} \tilde{T}_2(\nu_2 | \nu_1) \leq \frac{1}{s} H(\nu_1 | \mu) + \frac{1}{1 - s} H(\nu_2 | \mu), \quad \forall s \in (0,1).
\]

Transport-entropy inequalities tensorize: setting \( \mu^n = \mu \times \cdots \times \mu \in \mathcal{P}(\mathcal{X}^n), \)

\[
\frac{1}{2} \tilde{T}_2(\nu_2 | \nu_1) \leq \frac{1}{s} H(\nu_1 | \mu^n) + \frac{1}{1 - s} H(\nu_2 | \mu^n), \quad \forall s \in (0,1), \forall \nu_1, \nu_2 \in \mathcal{P}(\mathcal{X}^n).
\]

where

\[
\tilde{T}_2(\nu_2 | \nu_1) := \inf_{\pi \in \Pi(\nu_1, \nu_2)} \int c^n(x, \rho_x) \, d\nu_1(x),
\]

with for \( x = (x_1, \ldots, x_n) \in \mathcal{X}^n, \)

\[
c^n(x, \rho) := \sum_{i=1}^{n} c(x_i, \rho_i), \quad c(x_i, \rho_i) = \left( \int 1_{x_i \neq y_i} \, d\rho_i(y_i) \right)^2.
\]

\( \rho_i \) denotes the \( i \)-th marginal of \( \rho \).
Symmetric version of Marton’s transport inequalities:

\[
\frac{1}{2} \tilde{T}_2(\nu_2 | \nu_1) \leq \left( \sqrt{H(\nu_1 | \mu)} + \sqrt{H(\nu_2 | \mu)} \right)^2, \quad \forall \mu \in \mathcal{P}(\mathcal{X}), \nu_1, \nu_2 \in \mathcal{P}(\mathcal{X}),
\]

or equivalently, since \( \left( \sqrt{H_1} + \sqrt{H_2} \right)^2 = \inf_{s \in (0,1)} \{ H_1/s + H_2/(1-s) \}, \)

\[
\frac{1}{2} \tilde{T}_2(\nu_2 | \nu_1) \leq \frac{1}{s} H(\nu_1 | \mu) + \frac{1}{1-s} H(\nu_2 | \mu), \quad \forall s \in (0,1).
\]

Transport-entropy inequalities tensorize: setting \( \mu^n = \mu \times \cdots \times \mu \in \mathcal{P}(\mathcal{X}^n), \)

\[
\frac{1}{2} \tilde{T}_2(\nu_2 | \nu_1) \leq \frac{1}{s} H(\nu_1 | \mu^n) + \frac{1}{1-s} H(\nu_2 | \mu^n), \quad \forall s \in (0,1), \quad \forall \nu_1, \nu_2 \in \mathcal{P}(\mathcal{X}^n).
\]

where

\[
\tilde{T}_2(\nu_2 | \nu_1) := \inf_{\pi \in \Pi(\nu_1, \nu_2)} \int c^n(x, \rho_x) \, d\nu_1(x),
\]

with for \( x = (x_1, \ldots, x_n) \in \mathcal{X}^n, \)

\[
c^n(x, \rho) := \sum_{i=1}^n c(x_i, \rho_i), \quad c(x_i, \rho_i) = \left( \int 1_{x_i \neq y_i} \, d\rho_i(y_i) \right)^2.
\]

\( p_i \) denotes the \( i \)-th marginal of \( p \).
How to recover the Talagrand’s concentration inequality?

\[ \frac{1}{2} \tilde{T}_2(\nu_2|\nu_1) \leq \frac{1}{s} H(\nu_1|\mu^n) + \frac{1}{1-s} H(\nu_2|\mu^n), \quad \forall s \in (0, 1). \]
How to recover the Talagrand’s concentration inequality?

\[ \frac{1}{2} \tilde{T}_2(\nu_2|\nu_1) \leq \frac{1}{s} H(\nu_1|\mu^n) + \frac{1}{1-s} H(\nu_2|\mu^n), \quad \forall s \in (0, 1). \]

First method, the Marton’s argument:
How to recover the Talagrand’s concentration inequality?

\[ \frac{1}{2} \mathcal{T}_2(\nu_2|\nu_1) \leq \frac{1}{s} \mathcal{H}(\nu_1|\mu^n) + \frac{1}{1-s} \mathcal{H}(\nu_2|\mu^n), \quad \forall s \in (0, 1). \]

First method, the Marton’s argument: \( x \in \mathcal{X}^n, A \subset \mathcal{X}^n, \)

\[ c^n(x, A) := \inf_{\rho, p(A)=1} c^n(x, \rho), \]
How to recover the Talagrand’s concentration inequality?

\[
\frac{1}{2} \tilde{T}_2(\nu_2|\nu_1) \leq \frac{1}{s} H(\nu_1|\mu^n) + \frac{1}{1-s} H(\nu_2|\mu^n), \quad \forall s \in (0, 1).
\]

First method, the Marton’s argument: \( x \in \mathcal{X}^n, A \subset \mathcal{X}^n, \)

\[
c^n(x, A) := \inf_{p, p(A) = 1} c^n(x, p), \quad \text{and} \quad A_t := \{ x \in \mathcal{X}, c^n(x, A) \leq t \}.
\]
How to recover the Talagrand's concentration inequality?

\[
\frac{1}{2} \tilde{T}_2(\nu_2 | \nu_1) \leq \frac{1}{s} H(\nu_1 | \mu^n) + \frac{1}{1-s} H(\nu_2 | \mu^n), \quad \forall s \in (0, 1).
\]

First method, the Marton's argument: \( x \in \mathcal{X}^n, A \subset \mathcal{X}^n, \)

\[
c^n(x, A) := \inf_{\rho, \rho(A) = 1} c^n(x, \rho), \quad \text{and} \quad A_t := \{ x \in \mathcal{X}, c^n(x, A) \leq t \}.
\]

Choose \( \frac{d\nu_1}{d\mu} = \frac{1_A}{\mu(A)} \) and \( \frac{d\nu_2}{d\mu} = \frac{1_{\mathcal{X} \setminus A_t}}{\mu(\mathcal{X} \setminus A_t)} \),
How to recover the Talagrand’s concentration inequality?

\[ \frac{1}{2} \tilde{T}_2(\nu_2|\nu_1) \leq \frac{1}{s} H(\nu_1|\mu^n) + \frac{1}{1-s} H(\nu_2|\mu^n), \quad \forall s \in (0, 1). \]

First method, the Marton’s argument: \( x \in \mathcal{X}^n, A \subset \mathcal{X}^n, \)

\[ c^n(x, A) := \inf_{p, p(A)=1} c^n(x, p), \quad \text{and} \quad A_t := \{x \in \mathcal{X}, c^n(x, A) \leq t\}. \]

Choose \( \frac{d\nu_1}{d\mu} = \frac{1_A}{\mu(A)} \) \quad \text{and} \quad \frac{d\nu_2}{d\mu} = \frac{1_{\mathcal{X}\setminus A_t}}{\mu(\mathcal{X}\setminus A_t)}, \quad \text{so that} \quad \tilde{T}_2(\nu_2|\nu_1) \geq t. \)
How to recover the Talagrand’s concentration inequality?

\[
\frac{1}{2} \tilde{T}_2(\nu_2|\nu_1) \leq \frac{1}{s} H(\nu_1|\mu^n) + \frac{1}{1-s} H(\nu_2|\mu^n), \quad \forall s \in (0, 1).
\]

First method, the Marton’s argument: \( x \in \mathcal{X}^n, A \subset \mathcal{X}^n \),

\[
c^n(x, A) := \inf_{p, p(A) = 1} c^n(x, p), \quad \text{and} \quad A_t := \{ x \in \mathcal{X}, c^n(x, A) \leq t \}.
\]

Choose

\[
\frac{d\nu_1}{d\mu} = \frac{1_A}{\mu(A)} \quad \text{and} \quad \frac{d\nu_2}{d\mu} = \frac{1_{\mathcal{X}\setminus A_t}}{\mu(\mathcal{X}\setminus A_t)}, \quad \text{so that} \quad \tilde{T}_2(\nu_2|\nu_1) \geq t.
\]

We get

\[
\frac{t}{2} \leq \frac{1}{s} \log \left( \frac{1}{\mu^n(A)} \right) + \frac{1}{1-s} \log \left( \frac{1}{\mu^n(\mathcal{X}\setminus A_t)} \right),
\]
How to recover the Talagrand’s concentration inequality?

\[
\frac{1}{2} \tilde{T}_2(\nu_2|\nu_1) \leq \frac{1}{s} H(\nu_1|\mu^n) + \frac{1}{1-s} H(\nu_2|\mu^n), \quad \forall s \in (0, 1).
\]

First method, the Marton’s argument: \( x \in \mathcal{X}^n, A \subset \mathcal{X}^n, \)

\[
c^n(x, A) := \inf_{\rho, p(A)=1} c^n(x, \rho), \quad \text{and} \quad A_t := \{ x \in \mathcal{X}, c^n(x, A) \leq t \}.
\]

Choose \( \frac{d\nu_1}{d\mu} = \frac{1_A}{\mu(A)} \) and \( \frac{d\nu_2}{d\mu} = \frac{1_{\mathcal{X}\setminus A_t}}{\mu(\mathcal{X}\setminus A_t)} \), so that \( \tilde{T}_2(\nu_2|\nu_1) \geq t \).

We get

\[
\frac{t}{2} \leq \frac{1}{s} \log \left( \frac{1}{\mu^n(A)} \right) + \frac{1}{1-s} \log \left( \frac{1}{\mu^n(\mathcal{X}\setminus A_t)} \right),
\]

or equivalently

\[
\mu^n(\mathcal{X}^n \setminus A_t)^{1/s} \mu^n(A)^{(1-s)} \leq e^{-t/2}, \quad \forall t \geq 0, s \in (0, 1),
\]
How to recover the Talagrand’s concentration inequality?

\[
\frac{1}{2} \mathcal{T}_2(\nu_2 | \nu_1) \leq \frac{1}{s} H(\nu_1 | \mu^n) + \frac{1}{1 - s} H(\nu_2 | \mu^n), \quad \forall s \in (0, 1).
\]

First method, the Marton’s argument: \( x \in \mathcal{X}^n, A \subset \mathcal{X}^n, \)

\[
c^n(x, A) := \inf_{\rho, \rho(A) = 1} c^n(x, \rho), \quad \text{and} \quad A_t := \{ x \in \mathcal{X}, c^n(x, A) \leq t \}.
\]

Choose \( \frac{d\nu_1}{d\mu} = \frac{1_A}{\mu(A)} \) and \( \frac{d\nu_2}{d\mu} = \frac{1_{\mathcal{X}\setminus A_t}}{\mu(\mathcal{X}\setminus A_t)} \), so that \( \mathcal{T}_2(\nu_2 | \nu_1) \geq t \).

We get

\[
\frac{t}{2} \leq \frac{1}{s} \log \left( \frac{1}{\mu^n(A)} \right) + \frac{1}{1 - s} \log \left( \frac{1}{\mu^n(\mathcal{X}\setminus A_t)} \right),
\]

or equivalently

\[
\mu^n(\mathcal{X}^n \setminus A_t)^{1/s} \mu^n(A)^{1/(1-s)} \leq e^{-t/2}, \quad \forall t \geq 0, s \in (0, 1),
\]

Links with Talagrand’s convex-hull distance:
How to recover the Talagrand’s concentration inequality?

\[ \frac{1}{2} \tilde{T}_2(\nu_2|\nu_1) \leq \frac{1}{s} H(\nu_1|\mu^n) + \frac{1}{1-s} H(\nu_2|\mu^n), \quad \forall s \in (0, 1). \]

First method, the Marton’s argument: \( x \in \mathcal{X}^n, A \subset \mathcal{X}^n \),

\[ c^n(x, A) := \inf_{p, p(A)=1} c^n(x, p), \quad \text{and} \quad A_t := \{ x \in \mathcal{X}, c^n(x, A) \leq t \}. \]

Choose \( \frac{d\nu_1}{d\mu} = \frac{1_A}{\mu(A)} \) and \( \frac{d\nu_2}{d\mu} = \frac{1_{\mathcal{X}\setminus A_t}}{\mu(\mathcal{X}\setminus A_t)} \), so that \( \tilde{T}_2(\nu_2|\nu_1) \geq t \).

We get

\[ \frac{t}{2} \leq \frac{1}{s} \log \left( \frac{1}{\mu^n(A)} \right) + \frac{1}{1-s} \log \left( \frac{1}{\mu^n(\mathcal{X}\setminus A_t)} \right), \]

or equivalently

\[ \mu^n(\mathcal{X}^n\setminus A_t)^{1/s} \mu^n(A)^{1/(1-s)} \leq e^{-t/2}, \quad \forall t \geq 0, s \in (0, 1), \]

Links with Talagrand’s convex-hull distance:

\[ D_{\text{Tal}}(x, A) = \sup_{\alpha \in B_1} \inf_{y \in A} \sum_{i=1}^n \alpha_i 1_{x_i \neq y_i} \]
How to recover the Talagrand’s concentration inequality？

\[
\frac{1}{2} \tilde{T}_2(\nu_2 | \nu_1) \leq \frac{1}{s} H(\nu_1 | \mu^n) + \frac{1}{1-s} H(\nu_2 | \mu^n), \quad \forall s \in (0, 1).
\]

First method, the Marton’s argument：

\[
c^n(x, A) := \inf_{p, p(A) = 1} c^n(x, p), \quad \text{and} \quad A_t := \{ x \in X, c^n(x, A) \leq t \}.
\]

Choose \( \frac{d\nu_1}{d\mu} = \frac{1_A}{\mu(A)} \) and \( \frac{d\nu_2}{d\mu} = \frac{1_{X \setminus A_t}}{\mu(X \setminus A_t)} \), so that \( \tilde{T}_2(\nu_2 | \nu_1) \geq t \).

We get

\[
\frac{t}{2} \leq \frac{1}{s} \log \left( \frac{1}{\mu^n(A)} \right) + \frac{1}{1-s} \log \left( \frac{1}{\mu^n(X \setminus A_t)} \right),
\]

or equivalently

\[
\mu^n(X^n \setminus A_t)^{1/s} \mu^n(A)^{1/(1-s)} \leq e^{-t/2}, \quad \forall t \geq 0, s \in (0, 1).
\]

Links with Talagrand’s convex-hull distance :

\[
D_{Tal}(x, A) = \sup_{\alpha \in B_1} \inf_{y \in A} \sum_{i=1}^n \alpha_i 1_{x_i \neq y_i} \quad B_1 : \text{the Euclidean ball in } \mathbb{R}^n.
\]
How to recover the Talagrand’s concentration inequality?

\[
\frac{1}{2} \tilde{T}_2(\nu_2|\nu_1) \leq \frac{1}{s} H(\nu_1|\mu^n) + \frac{1}{1-s} H(\nu_2|\mu^n), \quad \forall s \in (0, 1).
\]

First method, the Marton’s argument: \(x \in \mathcal{X}^n, A \subset \mathcal{X}^n,\)

\[
c^n(x, A) := \inf_{p, p(A) = 1} c^n(x, p), \quad \text{and} \quad A_t := \{x \in \mathcal{X}, c^n(x, A) \leq t\}.
\]

Choose \(\frac{d\nu_1}{d\mu} = \frac{1}{\mu(A)}\) and \(\frac{d\nu_2}{d\mu} = \frac{\1 \mathcal{X} \setminus A_t}{\mu(\mathcal{X} \setminus A_t)},\) so that \(\tilde{T}_2(\nu_2|\nu_1) \geq t.\)

We get

\[
\frac{t}{2} \leq \frac{1}{s} \log \left(\frac{1}{\mu^n(A)}\right) + \frac{1}{1-s} \log \left(\frac{1}{\mu^n(\mathcal{X} \setminus A_t)}\right),
\]

or equivalently

\[
\mu^n(\mathcal{X}^n \setminus A_t)^{1/s} \mu^n(A)^{(1-s)} \leq e^{-t/2}, \quad \forall t \geq 0, s \in (0, 1).
\]

Links with Talagrand’s convex-hull distance:

\[
D_{Tal}(x, A) = \sup_{A_0} \inf_{y \in A} \sum_{i=1}^{n} \alpha_i 1_{x_i \neq y_i}, \quad B_1 : \text{the Euclidean ball in } \mathbb{R}^n.
\]

\[
\sqrt{c^n(x, A)} =
\]
How to recover the Talagrand’s concentration inequality?

\[ \frac{1}{2} \tilde{T}_2(\nu_2|\nu_1) \leq \frac{1}{s} H(\nu_1|\mu^n) + \frac{1}{1-s} H(\nu_2|\mu^n), \quad \forall s \in (0, 1). \]

First method, the Marton’s argument: \( x \in \mathcal{X}^n, A \subset \mathcal{X}^n, \)

\[ c^n(x, A) := \inf_{p, p(A) = 1} c^n(x, p), \quad \text{and} \quad A_t := \{ x \in \mathcal{X}, c^n(x, A) \leq t \}. \]

Choose \( \frac{d\nu_1}{d\mu} = \frac{1_A}{\mu(A)} \) and \( \frac{d\nu_2}{d\mu} = \frac{1_{\mathcal{X}\setminus A_t}}{\mu(\mathcal{X}\setminus A_t)}, \) so that \( \tilde{T}_2(\nu_2|\nu_1) \geq t. \)

We get

\[ \frac{t}{2} \leq \frac{1}{s} \log \left( \frac{1}{\mu^n(A)} \right) + \frac{1}{1-s} \log \left( \frac{1}{\mu^n(\mathcal{X}\setminus A_t)} \right), \]

or equivalently

\[ \mu^n(\mathcal{X}^n\setminus A_t)^{1/s} \mu^n(A)^{1/(1-s)} \leq e^{-t/2}, \quad \forall t \geq 0, s \in (0, 1), \]

Links with Talagrand’s convex-hull distance:

\[ D_{\text{Tal}}(x, A) = \sup_{\alpha \in B_1} \inf_{y \in A} \sum_{i=1}^n \alpha_i 1_{x_i \neq y_i}, \quad B_1 : \text{the Euclidean ball in } \mathbb{R}^n. \]

\[ \sqrt{c^n(x, A)} = \inf_{p \in \mathcal{P}(A)} \sup_{\alpha \in B_1} \sum_{i=1}^n \alpha_i \int 1_{x_i \neq y_i} dp(y) \]
How to recover the Talagrand’s concentration inequality?

\[
\frac{1}{2} \tilde{T}_2(\nu_2|\nu_1) \leq \frac{1}{s} H(\nu_1|\mu^n) + \frac{1}{1 - s} H(\nu_2|\mu^n), \quad \forall s \in (0, 1).
\]

First method, the Marton’s argument: \( x \in \mathcal{X}^n, A \subset \mathcal{X}^n, \)
\[
c^n(x, A) := \inf_{p, p(A) = 1} c^n(x, p), \quad \text{and} \quad A_t := \{ x \in \mathcal{X}, c^n(x, A) \leq t \}.
\]

Choose \( \frac{d\nu_1}{d\mu} = \frac{1}{\mu(A)} \) and \( \frac{d\nu_2}{d\mu} = \frac{1}{\mu(\mathcal{X}\setminus A_t)} \), so that \( \tilde{T}_2(\nu_2|\nu_1) \geq t. \)

We get
\[
\frac{t}{2} \leq \frac{1}{s} \log \left( \frac{1}{\mu^n(A)} \right) + \frac{1}{1 - s} \log \left( \frac{1}{\mu^n(\mathcal{X}\setminus A_t)} \right),
\]
or equivalently
\[
\mu^n(\mathcal{X}^n\setminus A_t)^{1/s} \mu^n(A)^{1/(1-s)} \leq e^{-t/2}, \quad \forall t \geq 0, s \in (0, 1),
\]

Links with Talagrand’s convex-hull distance:

\[
D_{Tal}(x, A) = \sup_{\alpha \in B_1} \inf_{y \in A} \sum_{i=1}^n \alpha_i \mathbb{1}_{x_i \neq y_i} \quad B_1 : \text{the Euclidean ball in } \mathbb{R}^n.
\]

\[
\sqrt{c^n(x, A)} = \inf_{p \in \mathcal{P}(A)} \sup_{\alpha \in B_1} \sum_{i=1}^n \alpha_i \int 1_{x_i \neq y_i} d\rho(y) = \inf_{p \in \mathcal{P}(A)} \sup_{\alpha \in B_1} F(\alpha, p).
\]
**How to recover the Talagrand's concentration inequality?**

\[
\frac{1}{2} \tilde{T}_2(\nu_2|\nu_1) \leq \frac{1}{s} H(\nu_1|\mu^n) + \frac{1}{1-s} H(\nu_2|\mu^n), \quad \forall s \in (0, 1).
\]

**First method, the Marton’s argument:**

\[c^n(x, A) := \inf_{p, p(A) = 1} c^n(x, p), \quad \text{and} \quad A_t := \{x \in \mathcal{X}, c^n(x, A) \leq t\}.
\]

Choose \( \frac{d\nu_1}{d\mu} = \frac{1_A}{\mu(A)} \) and \( \frac{d\nu_2}{d\mu} = \frac{1_{\mathcal{X} \setminus A_t}}{\mu(\mathcal{X} \setminus A_t)} \), so that \( \tilde{T}_2(\nu_2|\nu_1) \geq t \).

We get

\[
\frac{t}{2} \leq \frac{1}{s} \log \left( \frac{1}{\mu^n(A)} \right) + \frac{1}{1-s} \log \left( \frac{1}{\mu^n(\mathcal{X} \setminus A_t)} \right),
\]

or equivalently

\[
\mu^n(\mathcal{X} \setminus A_t)^{1/s} \mu^n(A)^{1/(1-s)} \leq e^{-t/2}, \quad \forall t \geq 0, s \in (0, 1),
\]

**Links with Talagrand’s convex-hull distance:**

\[
D_{Tal}(x, A) = \sup_{\alpha \in B_1} \inf_{y \in A} \sum_{i=1}^n \alpha_i 1_{x_i \neq y_i} \quad B_1 : \text{the Euclidean ball in } \mathbb{R}^n.
\]

\[
\sqrt{c^n(x, A)} = \inf_{p \in \mathcal{P}(A)} \sup_{\alpha \in B_1} \sum_{i=1}^n \alpha_i \int |x_i \neq y_i| d\mu(y) = \inf_{p \in \mathcal{P}(A)} \sup_{\alpha \in B_1} F(\alpha, p).
\]

The function \( F \) is convex in \( p \).
How to recover the Talagrand’s concentration inequality?

\[ \frac{1}{2} \mathcal{T}_2(\nu_2|\nu_1) \leq \frac{1}{s} H(\nu_1|\mu^n) + \frac{1}{1-s} H(\nu_2|\mu^n), \quad \forall s \in (0, 1). \]

First method, the Marton’s argument: \( x \in \mathcal{X}^n, A \subset \mathcal{X}^n, \)

\[ c^n(x, A) := \inf_{p, p(A) = 1} c^n(x, p), \quad \text{and} \quad A_t := \{ x \in \mathcal{X}, c^n(x, A) \leq t \}. \]

Choose \( \frac{d\nu_1}{d\mu} = \frac{1_A}{\mu(A)} \) and \( \frac{d\nu_2}{d\mu} = \frac{1_{\mathcal{X}\setminus A_t}}{\mu(\mathcal{X}\setminus A_t)}, \) so that \( \mathcal{T}_2(\nu_2|\nu_1) \geq t. \)

We get

\[ \frac{t}{2} \leq \frac{1}{s} \log \left( \frac{1}{\mu^n(A)} \right) + \frac{1}{1-s} \log \left( \frac{1}{\mu^n(\mathcal{X}\setminus A_t)} \right), \]

or equivalently

\[ \mu^n(\mathcal{X}^n\setminus A_t)^{1/s} \mu^n(A)^{1/(1-s)} \leq e^{-t/2}, \quad \forall t \geq 0, s \in (0, 1), \]

Links with Talagrand’s convex-hull distance:

\[ D_{Tal}(x, A) = \sup_{\alpha \in B_1} \inf_{\sum_{i=1}^n \alpha_i 1_{x_i \neq y_i}} \sum_{i=1}^n \alpha_i 1_{x_i \neq y_i} \quad B_1 : \text{the Euclidean ball in } \mathbb{R}^n. \]

\[ \sqrt{c^n(x, A)} = \inf_{p \in \mathcal{P}(A)} \sup_{\sum_{i=1}^n \alpha_i \int 1_{x_i \neq y_i} d\rho(y)} \sum_{i=1}^n \alpha_i \int 1_{x_i \neq y_i} d\rho(y) = \inf_{p \in \mathcal{P}(A)} \sup_{\alpha \in B_1} F(\alpha, p). \]

The function \( F \) is convex in \( p \) and concave in \( \alpha \).
How to recover the Talagrand’s concentration inequality?

\[ \frac{1}{2} \tilde{T}_2(\nu_2|\nu_1) \leq \frac{1}{s} H(\nu_1|\mu_n) + \frac{1}{1-s} H(\nu_2|\mu_n), \quad \forall s \in (0, 1). \]

First method, the Marton’s argument: \( x \in \mathcal{X}^n, A \subset \mathcal{X}^n, \)

\[ c^n(x, A) := \inf_{p, p(A)=1} c^n(x, p), \quad \text{and} \quad A_t := \{ x \in \mathcal{X}, c^n(x, A) \leq t \}. \]

Choose \( \frac{d\nu_1}{d\mu} = \frac{1}{\mu(A)} \) and \( \frac{d\nu_2}{d\mu} = \frac{1}{\mu(\mathcal{X}\setminus A_t)} \), so that \( \tilde{T}_2(\nu_2|\nu_1) \geq t. \)

We get

\[ \frac{t}{2} \leq \frac{1}{s} \log \left( \frac{1}{\mu^n(A)} \right) + \frac{1}{1-s} \log \left( \frac{1}{\mu^n(\mathcal{X}\setminus A_t)} \right), \]

or equivalently

\[ \mu^n(\mathcal{X}^n\setminus A_t)^{1/s} \mu^n(A)^{1/(1-s)} \leq e^{-t/2}, \quad \forall t \geq 0, s \in (0, 1). \]

Links with Talagrand’s convex-hull distance:

\[ D_{\text{Tal}}(x, A) = \sup_{A_1} \inf_{\sum \alpha_i 1_{x_i \neq y_i}} \sum_{i=1}^n \alpha_i 1_{x_i \neq y_i} B_1 : \text{the Euclidean ball in } \mathbb{R}^n. \]

\[ \sqrt{c^n(x, A)} = \inf_{\sum \alpha \in B_1} \sup_{\sum \alpha} \sum_{i=1}^n \alpha_i \int 1_{x_i \neq y_i} d\rho(y) = \inf_{\sum \alpha \in B_1} \sup_{\sum \alpha} F(\alpha, p). \]

The function \( F \) is convex in \( p \) and concave in \( \alpha, B_1 \) is convex,
How to recover the Talagrand’s concentration inequality?

\[
\frac{1}{2} \widetilde{T}_2(\nu_2|\nu_1) \leq \frac{1}{s} H(\nu_1|\mu^n) + \frac{1}{1-s} H(\nu_2|\mu^n), \quad \forall s \in (0, 1).
\]

First method, the Marton’s argument:

\[ c^n(x, A) := \inf_{p, p(A) = 1} c^n(x, p), \quad \text{and} \quad A_t := \{x \in X, c^n(x, A) \leq t\}. \]

Choose \( \frac{d\nu_1}{d\mu} = \frac{1}{\mu(A)} \) and \( \frac{d\nu_2}{d\mu} = \frac{1}{\mu(X \setminus A_t)} \), so that \( \widetilde{T}_2(\nu_2|\nu_1) \geq t \).

We get

\[
\frac{t}{2} \leq \frac{1}{s} \log \left( \frac{1}{\mu^n(A)} \right) + \frac{1}{1-s} \log \left( \frac{1}{\mu^n(X \setminus A_t)} \right),
\]

or equivalently

\[
\mu^n(X \setminus A_t)^{1/s} \mu^n(A)^{1/(1-s)} \leq e^{-t/2}, \quad \forall t \geq 0, s \in (0, 1),
\]

Links with Talagrand’s convex-hull distance:

\[
D_{\text{Tal}}(x, A) = \sup_{\alpha \in B_1} \inf_{y \in A} \sum_{i=1}^{n} \alpha_i 1_{x_i \neq y_i} \quad B_1 : \text{the Euclidean ball in } \mathbb{R}^n.
\]

\[
\sqrt{c^n(x, A)} = \inf_{p \in \mathcal{P}(A)} \sup_{x \in B_1} \sum_{i=1}^{n} \alpha_i \int 1_{x_i \neq y_i} d\mu(y) = \inf_{p \in \mathcal{P}(A)} \sup_{x \in B_1} F(\alpha, p).
\]

The function \( F \) is convex in \( p \) and concave in \( \alpha \), \( B_1 \) is convex, \( \mathcal{P}(A) \) is compact convex,
How to recover the Talagrand’s concentration inequality?

\[ \frac{1}{2} \tilde{T}_2(\nu_2|\nu_1) \leq \frac{1}{s} H(\nu_1|\mu^n) + \frac{1}{1-s} H(\nu_2|\mu^n), \quad \forall s \in (0, 1). \]

First method, the Marton’s argument: \( x \in \mathcal{X}^n, A \subset \mathcal{X}^n, \)

\[ c^n(x, A) := \inf_{p, p(A) = 1} c^n(x, p), \quad \text{and} \quad A_t := \{ x \in \mathcal{X}, c^n(x, A) \leq t \}. \]

Choose \( \frac{d\nu_1}{d\mu} = \frac{1_A}{\mu(A)} \) and \( \frac{d\nu_2}{d\mu} = \frac{1_{\mathcal{X} \setminus A_t}}{\mu(\mathcal{X} \setminus A_t)} \), so that \( \tilde{T}_2(\nu_2|\nu_1) \geq t \).

We get

\[ \frac{t}{2} \leq \frac{1}{s} \log \left( \frac{1}{\mu^n(A)} \right) + \frac{1}{1-s} \log \left( \frac{1}{\mu^n(\mathcal{X} \setminus A_t)} \right), \]

or equivalently

\[ \mu^n(\mathcal{X}^n \setminus A_t)^{1/s} \mu^n(A)^{(1-s)} \leq e^{-t/2}, \quad \forall t \geq 0, s \in (0, 1), \]

Links with Talagrand’s convex-hull distance:

\[ D_{\text{Tal}}(x, A) = \sup_{\alpha \in B_1} \inf_{\sum_{\alpha} \alpha_i 1_{x_i \neq y_i} \leq 1} \sum_{i=1}^n \alpha_i 1_{x_i \neq y_i} B_1 : \text{the Euclidean ball in } \mathbb{R}^n. \]

\[ \sqrt{c^n(x, A)} = \inf_{p \in \mathcal{P}(A)} \sup_{\alpha \in B_1} \sum_{i=1}^n \alpha_i \int 1_{x_i \neq y_i} dp(y) = \inf_{p \in \mathcal{P}(A)} \sup_{\alpha \in B_1} F(\alpha, p). \]

The function \( F \) is convex in \( p \) and concave in \( \alpha \), \( B_1 \) is convex, \( \mathcal{P}(A) \) is compact convex, by the Minimax Theorem,
How to recover the Talagrand’s concentration inequality?

\[ \frac{1}{2} \widetilde{T}_2(\nu_2|\nu_1) \leq \frac{1}{s} H(\nu_1|\mu^n) + \frac{1}{1-s} H(\nu_2|\mu^n), \forall s \in (0, 1). \]

First method, the Marton’s argument: \( x \in \mathcal{X}^n, A \subset \mathcal{X}^n, \)
\[ c^n(x, A) := \inf_{p, p(A) = 1} c^n(x, p), \text{ and } A_t := \{ x \in \mathcal{X}, c^n(x, A) \leq t \}. \]

Choose \( \frac{d\nu_1}{d\mu} = \frac{1_A}{\mu(A)} \) and \( \frac{d\nu_2}{d\mu} = \frac{1_{\mathcal{X} \setminus A_t}}{\mu(\mathcal{X} \setminus A_t)} \), so that \( \widetilde{T}_2(\nu_2|\nu_1) \geq t \).

We get
\[ \frac{t}{2} \leq \frac{1}{s} \log \left( \frac{1}{\mu^n(A)} \right) + \frac{1}{1-s} \log \left( \frac{1}{\mu^n(\mathcal{X} \setminus A_t)} \right), \]

or equivalently
\[ \mu^n(\mathcal{X}^n \setminus A_t)^{1/s} \mu^n(A)^{1/(1-s)} \leq e^{-t/2}, \forall t \geq 0, s \in (0, 1), \]

Links with Talagrand’s convex-hull distance:
\[ D_{\text{Tal}}(x, A) = \sup_{\alpha \in B_1} \inf_{y \in A} \sum_{i=1}^n \alpha_i 1_{x_i \neq y_i} \quad B_1 : \text{the Euclidean ball in } \mathbb{R}^n. \]

\[ \sqrt{c^n(x, A)} = \inf_{p \in \mathcal{P}(A)} \sup_{x \in B_1} \sum_{i=1}^n \alpha_i \int 1_{x_i \neq y_i} dp(y) = \inf_{p \in \mathcal{P}(A)} \sup_{\alpha \in B_1} F(\alpha, p). \]

The function \( F \) is convex in \( p \) and concave in \( \alpha \), \( B_1 \) is convex, \( \mathcal{P}(A) \) is compact convex, by the Minimax Theorem,
\[ \sqrt{c^n(x, A)} = \sup_{\alpha \in B_1} \inf_{p \in \mathcal{P}(A)} F(\alpha, p) \]
How to recover the Talagrand’s concentration inequality?

$$\frac{1}{2} \tilde{T}_2(\nu_2|\nu_1) \leq \frac{1}{s} H(\nu_1|\mu^n) + \frac{1}{1-s} H(\nu_2|\mu^n), \quad \forall s \in (0, 1).$$

First method, the Marton’s argument: \( x \in \mathcal{X}^n, A \subset \mathcal{X}^n, \)

\[ c^n(x, A) := \inf_{p, p(A) = 1} c^n(x, p), \quad \text{and} \quad \mathcal{A}_t := \{x \in \mathcal{X}, c^n(x, A) \leq t\}. \]

Choose \( \frac{d\nu_1}{d\mu} = \frac{1_A}{\mu(A)} \) and \( \frac{d\nu_2}{d\mu} = \frac{1_{\mathcal{X} \setminus \mathcal{A}_t}}{\mu(\mathcal{X} \setminus \mathcal{A}_t)}, \) so that \( \tilde{T}_2(\nu_2|\nu_1) \geq t. \)

We get

\[ \frac{t}{2} \leq \frac{1}{s} \log \left( \frac{1}{\mu^n(A)} \right) + \frac{1}{1-s} \log \left( \frac{1}{\mu^n(\mathcal{X} \setminus \mathcal{A}_t)} \right), \]

or equivalently

\[ \mu^n(\mathcal{X}^n \setminus \mathcal{A}_t)^{1/s} \mu^n(\mathcal{A})^{1/(1-s)} \leq e^{-t/2}, \quad \forall t \geq 0, s \in (0, 1), \]

Links with Talagrand’s convex-hull distance:

\[ D_{Tal}(x, A) = \sup_{\alpha \in B_1} \inf_{y \in A} \sum_{i=1}^n \alpha_i 1_{x_i \neq y_i}, \quad B_1 : \text{the Euclidean ball in } \mathbb{R}^n. \]

\[ \sqrt{c^n(x, A)} = \inf_{p \in \mathcal{P}(A)} \sup_{\alpha \in B_1} \sum_{i=1}^n \alpha_i \int 1_{x_i \neq y_i} dp(y) = \inf_{p \in \mathcal{P}(A)} \sup_{\alpha \in B_1} F(\alpha, p). \]

The function \( F \) is convex in \( p \) and concave in \( \alpha \), \( B_1 \) is convex, \( \mathcal{P}(A) \) is compact convex, by the Minimax Theorem,

\[ \sqrt{c^n(x, A)} = \sup_{\alpha \in B_1} \inf_{p \in \mathcal{P}(A)} F(\alpha, p) \]

\( (F \) is linear in \( p \), the infimum is reached at the extremal points of \( \mathcal{P}(A) \).)
How to recover the Talagrand’s concentration inequality?

\[ \frac{1}{2} \tilde{T}_2(\nu_2|\nu_1) \leq \frac{1}{s} H(\nu_1|\mu^n) + \frac{1}{1-s} H(\nu_2|\mu^n), \quad \forall s \in (0, 1). \]

First method, the Marton’s argument: \( x \in \mathcal{X}^n, A \subset \mathcal{X}^n, \)

\[ c^n(x, A) := \inf_{p, p(A) = 1} c^n(x, p), \quad \text{and} \quad A_t := \{ x \in \mathcal{X}, c^n(x, A) \leq t \}. \]

Choose \( \frac{d\nu_1}{d\mu} = \frac{1 \mathbb{I}_A}{\mu(A)} \) and \( \frac{d\nu_2}{d\mu} = \frac{\mathbb{1}_{\mathcal{X} \setminus A_t}}{\mu(\mathcal{X} \setminus A_t)} \), so that \( \tilde{T}_2(\nu_2|\nu_1) \geq t. \)

We get

\[ \frac{t}{2} \leq \frac{1}{s} \log \left( \frac{1}{\mu^n(A)} \right) + \frac{1}{1-s} \log \left( \frac{1}{\mu^n(\mathcal{X}\setminus A_t)} \right), \]

or equivalently

\[ \mu^n(\mathcal{X}^n \setminus A_t)^{1/s} \mu^n(A)^{1/(1-s)} \leq e^{-t/2}, \quad \forall t \geq 0, s \in (0, 1), \]

Links with Talagrand’s convex-hull distance:

\[ D_{\text{Tal}}(x, A) = \sup_{\alpha \in B_1} \inf_{y \in A} \sum_{i=1}^{n} \alpha_i 1_{x_i \neq y_i} \quad B_1 : \text{the Euclidean ball in } \mathbb{R}^n. \]

\[ \sqrt{c^n(x, A)} = \inf_{p \in \mathcal{P}(A)} \sup_{\alpha \in B_1} \sum_{i=1}^{n} \alpha_i \int_{\mathcal{X}} 1_{x_i \neq y_i} dp(y) = \inf_{p \in \mathcal{P}(A)} \sup_{\alpha \in B_1} F(\alpha, p). \]

The function \( F \) is convex in \( p \) and concave in \( \alpha \), \( B_1 \) is convex, \( \mathcal{P}(A) \) is compact convex, by the Minimax Theorem,

\[ \sqrt{c^n(x, A)} = \sup_{\alpha \in B_1} \inf_{p \in \mathcal{P}(A)} F(\alpha, p) = \sup_{\alpha \in B_1} \inf_{\delta y, y \in A} F(\alpha, \delta y). \]
How to recover the Talagrand’s concentration inequality?

\[
\frac{1}{2} \widetilde{T}_2(\nu_2|\nu_1) \leq \frac{1}{s} H(\nu_1|\mu^n) + \frac{1}{1 - s} H(\nu_2|\mu^n), \quad \forall s \in (0, 1).
\]

First method, the Marton’s argument: \( x \in \mathcal{X}^n, A \subset \mathcal{X}^n, \)

\[
c^n(x, A) := \inf_{p, p(A) = 1} c^n(x, p), \quad \text{and} \quad A_t := \{ x \in \mathcal{X}, c^n(x, A) \leq t \}.
\]

Choose \( \frac{d\nu_1}{d\mu} = \frac{1_A}{\mu(A)} \) and \( \frac{d\nu_2}{d\mu} = \frac{1_{\mathcal{X}\setminus A_t}}{\mu(\mathcal{X}\setminus A_t)} \), so that \( \widetilde{T}_2(\nu_2|\nu_1) \geq t \).

We get

\[
\frac{t}{2} \leq \frac{1}{s} \log \left( \frac{1}{\mu^n(A)} \right) + \frac{1}{1 - s} \log \left( \frac{1}{\mu^n(\mathcal{X}\setminus A_t)} \right),
\]

or equivalently

\[
\mu^n(\mathcal{X}^n\setminus A_t)^{1/s} \mu^n(A)^{1/(1-s)} \leq e^{-t/2}, \quad \forall t \geq 0, s \in (0, 1),
\]

Links with Talagrand’s convex-hull distance:

\[
D_{\text{Tal}}(x, A) = \sup_{\alpha \in B_1} \inf_{y \in A} \sum_{i=1}^{n} \alpha_i 1_{x_i \neq y_i}, \quad B_1 : \text{the Euclidean ball in } \mathbb{R}^n.
\]

\[
\sqrt{c^n(x, A)} = \inf_{p \in \mathcal{P}(A)} \sup_{\alpha \in B_1} \sum_{i=1}^{n} \alpha_i \int 1_{x_i \neq y_i} dp(y) = \inf_{p \in \mathcal{P}(A)} \sup_{\alpha \in B_1} F(\alpha, p).
\]

The function \( F \) is convex in \( p \) and concave in \( \alpha \), \( B_1 \) is convex, \( \mathcal{P}(A) \) is compact convex, by the Minimax Theorem,

\[
\sqrt{c^n(x, A)} = \sup_{\alpha \in B_1} \inf_{p \in \mathcal{P}(A)} F(\alpha, p) = \sup_{\alpha \in B_1} \inf_{\delta, y \in A} F(\alpha, \delta, y).
\]
How to recover the Talagrand’s concentration inequality?

\[
\frac{1}{2} \tilde{T}_2(\nu_2|\nu_1) \leq \frac{1}{s} H(\nu_1|\mu^n) + \frac{1}{1-s} H(\nu_2|\mu^n), \quad \forall s \in (0, 1).
\]

First method, the Marton’s argument: \( x \in X^n, A \subset X^n, \)

\[
c^n(x, A) := \inf_{p, p(A) = 1} c^n(x, p), \quad \text{and} \quad A_t := \{ x \in X, c^n(x, A) \leq t \}.
\]

Choose \( \frac{d\nu_1}{d\mu} = \frac{1_A}{\mu(A)} \) and \( \frac{d\nu_2}{d\mu} = \frac{1_{X\setminus A_t}}{\mu(X\setminus A_t)}, \) so that \( \tilde{T}_2(\nu_2|\nu_1) \geq t. \)

We get

\[
\frac{t}{2} \leq \frac{1}{s} \log \left( \frac{1}{\mu^n(A)} \right) + \frac{1}{1-s} \log \left( \frac{1}{\mu^n(X\setminus A_t)} \right),
\]

or equivalently

\[
\mu^n(X^n \setminus A_t)^{1/s} \mu^n(A)^{1/(1-s)} \leq e^{-t/2}, \quad \forall t \geq 0, s \in (0, 1),
\]

Links with Talagrand’s convex-hull distance:

\[
D_{Tal}(x, A) = \sup_{\alpha \in B_1} \inf_{y \in A} \sum_{i=1}^n \alpha_i 1_{x_i \neq y_i} \quad B_1 : \text{the Euclidean ball in } \mathbb{R}^n.
\]

\[
\sqrt{c^n(x, A)} = \inf_{p \in P(A)} \sup_{\alpha \in B_1} \sum_{i=1}^n \alpha_i \int 1_{x_i \neq y_i} dp(y) = \inf_{p \in P(A)} \sup_{\alpha \in B_1} F(\alpha, p).
\]

The function \( F \) is convex in \( p \) and concave in \( \alpha \), \( B_1 \) is convex, \( P(A) \) is compact convex, by the Minimax Theorem,

\[
\sqrt{c^n(x, A)} = \sup_{\alpha \in B_1} \inf_{p \in P(A)} F(\alpha, p) = D_{Tal}(x, A),
\]
How to recover the Talagrand’s concentration inequality?

\[
\frac{1}{2} \tilde{T}_2(\nu_2|\nu_1) \leq \frac{1}{s} H(\nu_1|\mu^n) + \frac{1}{1-s} H(\nu_2|\mu^n), \quad \forall s \in (0, 1).
\]

First method, the Marton’s argument: \( x \in \mathcal{X}^n, A \subset \mathcal{X}^n, \)

\[
c^n(x, A) := \inf_{p, p(A) = 1} c^n(x, p), \quad \text{and} \quad A_t := \{x \in \mathcal{X}, c^n(x, A) \leq t\}.
\]

Choose \( \frac{d\nu_1}{d\mu} = \frac{1_A}{\mu(A)} \) and \( \frac{d\nu_2}{d\mu} = \frac{1_{\mathcal{X}\setminus A_t}}{\mu(\mathcal{X}\setminus A_t)} \), so that \( \tilde{T}_2(\nu_2|\nu_1) \geq t \).

We get

\[
\frac{t}{2} \leq \frac{1}{s} \log \left( \frac{1}{\mu^n(A)} \right) + \frac{1}{1-s} \log \left( \frac{1}{\mu^n(\mathcal{X}\setminus A_t)} \right),
\]

or equivalently

\[
\mu^n(\mathcal{X}^n\setminus A_t)^{1/s} \mu^n(A)^{1/(1-s)} \leq e^{-t/2}, \quad \forall t \geq 0, s \in (0, 1),
\]

Links with Talagrand’s convex-hull distance:

\[
D_{\text{Tal}}(x, A) = \sup_{\alpha \in B_1} \inf_{y \in A} \sum_{i=1}^n \alpha_i 1_{x_i \neq y_i} \quad B_1 : \text{the Euclidean ball in } \mathbb{R}^n.
\]

\[
\sqrt{c^n(x, A)} = \inf_{p \in \mathcal{P}(A)} \sup_{\alpha \in B_1} \sum_{i=1}^n \alpha_i \int 1_{x_i \neq y_i} dp(y) = \inf_{p \in \mathcal{P}(A)} \sup_{\alpha \in B_1} F(\alpha, p).
\]

The function \( F \) is convex in \( p \) and concave in \( \alpha \), \( B_1 \) is convex, \( \mathcal{P}(A) \) is compact convex, by the Minimax Theorem,

\[
\sqrt{c^n(x, A)} = \sup_{\alpha \in B_1} \inf_{p \in \mathcal{P}(A)} F(\alpha, p) = D_{\text{Tal}}(x, A), \quad A_t = A_{\text{Tal}t}.
\]
How to recover the Talagrand’s concentration inequality?
Second method, duality arguments:

Introduction
Marton’s inequality
Talagrand’s concentration
Kantorovich duality
for classical costs
for weak costs
Examples of weak cost
Marton’s type of cost
Barycentric cost
Strassen’s result
Martingale costs
Weak transport inequalities
Dual characterization to concentration
Universal transport inequalities
Barycentric transport inequalities
Examples
characterisation on \( \mathbb{R} \)
Transport inequality on the symmetric group
Ewens distribution
deviation inequalities
The Schrödinger minimization problem
definition
curvature in discrete spaces
functional inequalities
Examples in discrete
Weak transport costs
How to recover the Talagrand’s concentration inequality?
Second method, duality arguments: based on a generalized Kantorovich duality theorem for weak transport costs.
How to recover the Talagrand’s concentration inequality?

Second method, duality arguments: based on a generalized Kantorovich duality theorem for weak transport costs.

The classical Kantorovich dual theorem

How to recover the Talagrand’s concentration inequality?

Second method, duality arguments: based on a generalized Kantorovich duality theorem for weak transport costs.

The classical Kantorovich dual theorem

If $\omega: \hat{X} \rightarrow \mathbb{R}_+$ is lower semi-continuous, then

$$
\inf_{\pi} \mathbb{E}_{\pi} \omega_x, y = \sup_{\phi, \psi} \mathbb{E}_\mu \phi \quad \text{s.t.} \quad \mathbb{E}_\nu \psi \geq \omega_x, y.
$$

Given $\phi$, we may replace $\psi$ by the optimal function

$$
Q_{\omega} \phi = \inf_{y} \phi_y \quad \text{s.t.} \quad \mathbb{E}_{\pi} \omega_x, y.
$$

This yields

$$
\inf_{\pi} \mathbb{E}_{\pi} \omega_x, y = \sup_{\phi} \mathbb{E}_\mu \phi \quad \text{s.t.} \quad \mathbb{E}_\nu \psi \geq \omega_x, y.
$$

Usual example: the Wasserstein metric $W_{\mu, \nu}$

$$
W_{\mu, \nu} = \inf_{\pi} \mathbb{E}_{\pi} d_x, y = \inf_{\pi} \mathbb{E}_{\pi} \omega_x, y.
$$
How to recover the Talagrand’s concentration inequality?

Second method, duality arguments: based on a generalized Kantorovich duality theorem for weak transport costs.

**The classical Kantorovich dual theorem**

If $\omega : \mathcal{X} \times \mathcal{X} \to [0, + \infty]$ is lower semi-continuous,
How to recover the Talagrand’s concentration inequality?

Second method, duality arguments: based on a generalized Kantorovich duality theorem for weak transport costs.

The classical Kantorovich dual theorem

If \( \omega : \mathcal{X} \times \mathcal{X} \to [0, +\infty] \) is lower semi-continuous, then

\[
T_\omega(\mu, \nu) := \inf_{\pi \in \Pi(\mu, \nu)} \int \int \omega(x, y) d\pi(x, y)
\]

video iconic image reference
How to recover the Talagrand’s concentration inequality?

Second method, duality arguments: based on a generalized Kantorovich duality theorem for weak transport costs.

The classical Kantorovich dual theorem

If \( \omega : X \times X \to [0. + \infty] \) is lower semi-continuous, then

\[
T_\omega (\mu, \nu) := \inf_{\pi \in \Pi (\mu, \nu)} \int \int \omega (x, y) d\pi (x, y)
\]

\[
= \sup_{(\varphi, \psi)} \left\{ \int \psi \ d\mu - \int \varphi \ d\nu \right\},
\]

where \( \Pi (\mu, \nu) \) is the set of all measures \( \pi \) on \( X^2 \) such that \( \pi (A) = \mu (A) \) and \( \pi (B) = \nu (B) \) for each measurable set \( A \subset X \) and \( B \subset Y \).

Usual example: the Wasserstein metric

\[
T_\omega (\mu, \nu) = W_1 (\mu, \nu) := \inf_{\pi \in \Pi (\mu, \nu)} \int d\pi (x, y)
\]

where the infimum runs over all measures \( \pi \) such that \( \pi (A \times Y) = \mu (A) \) and \( \pi (X \times B) = \nu (B) \) for each measurable set \( A \subset X \) and \( B \subset Y \).
How to recover the Talagrand’s concentration inequality?

Second method, duality arguments: based on a generalized Kantorovich duality theorem for weak transport costs.

The classical Kantorovich dual theorem

If $\omega : \mathcal{X} \times \mathcal{X} \rightarrow [0. + \infty]$ is lower semi-continuous, then

$$T_\omega (\mu, \nu) := \inf_{\pi \in \Pi(\mu, \nu)} \iint \omega(x, y) d\pi(x, y)$$

$$= \sup_{(\varphi, \psi)} \left\{ \int \psi d\mu - \int \varphi d\nu \right\},$$

where the supremum runs over all bounded continuous functions $\psi, \varphi$ on $\mathcal{X}$ such that

$$\psi(x) - \varphi(y) \leq \omega(x, y), \quad \forall x, y \in \mathcal{X}.$$
How to recover the Talagrand’s concentration inequality?
Second method, duality arguments: based on a generalized Kantorovich duality theorem for weak transport costs.

**The classical Kantorovich dual theorem**

If \( \omega : \mathcal{X} \times \mathcal{X} \to [0, +\infty] \) is lower semi-continuous, then

\[
\mathcal{T}_\omega(\mu, \nu) := \inf_{\pi \in \Pi(\mu, \nu)} \int \int \omega(x, y) d\pi(x, y)
\]

\[
= \sup_{(\varphi, \psi)} \left\{ \int \psi \, d\mu - \int \varphi \, d\nu \right\},
\]

where the supremum runs over all bounded continuous functions \( \psi, \varphi \) on \( \mathcal{X} \) such that

\[
\psi(x) - \varphi(y) \leq \omega(x, y), \quad \forall x, y \in \mathcal{X}.
\]

Given \( \varphi \),

\[
\text{Ewens distribution}
\]

\[
\text{Schrödinger minimization problem}
\]
How to recover the Talagrand’s concentration inequality?

Second method, duality arguments: based on a generalized Kantorovich duality theorem for weak transport costs.

The classical Kantorovich dual theorem

If $\omega : \mathcal{X} \times \mathcal{X} \to [0, + \infty]$ is lower semi-continuous, then

$$
 T_\omega(\mu, \nu) := \inf_{\pi \in \Pi(\mu, \nu)} \int \int \omega(x, y) d\pi(x, y)
$$

$$
 = \sup_{(\varphi, \psi)} \left\{ \int \psi d\mu - \int \varphi d\nu \right\},
$$

where the supremum runs over all bounded continuous functions $\psi, \varphi$ on $\mathcal{X}$ such that

$$
 \psi(x) - \varphi(y) \leq \omega(x, y), \quad \forall x, y \in \mathcal{X}.
$$

Given $\varphi$, we may replace $\psi$ by the optimal function

$$
 Q_\omega \varphi(x) = \inf_{y \in \mathcal{X}} \{ \varphi(y) + \omega(x, y) \}. 
$$

How to recover the Talagrand’s concentration inequality?
Second method, duality arguments: based on a generalized Kantorovich duality theorem for weak transport costs.

The classical Kantorovich dual theorem

If $\omega : \mathcal{X} \times \mathcal{X} \to [0, + \infty]$ is lower semi-continuous, then

$$
 T_\omega(\mu, \nu) := \inf_{\pi \in \Pi(\mu, \nu)} \int \int \omega(x, y) d\pi(x, y)
$$

$$
 = \sup_{(\varphi, \psi)} \left\{ \int \psi d\mu - \int \varphi d\nu \right\},
$$

where the supremum runs over all bounded continuous functions $\psi, \varphi$ on $\mathcal{X}$ such that

$$
 \psi(x) - \varphi(y) \leq \omega(x, y), \quad \forall x, y \in \mathcal{X}.
$$

Given $\varphi$, we may replace $\psi$ by the optimal function

$$
 Q_\omega \varphi(x) = \inf_{y \in \mathcal{X}} \{ \varphi(y) + \omega(x, y) \}. 
$$
How to recover the Talagrand’s concentration inequality?

Second method, duality arguments: based on a generalized Kantorovich duality theorem for weak transport costs.

The classical Kantorovich dual theorem

If $\omega : \mathcal{X} \times \mathcal{X} \to [0, + \infty]$ is lower semi-continuous, then

$$T_\omega (\mu, \nu) := \inf_{\pi \in \Pi (\mu, \nu)} \int \int \omega (x, y) d\pi (x, y)$$

$$= \sup_{(\varphi, \psi)} \left\{ \int \psi \, d\mu - \int \varphi \, d\nu \right\},$$

where the supremum runs over all bounded continuous functions $\psi, \varphi$ on $\mathcal{X}$ such that

$$\psi (x) - \varphi (y) \leq \omega (x, y), \quad \forall x, y \in \mathcal{X}.$$

Given $\varphi$, we may replace $\psi$ by the optimal function

$$Q_\omega \varphi (x) = \inf_{y \in \mathcal{X}} \{ \varphi (y) + \omega (x, y) \}.$$

This yields

$$T_\omega (\mu, \nu) = \sup_\varphi \left\{ \int Q_\omega \varphi \, d\mu - \int \varphi \, d\nu \right\},$$
How to recover the Talagrand’s concentration inequality?

Second method, duality arguments: based on a generalized Kantorovich duality theorem for weak transport costs.

The classical Kantorovich dual theorem

If $\omega : \mathcal{X} \times \mathcal{X} \rightarrow [0. + \infty]$ is lower semi-continuous, then

$$
T_\omega (\mu, \nu) := \inf_{\pi \in \Pi(\mu, \nu)} \int \int \omega(x, y) d\pi(x, y)
$$

$$
= \sup_{(\varphi, \psi)} \left\{ \int \psi d\mu - \int \varphi d\nu \right\} ,
$$

where the supremum runs over all bounded continuous functions $\psi, \varphi$ on $\mathcal{X}$ such that

$$
\psi(x) - \varphi(y) \leq \omega(x, y), \quad \forall x, y \in \mathcal{X}.
$$

Given $\varphi$, we may replace $\psi$ by the optimal function

$$
Q_\omega \varphi(x) = \inf_{y \in \mathcal{X}} \{ \varphi(y) + \omega(x, y) \}.
$$

This yields

$$
T_\omega (\mu, \nu) = \sup_{\varphi} \left\{ \int Q_\omega \varphi d\mu - \int \varphi d\nu \right\} ,
$$

where the supremum runs over all bounded continuous functions $\varphi$ on $\mathcal{X}$.
How to recover the Talagrand’s concentration inequality?

**Second method, duality arguments:** based on a generalized Kantorovich duality theorem for weak transport costs.

### The classical Kantorovich dual theorem

If \( \omega : \mathcal{X} \times \mathcal{X} \to [0. + \infty] \) is lower semi-continuous, then

\[
\mathcal{T}_\omega(\mu, \nu) := \inf_{\pi \in \Pi(\mu, \nu)} \iint \omega(x, y) d\pi(x, y)
\]

\[
= \sup_{(\varphi, \psi)} \left\{ \int \psi \, d\mu - \int \varphi \, d\nu \right\},
\]

where the supremum runs over all bounded continuous functions \( \psi, \varphi \) on \( \mathcal{X} \) such that

\[
\psi(x) - \varphi(y) \leq \omega(x, y), \quad \forall x, y \in \mathcal{X}.
\]

Given \( \varphi \), we may replace \( \psi \) by the optimal function

\[
Q_\omega \varphi(x) = \inf_{y \in \mathcal{X}} \{ \varphi(y) + \omega(x, y) \}.
\]

This yields

\[
\mathcal{T}_\omega(\mu, \nu) = \sup_{\varphi} \left\{ \int Q_\omega \varphi \, d\mu - \int \varphi \, d\nu \right\},
\]

where the supremum runs over all bounded continuous functions \( \varphi \) on \( \mathcal{X} \).

### Usual example: the Wasserstein metric \( W_q \),

**Usual example:** the Wasserstein metric \( W_q \),
How to recover the Talagrand’s concentration inequality?

Second method, duality arguments: based on a generalized Kantorovich duality theorem for weak transport costs.

The classical Kantorovich dual theorem

If $\omega: \mathcal{X} \times \mathcal{X} \to [0, +\infty]$ is lower semi-continuous, then

$$ T_\omega(\mu, \nu) := \inf_{\pi \in \Pi(\mu, \nu)} \int \int \omega(x, y) d\pi(x, y) $$

$$ = \sup_{(\varphi, \psi)} \left\{ \int \psi d\mu - \int \varphi d\nu \right\}, $$

where the supremum runs over all bounded continuous functions $\psi, \varphi$ on $\mathcal{X}$ such that

$$ \psi(x) - \varphi(y) \leq \omega(x, y), \quad \forall x, y \in \mathcal{X}. $$

Given $\varphi$, we may replace $\psi$ by the optimal function

$$ Q_\omega \varphi(x) = \inf_{y \in \mathcal{X}} \{ \varphi(y) + \omega(x, y) \}. $$

This yields

$$ T_\omega(\mu, \nu) = \sup_{\varphi} \left\{ \int Q_\omega \varphi d\mu - \int \varphi d\nu \right\}, $$

where the supremum runs over all bounded continuous functions $\varphi$ on $\mathcal{X}$.

Usual example: the Wasserstein metric $W_q$, $q \geq 1$, 

---

**Notes:**

- How to recover the Talagrand’s concentration inequality?
- Second method, duality arguments: based on a generalized Kantorovich duality theorem for weak transport costs.
- The classical Kantorovich dual theorem
- If $\omega: \mathcal{X} \times \mathcal{X} \to [0, +\infty]$ is lower semi-continuous, then
  $$ T_\omega(\mu, \nu) := \inf_{\pi \in \Pi(\mu, \nu)} \int \int \omega(x, y) d\pi(x, y) $$
  $$ = \sup_{(\varphi, \psi)} \left\{ \int \psi d\mu - \int \varphi d\nu \right\}, $$
  where the supremum runs over all bounded continuous functions $\psi, \varphi$ on $\mathcal{X}$ such that
  $$ \psi(x) - \varphi(y) \leq \omega(x, y), \quad \forall x, y \in \mathcal{X}. $$
- Given $\varphi$, we may replace $\psi$ by the optimal function
  $$ Q_\omega \varphi(x) = \inf_{y \in \mathcal{X}} \{ \varphi(y) + \omega(x, y) \}. $$
- This yields
  $$ T_\omega(\mu, \nu) = \sup_{\varphi} \left\{ \int Q_\omega \varphi d\mu - \int \varphi d\nu \right\}, $$
  where the supremum runs over all bounded continuous functions $\varphi$ on $\mathcal{X}$.
How to recover the Talagrand’s concentration inequality?

Second method, duality arguments: based on a generalized Kantorovich duality theorem for weak transport costs.

**The classical Kantorovich dual theorem**

If $\omega : \mathcal{X} \times \mathcal{X} \to [0. \ + \infty]$ is lower semi-continuous, then

$$
\mathcal{T}_\omega(\mu, \nu) := \inf_{\pi \in \Pi(\mu, \nu)} \iint \omega(x, y) d\pi(x, y)
$$

$$
= \sup_{(\varphi, \psi)} \left\{ \int \psi \ d\mu - \int \varphi \ d\nu \right\},
$$

where the supremum runs over all bounded continuous functions $\psi, \varphi$ on $\mathcal{X}$ such that

$$
\psi(x) - \varphi(y) \leq \omega(x, y), \quad \forall x, y \in \mathcal{X}.
$$

Given $\varphi$, we may replace $\psi$ by the optimal function

$$
Q_\omega \varphi(x) = \inf_{y \in \mathcal{X}} \{ \varphi(y) + \omega(x, y) \}.
$$

This yields

$$
\mathcal{T}_\omega(\mu, \nu) = \sup_{\varphi} \left\{ \int Q_\omega \varphi \ d\mu - \int \varphi \ d\nu \right\},
$$

where the supremum runs over all bounded continuous functions $\varphi$ on $\mathcal{X}$.

**Usual example**: the Wasserstein metric $W_q$, $q \geq 1$, $\mu, \nu \in \mathcal{P}_q(\mathcal{X})$, where $\mathcal{P}_q(\mathcal{X})$ denotes the set of probability measures on $\mathcal{X}$.
How to recover the Talagrand’s concentration inequality?

Second method, duality arguments: based on a generalized Kantorovich duality theorem for weak transport costs.

The classical Kantorovich dual theorem

If \( \omega : X \times X \to [0. + \infty] \) is lower semi-continuous, then

\[
\mathcal{T}_\omega(\mu, \nu) := \inf_{\pi \in \Pi(\mu, \nu)} \int \int \omega(x, y) d\pi(x, y)
\]

\[= \sup_{(\varphi, \psi)} \left\{ \int \psi \, d\mu - \int \varphi \, d\nu \right\}, \]

where the supremum runs over all bounded continuous functions \( \psi, \varphi \) on \( X \) such that

\[\psi(x) - \varphi(y) \leq \omega(x, y), \quad \forall x, y \in X.\]

Given \( \varphi \), we may replace \( \psi \) by the optimal function

\[Q_\omega \varphi(x) = \inf_{y \in X} \{ \varphi(y) + \omega(x, y) \}.\]

This yields

\[\mathcal{T}_\omega(\mu, \nu) = \sup_{\varphi} \left\{ \int Q_\omega \varphi \, d\mu - \int \varphi \, d\nu \right\}, \]

where the supremum runs over all bounded continuous functions \( \varphi \) on \( X \).

Usual example: the Wasserstein metric \( W_q, q \geq 1, \mu, \nu \in \mathcal{P}_q(X) \),

\[\mathcal{T}_q(\mu, \nu) = W_q^q(\mu, \nu) := \inf_{\pi \in \Pi(\mu, \nu)} \int \int d^q(x, y) d\pi(x, y)\]
How to recover the Talagrand’s concentration inequality?

Second method, duality arguments: based on a generalized Kantorovich duality theorem for weak transport costs.

The classical Kantorovich dual theorem

If \( \omega : \mathcal{X} \times \mathcal{X} \to [0. + \infty] \) is lower semi-continuous, then

\[
\mathcal{T}_\omega(\mu, \nu) := \inf_{\pi \in P(\mu, \nu)} \iint \omega(x, y) d\pi(x, y)
= \sup_{(\psi, \varphi)} \left\{ \int \psi d\mu - \int \varphi d\nu \right\},
\]

where the supremum runs over all bounded continuous functions \( \psi, \varphi \) on \( \mathcal{X} \) such that

\[
\psi(x) - \varphi(y) \leq \omega(x, y), \quad \forall x, y \in \mathcal{X}.
\]

Given \( \varphi \), we may replace \( \psi \) by the optimal function

\[
Q_\omega \varphi(x) = \inf_{y \in \mathcal{X}} \{ \varphi(y) + \omega(x, y) \}.
\]

This yields \( \mathcal{T}_\omega(\mu, \nu) = \sup_{\varphi} \left\{ \int Q_\omega \varphi d\mu - \int \varphi d\nu \right\} \),

where the supremum runs over all bounded continuous functions \( \varphi \) on \( \mathcal{X} \).

Usual example: the Wasserstein metric \( W_q, q \geq 1, \mu, \nu \in \mathcal{P}_q(\mathcal{X}) \),

\[
\mathcal{T}_q(\mu, \nu) = W^q_q(\mu, \nu) := \inf_{\pi \in P(\mu, \nu)} \iint d^q(x, y) d\pi(x, y) = \inf_{(X, Y)} \mathbb{E}[d(X, Y)^q],
\]

\( X \sim \mu, Y \sim \nu \).
How to recover the Talagrand’s concentration inequality?

Second method, duality arguments: based on a generalized Kantorovich duality theorem for weak transport costs.

The classical Kantorovich dual theorem

If $\omega : \mathcal{X} \times \mathcal{X} \to [0, +\infty]$ is lower semi-continuous, then

$$
T_{\omega}(\mu, \nu) := \inf_{\pi \in \Pi(\mu, \nu)} \iint \omega(x, y) d\pi(x, y)
= \sup_{(\psi, \varphi)} \left\{ \int \psi d\mu - \int \varphi d\nu \right\},
$$

where the supremum runs over all bounded continuous functions $\psi, \varphi$ on $\mathcal{X}$ such that

$$
\psi(x) - \varphi(y) \leq \omega(x, y), \quad \forall x, y \in \mathcal{X}.
$$

Given $\varphi$, we may replace $\psi$ by the optimal function

$$
Q_\omega \varphi(x) = \inf_{y \in \mathcal{X}} \{ \varphi(y) + \omega(x, y) \}.
$$

This yields

$$
T_{\omega}(\mu, \nu) = \sup_{\varphi} \left\{ \int Q_\omega \varphi d\mu - \int \varphi d\nu \right\},
$$

where the supremum runs over all bounded continuous functions $\varphi$ on $\mathcal{X}$.

Usual example: the Wasserstein metric $W_q$, $q \geq 1$, $\mu, \nu \in \mathcal{P}_q(\mathcal{X})$,

$$
T_q(\mu, \nu) = W_q^q(\mu, \nu) := \inf_{\pi \in \Pi(\mu, \nu)} \iint d^q(x, y) d\pi(x, y) = \inf_{(X, Y)} \mathbb{E}[d(X, Y)^q],
$$

$X \sim \mu$, $Y \sim \nu$. Duality holds with $Q_\varphi(x) = \inf_{y \in \mathcal{X}} \{ \varphi(y) + d^q(x, y) \}.$
Extension of Kantorovich duality to weak transport cost
Extension of Kantorovich duality to weak transport cost

<table>
<thead>
<tr>
<th>Definition</th>
<th>Weak optimal transport cost</th>
</tr>
</thead>
</table>

**Definition : Weak optimal transport cost**

Let us consider a measurable function $\hat{X}$ on $\mathbb{R}^d$. The weak optimal cost, $T_{\gamma}^{\mu, \nu}$, associated to $c$ is defined by

$$
T_{\gamma}^{\mu, \nu} := \inf_{\pi} \int \gamma(p, q) \, d\mu \, p + \int \gamma(q, p) \, d\nu \, q
$$

where $\gamma(p, q)$ is the cost function.

**Examples of weak cost**
- Marton's type of cost
- Barycentric cost
- Strassen's result
- Martingale costs

**Weak transport inequalities**
- Dual characterization to concentration

**Universal transport inequalities**

**Barycentric transport inequalities**

**Examples in discrete**

**Transport inequality on the symmetric group**
- Introduction
- Ewens distribution
- Deviation inequalities

**The Schrödinger minimization problem**
- Definition
- Curvature in discrete spaces
- Functional inequalities
- Examples in discrete
Extension of Kantorovich duality to weak transport cost

**Definition: Weak optimal transport cost**

Let us consider a measurable function

\[ \mathcal{X} \times \mathcal{P}_{\gamma}(\mathcal{X}) \rightarrow [0, +\infty] \]

\[ c : (x, p) \mapsto c(x, p), \]

where \( \mathcal{X} \) is a space, \( \mathcal{P}_{\gamma}(\mathcal{X}) \) denotes the set of probability measures on \( \mathcal{X} \) with some additional properties, and \( c(x, p) \) is the cost function.
Extension of Kantorovich duality to weak transport cost

**Definition: Weak optimal transport cost**

Let us consider a measurable function

\[ \mathcal{X} \times \mathcal{P}_\gamma(\mathcal{X}) \rightarrow [0, +\infty] \]

\[ c: (x, p) \mapsto c(x, p), \]

The weak optimal cost, \( T_c(\nu|\mu) \), associated to \( c \) is defined by

\[ T_c(\nu|\mu) = \inf_{\pi P \Pi \mu, \nu} \int_{\mathcal{X}} \hat{\gamma}_p x, q \, d\mu(x, y), \]

Example 0: For \( c(x, p) = \int_{\mathcal{X}} \omega(x, y) \, d\mu(x, y) \), with \( \omega: X \rightarrow \mathbb{R} \),

\[ T_c(\nu|\mu) = \inf_{\pi P \Pi \mu, \nu} \int_{\mathcal{X}} \omega(x, y) \, d\mu(x, y). \]

This is the usual Kantorovich optimal transport cost.

Example 1: For \( c(x, p) = \alpha \int_{\mathcal{X}} \gamma_p x, q \, d\mu(x, y) \), with \( \alpha: \mathbb{R} \rightarrow \mathbb{R} \),

\[ T_c(\nu|\mu) = \inf_{\pi P \Pi \mu, \nu} \int_{\mathcal{X}} \alpha \int_{\mathcal{X}} \gamma_p x, q \, d\mu(x, y). \]

For \( \gamma = 0 \) and \( \alpha = 1 \) for other convex functions \( \alpha \), \( T_c(\nu|\mu) \) is Marton's cost (1996) (or Dembo's cost (1997) for other convex functions \( \alpha \)).
### Extension of Kantorovich duality to weak transport cost

**Definition: Weak optimal transport cost**

Let us consider a measurable function

\[
\begin{align*}
\mathcal{X} \times \mathcal{P}_\gamma(\mathcal{X}) & \to [0, +\infty] \\
(c : (x, p)) & \mapsto c(x, p),
\end{align*}
\]

The weak optimal cost, \( T_c(\nu|\mu) \), associated to \( c \) is defined by

\[
T_c(\nu|\mu) := \inf_{\pi \in \Pi(\mu, \nu)} \int c(x, p_x) d\mu(x), \quad \mu, \nu \in \mathcal{P}_\gamma(\mathcal{X}),
\]
Extension of Kantorovich duality to weak transport cost

**Definition : Weak optimal transport cost**

Let us consider a measurable function

\[ \mathcal{X} \times \mathcal{P}_\gamma(\mathcal{X}) \rightarrow [0, +\infty], \]
\[ c : (x, p) \mapsto c(x, p), \]

The weak optimal cost, \( T_c(\nu|\mu) \), associated to \( c \) is defined by

\[ T_c(\nu|\mu) := \inf_{\pi \in \Pi(\mu, \nu)} \int c(x, p_x) d\mu(x), \quad \mu, \nu \in \mathcal{P}_\gamma(\mathcal{X}), \]
\[ \pi = \mu \otimes p \]

---

Example 0:
For \( c_{p|x,q} \),
\[ \omega_{p|x,y} dp_y \]
with \( \omega : \mathcal{X} \rightarrow \mathbb{R}^+ \),
\[ T_{\omega_{p|x,q}}(\nu|\mu) \]
is the usual Kantorovich optimal transport cost.

Example 1:
For \( c_{p|x,q} \),
\[ \alpha \hat{x}_{\gamma} dp_y \]
with \( \alpha : \mathbb{R} \rightarrow \mathbb{R}^+ \),
\[ T_{\alpha_{p|x,q}}(\nu|\mu) \]
is Marton's cost (1996) (or Dembo's cost (1997) for other convex functions \( \alpha \)).

For \( \gamma_0 p|x,q \)
\[ 1 \leq x \equiv y \] and \( \alpha_{p|h,q} = h^2 \),
\[ T_{\alpha_{p|h,q}}(\nu|\mu) \] is Marton's type of cost.
Extension of Kantorovich duality to weak transport cost

Definition: Weak optimal transport cost

Let us consider a measurable function

\[ \mathcal{X} \times \mathcal{P}_\gamma(\mathcal{X}) \rightarrow [0, +\infty] \]

\[ c : (x, p) \mapsto c(x, p), \]

The weak optimal cost, \( T_c(\nu|\mu) \), associated to \( c \) is defined by

\[ T_c(\nu|\mu) := \inf_{\pi \in \Pi(\mu, \nu)} \int c(x, p_x) d\mu(x), \quad \mu, \nu \in \mathcal{P}_\gamma(\mathcal{X}), \]

\[ \pi = \mu \otimes p \]

Example 0: For \( c(x, p) = \int \omega(x, y) dp(y) \), with \( \omega : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}^+ \),
Extension of Kantorovich duality to weak transport cost

**Definition : Weak optimal transport cost**

Let us consider a measurable function

\[ \mathcal{X} \times \mathcal{P}_\gamma(\mathcal{X}) \rightarrow [0, +\infty] \]
\[ c : (x, p) \mapsto c(x, p), \]

The weak optimal cost, \( T_c(\nu|\mu) \), associated to \( c \) is defined by

\[ T_c(\nu|\mu) := \inf_{\pi \in \Pi(\mu, \nu)} \int c(x, p_x) d\mu(x), \quad \mu, \nu \in \mathcal{P}_\gamma(\mathcal{X}), \]

\[ \pi = \mu \otimes p \]

**Example 0 :** For \( c(x, p) = \int \omega(x, y) \, dp(y) \), with \( \omega : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}^+ \),
**Extension of Kantorovich duality to weak transport cost**

**Definition : Weak optimal transport cost**

Let us consider a measurable function

\[ \mathcal{X} \times \mathcal{P}_\gamma(\mathcal{X}) \rightarrow [0, +\infty] \]
\[ c : (x, p) \mapsto c(x, p), \]

The weak optimal cost, \( T_c(\nu|\mu) \), associated to \( c \) is defined by

\[ T_c(\nu|\mu) := \inf_{\pi \in \Pi(\mu, \nu)} \int c(x, p_x) d\mu(x), \quad \mu, \nu \in \mathcal{P}_\gamma(\mathcal{X}), \]

\[ \pi = \mu \otimes p \]

**Example 0 :** For \( c(x, p) = \int \omega(x, y) dp(y) \), with \( \omega : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}^+ \),

\[ T_c(\nu|\mu) = \inf_{\pi \in \Pi(\mu, \nu)} \int \int \omega(x, y) p_x(y) d\mu(x) \]
Extension of Kantorovich duality to weak transport cost

**Definition : Weak optimal transport cost**

Let us consider a measurable function

\[ X \times \mathcal{P}_\gamma(X) \rightarrow [0, +\infty) \]

\[ c : (x, p) \mapsto c(x, p), \]

The weak optimal cost, \( T_c(\nu|\mu) \), associated to \( c \) is defined by

\[
T_c(\nu|\mu) := \inf_{\pi \in \Pi(\mu, \nu)} \int c(x, p_x) d\mu(x), \quad \mu, \nu \in \mathcal{P}_\gamma(X),
\]

\[ \pi = \mu \otimes p \]

**Example 0 :** For \( c(x, p) = \int \omega(x, y) dp(y) \), with \( \omega : X \times X \rightarrow \mathbb{R}^+ \),

\[
T_c(\nu|\mu) = \inf_{\pi \in \Pi(\mu, \nu)} \int \int \omega(x, y) dp_x(y) d\mu(x) \frac{d\pi(x, y)}{d\pi(x, y)}
\]
Extension of Kantorovich duality to weak transport cost

Definition: Weak optimal transport cost

Let us consider a measurable function

\[ \mathcal{X} \times \mathcal{P}_\gamma(\mathcal{X}) \rightarrow [0, +\infty], \]
\[ c : (x, p) \mapsto c(x, p), \]

The weak optimal cost, \( T_c(\nu|\mu) \), associated to \( c \) is defined by

\[ T_c(\nu|\mu) := \inf_{\pi \in \Pi(\mu, \nu)} \int c(x, p_x) d\mu(x), \quad \mu, \nu \in \mathcal{P}_\gamma(\mathcal{X}), \]

Example 0: For \( c(x, p) = \int \omega(x, y) \, dp(y) \), with \( \omega : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}^+ \),

\[ T_c(\nu|\mu) = \inf_{\pi \in \Pi(\mu, \nu)} \int \int \omega(x, y) \frac{dp(x)}{d\pi(x,y)} d\mu(x) = T_\omega(\mu, \nu), \]

is the usual Kantorovich optimal transport cost.
Extension of Kantorovich duality to weak transport cost

**Definition : Weak optimal transport cost**

Let us consider a measurable function

\[ c : \mathcal{X} \times \mathcal{P}_\gamma(\mathcal{X}) \rightarrow [0, +\infty] \]

\[ c(x, p) \mapsto c(x, p), \]

The weak optimal cost, \( T_c(\nu|\mu) \), associated to \( c \) is defined by

\[
T_c(\nu|\mu) := \inf_{\pi \in \Pi(\mu, \nu) \quad \pi = \mu \otimes p} \int c(x, p_x) d\mu(x), \quad \mu, \nu \in \mathcal{P}_\gamma(\mathcal{X}),
\]

**Example 0** : For \( c(x, p) = \int \omega(x, y) \, dp(y) \), with \( \omega : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}^+ \),

\[
T_c(\nu|\mu) = \inf_{\pi \in \Pi(\mu, \nu)} \iint \omega(x, y) \underbrace{dp_x(y) d\mu(x)}_{d\pi(x, y)} = T_\omega(\mu, \nu),
\]

is the usual Kantorovich optimal transport cost.

**Example 1** : For \( c(x, p) = \alpha \left( \int \gamma(d(x, y)) \, dp(y) \right) \), with \( \alpha : \mathbb{R} \rightarrow \mathbb{R} \),
Extension of Kantorovich duality to weak transport cost

Definition : Weak optimal transport cost

Let us consider a measurable function

\[ c : (x, p) \mapsto c(x, p), \]

\[ \mathcal{X} \times \mathcal{P}_\gamma(\mathcal{X}) \rightarrow [0, +\infty] \]

The weak optimal cost, \( T_c(\nu|\mu) \), associated to \( c \) is defined by

\[ T_c(\nu|\mu) := \inf_{\pi \in \Pi(\mu, \nu), \pi = \mu \otimes p} \int c(x, p_x) d\mu(x), \quad \mu, \nu \in \mathcal{P}_\gamma(\mathcal{X}), \]

Example 0 : For \( c(x, p) = \int \omega(x, y) \, dp(y) \), with \( \omega : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}^+ \),

\[ T_c(\nu|\mu) = \inf_{\pi \in \Pi(\mu, \nu)} \int \int \omega(x, y) \, dp_x(y) \, d\mu(x) \, d\pi(x, y) = T_\omega(\mu, \nu), \]

is the usual Kantorovich optimal transport cost.

Example 1 : For \( c(x, p) = \alpha \left( \int \gamma(d(x, y)) \, dp(y) \right) \), with \( \alpha : \mathbb{R} \rightarrow \mathbb{R} \),

\[ T_c(\nu|\mu) = \inf_{\pi \in \Pi(\mu, \nu)} \int \alpha \left( \int \gamma(d(x, y)) \, dp_x(y) \right) \, d\mu(x) \]
Extension of Kantorovich duality to weak transport cost

Definition: Weak optimal transport cost

Let us consider a measurable function

$$X \times \mathcal{P}_\gamma(X) \rightarrow [0, +\infty]$$

$$c : (x, p) \mapsto c(x, p),$$

The weak optimal cost, $T_c(\nu|\mu)$, associated to $c$ is defined by

$$T_c(\nu|\mu) := \inf_{\pi \in \Pi(\mu, \nu)} \int c(x, p_x) d\mu(x), \quad \mu, \nu \in \mathcal{P}_\gamma(X),$$

$$\pi = \mu \otimes p$$

Example 0: For $c(x, p) = \int \omega(x, y) dp(y)$, with $\omega : X \times X \rightarrow \mathbb{R}^+$,

$$T_c(\nu|\mu) = \inf_{\pi \in \Pi(\mu, \nu)} \int \int \omega(x, y) dp_x(y) d\mu(x) d\pi(x, y) = T_\omega(\mu, \nu),$$

is the usual Kantorovich optimal transport cost.

Example 1: For $c(x, p) = \alpha \left( \int \gamma(d(x, y)) dp(y) \right)$, with $\alpha : \mathbb{R} \rightarrow \mathbb{R}$,

$$T_c(\nu|\mu) = \inf_{\pi \in \Pi(\mu, \nu)} \alpha \left( \int \gamma(d(x, y)) dp_x(y) \right) d\mu(x) = \tilde{T}_\alpha(\nu|\mu),$$
Extension of Kantorovich duality to weak transport cost

**Definition: Weak optimal transport cost**

Let us consider a measurable function

\[
\mathcal{X} \times \mathcal{P}_\gamma(\mathcal{X}) \rightarrow [0, +\infty]
\]

\[
c : (x, p) \mapsto c(x, p),
\]

The weak optimal cost, \( T_c(\nu|\mu) \), associated to \( c \) is defined by

\[
T_c(\nu|\mu) := \inf_{\pi \in \Pi(\mu, \nu) \atop \pi = \mu \otimes \rho} \int c(x, p_x) d\mu(x), \quad \mu, \nu \in \mathcal{P}_\gamma(\mathcal{X}),
\]

**Example 0:** For \( c(x, p) = \int \omega(x, y) dp(y) \), with \( \omega : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}^+ \),

\[
T_c(\nu|\mu) = \inf_{\pi \in \Pi(\mu, \nu)} \int \int \omega(x, y) d\mu(x) \frac{dp_x(y)}{d\pi(x, y)} = T_\omega(\mu, \nu),
\]

is the usual Kantorovich optimal transport cost.

**Example 1:** For \( c(x, p) = \alpha \left( \int \gamma(d(x, y)) dp(y) \right) \), with \( \alpha : \mathbb{R} \rightarrow \mathbb{R} \),

\[
T_c(\nu|\mu) = \inf_{\pi \in \Pi(\mu, \nu)} \int \alpha \left( \int \gamma(d(x, y)) dp_x(y) \right) d\mu(x) = \tilde{T}_\alpha(\nu|\mu),
\]

For \( \gamma_0(d(x, y)) = 1_{x \neq y} \) and \( \alpha(h) = h^2 \),
Extension of Kantorovich duality to weak transport cost

**Definition : Weak optimal transport cost**

Let us consider a measurable function

$$c : (x, p) \rightarrow c(x, p),$$

$$\mathcal{X} \times \mathcal{P}_\gamma(\mathcal{X}) \rightarrow [0, +\infty]$$

The weak optimal cost, $\mathcal{T}_c(\nu|\mu)$, associated to $c$ is defined by

$$\mathcal{T}_c(\nu|\mu) := \inf_{\pi \in \Pi(\mu, \nu)} \int c(x, p_x) d\mu(x), \quad \mu, \nu \in \mathcal{P}_\gamma(\mathcal{X}),$$

with $\pi = \mu \otimes p$.

**Example 0** : For $c(x, p) = \int \omega(x, y) \, dp(y)$, with $\omega : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}^+$,

$$\mathcal{T}_c(\nu|\mu) = \inf_{\pi \in \Pi(\mu, \nu)} \iint \omega(x, y) \, dp_x(y) \, d\mu(x) = \mathcal{T}_\omega(\mu, \nu),$$

is the usual Kantorovich optimal transport cost.

**Example 1** : For $c(x, p) = \alpha \left( \int \gamma(d(x, y)) \, dp(y) \right)$, with $\alpha : \mathbb{R} \rightarrow \mathbb{R}$,

$$\mathcal{T}_c(\nu|\mu) = \inf_{\pi \in \Pi(\mu, \nu)} \int \alpha \left( \int \gamma(d(x, y)) \, dp_x(y) \right) \, d\mu(x) = \mathcal{T}_\alpha(\nu|\mu),$$

For $\gamma_0(d(x, y)) = 1_{x \neq y}$ and $\alpha(h) = h^2$, $\mathcal{T}_\alpha(\nu|\mu)$ is Marton’s cost (1996) (or Dembo’s cost (1997) for other convex functions $\alpha$).
Extension of Kantorovich duality to weak transport cost

Definition: Weak optimal transport cost

Let us consider a measurable function

\[ \mathcal{X} \times \mathcal{P}_{\gamma}(\mathcal{X}) \rightarrow [0, +\infty] \]

\[ c : (x, p) \mapsto c(x, p), \]

The weak optimal cost, \( T_c(\nu|\mu) \), associated to \( c \) is defined by

\[ T_c(\nu|\mu) := \inf_{\pi \in \Pi(\mu, \nu)} \int c(x, p_x) d\mu(x), \quad \mu, \nu \in \mathcal{P}_{\gamma}(\mathcal{X}), \]

\[ \pi = \mu \otimes p \]

Example 0: For \( c(x, p) = \int \omega(x, y) dp(y) \), with \( \omega : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}^+ \),

\[ T_c(\nu|\mu) = \inf_{\pi \in \Pi(\mu, \nu)} \int \omega(x, y) dp_x(y) d\mu(x) = T_\omega(\mu, \nu), \]

is the usual Kantorovich optimal transport cost.

Example 1: For \( c(x, p) = \alpha \left( \int \gamma(d(x, y)) dp(y) \right) \), with \( \alpha : \mathbb{R} \rightarrow \mathbb{R} \),

\[ T_c(\nu|\mu) = \inf_{\pi \in \Pi(\mu, \nu)} \int \alpha \left( \int \gamma(d(x, y)) dp_x(y) \right) d\mu(x) = \tilde{T}_\alpha(\nu|\mu), \]

For \( \gamma_0(d(x, y)) = 1_{x \neq y} \) and \( \alpha(h) = h^2 \), \( \tilde{T}_\alpha(\nu|\mu) \) is Marton’s cost (1996) (or Dembo’s cost (1997) for other convex functions \( \alpha \)).
Kantorovich duality for weak transport costs

introduction
Marton's inequality
Talagrand's concentration
Kantorovich duality
for classical costs
for weak costs
Examples of weak cost
Marton's type of cost
Barycentric cost
Strassen's result
Martingale costs
Weak transport inequalities
Dual characterization to concentration
Universal transport inequalities
Barycentric transport inequalities
Examples
characterisation on $\mathbb{R}$
Transport inequality on the symmetric group
introduction
Ewens distribution
deviation inequalities
The Schrödinger minimization problem
definition
curvature in discrete spaces
functional inequalities
Examples in discrete
Weak transport costs
Kantorovich duality for weak transport costs

\[ \Phi_{\gamma}(\mathcal{X}) : \text{the set of continuous functions } \varphi : \mathcal{X} \to \mathbb{R} \text{ such that} \]

\[ |\varphi(x)| \leq a + b \gamma(d(x, x_0)), \quad \forall x \in \mathcal{X}. \]
Kantorovich duality for weak transport costs

\( \Phi_{\gamma}(\mathcal{X}') : \) the set of continuous functions \( \varphi : \mathcal{X} \to \mathbb{R} \) such that

\[
|\varphi(x)| \leq a + b \gamma(d(x, x_0)), \quad \forall x \in \mathcal{X}.
\]

\( \Phi_{\gamma, b}(\mathcal{X}') : \) the set of functions in \( \Phi_{\gamma}(\mathcal{X}') \) bounded from below.
Kantorovich duality for weak transport costs

Φ_γ(\mathcal{X}) : the set of continuous functions \( \varphi : \mathcal{X} \rightarrow \mathbb{R} \) such that

\[ |\varphi(x)| \leq a + b \gamma(d(x, x_0)), \quad \forall x \in \mathcal{X}. \]

Φ_γ,b(\mathcal{X}) : the set of functions in Φ_γ(\mathcal{X}) bounded from below.

**Definition : duality for weak transport costs**
Kantorovich duality for weak transport costs

\[ \Phi_\gamma(\mathcal{X}) : \text{the set of continuous functions } \varphi : \mathcal{X} \to \mathbb{R} \text{ such that} \]

\[ |\varphi(x)| \leq a + b \gamma(d(x, x_0)), \quad \forall x \in \mathcal{X}. \]

\[ \Phi_\gamma, b(\mathcal{X}) : \text{the set of functions in } \Phi_\gamma(\mathcal{X}) \text{ bounded from below.} \]

**Definition : duality for weak transport costs**

One says that **duality holds** for the cost

\[ c : \mathcal{X} \times \mathcal{P}(\mathcal{X}) \to [0, +\infty], \]

\[ \Phi_\gamma(\mathcal{X}) : \text{the set of continuous functions } \varphi : \mathcal{X} \to \mathbb{R} \text{ such that} \]

\[ |\varphi(x)| \leq a + b \gamma(d(x, x_0)), \quad \forall x \in \mathcal{X}. \]
Kantorovich duality for weak transport costs

$$\Phi_\gamma(\mathcal{X}):$$ the set of continuous functions $\varphi : \mathcal{X} \to \mathbb{R}$ such that

$$|\varphi(x)| \leq a + b \gamma(d(x, x_0)), \quad \forall x \in \mathcal{X}.$$

$$\Phi_\gamma, b(\mathcal{X}):$$ the set of functions in $\Phi_\gamma(\mathcal{X})$ bounded from below.

**Definition: duality for weak transport costs**

One says that duality holds for the cost

$$c : \mathcal{X} \times \mathcal{P}_\gamma(\mathcal{X}) \to [0, +\infty],$$

if for all $\mu, \nu \in \mathcal{P}_\gamma(\mathcal{X})$, it holds

$$\mathcal{T}_c(\nu|\mu) := \inf_{\pi \in \Pi(\mu, \nu)} \int c(x, p_x) d\mu(x)$$

$$= \sup_{\varphi \in \Phi_\gamma, b(\mathcal{X})} \left\{ \int R_c \varphi d\mu - \int \varphi d\nu \right\},$$
### Kantorovich duality for weak transport costs

\( \Phi_{\gamma}(\mathcal{X}) \): the set of continuous functions \( \varphi : \mathcal{X} \rightarrow \mathbb{R} \) such that

\[
|\varphi(x)| \leq a + b \gamma(d(x, x_0)), \quad \forall x \in \mathcal{X}.
\]

\( \Phi_{\gamma,b}(\mathcal{X}) \): the set of functions in \( \Phi_{\gamma}(\mathcal{X}) \) bounded from below.

**Definition: duality for weak transport costs**

One says that duality holds for the cost

\[
c : \mathcal{X} \times \mathcal{P}(\mathcal{X}) \rightarrow [0, +\infty],
\]

if for all \( \mu, \nu \in \mathcal{P}(\mathcal{X}) \), it holds

\[
\mathcal{T}_c(\nu | \mu) := \inf_{\pi \in \Pi(\mu, \nu)} \int c(x, p_x) d\mu(x)
\]

\[
= \sup_{\varphi \in \Phi_{\gamma,b}(\mathcal{X})} \left\{ \int R_c \varphi \, d\mu - \int \varphi \, d\nu \right\},
\]

where for \( \varphi \in \Phi_{\gamma,b}(\mathcal{X}) \),

\[
R_c \varphi(x) = \inf_{\rho \in \mathcal{P}(\mathcal{X})} \left\{ \int \varphi \, d\rho + c(x, \rho) \right\}, \quad x \in \mathcal{X}.
\]
Kantorovich duality for weak transport costs

\( \Phi_{\gamma}(\mathcal{X}) \) : the set of continuous functions \( \varphi : \mathcal{X} \to \mathbb{R} \) such that

\[
|\varphi(x)| \leq a + b \gamma(d(x, x_0)), \quad \forall x \in \mathcal{X}.
\]

\( \Phi_{\gamma,b}(\mathcal{X}) \) : the set of functions in \( \Phi_{\gamma}(\mathcal{X}) \) bounded from below.

**Definition : duality for weak transport costs**

One says that **duality holds** for the cost

\[
c : \mathcal{X} \times \mathcal{P}_{\gamma}(\mathcal{X}) \to [0, +\infty],
\]

if for all \( \mu, \nu \in \mathcal{P}_{\gamma}(\mathcal{X}) \), it holds

\[
\mathcal{T}_c(\nu | \mu) := \inf_{\pi \in \Pi(\mu, \nu)} \int c(x, p_x) d\mu(x)
\]

\[
= \sup_{\varphi \in \Phi_{\gamma,b}(\mathcal{X})} \left\{ \int R_{c,\varphi} d\mu - \int \varphi d\nu \right\},
\]

where for \( \varphi \in \Phi_{\gamma,b}(\mathcal{X}) \),

\[
R_{c,\varphi}(x) = \inf_{p \in \mathcal{P}_{\gamma}(\mathcal{X})} \left\{ \int \varphi d\rho + c(x, p) \right\}, \quad x \in \mathcal{X}.
\]
### Kantorovich duality for weak transport costs

**Definition**: Kantorovich duality for weak transport costs

- **\( \Phi_\gamma(\mathcal{X}) \)**: the set of continuous functions \( \varphi : \mathcal{X} \to \mathbb{R} \) such that
  \[
  |\varphi(x)| \leq a + b \gamma(d(x, x_0)), \quad \forall x \in \mathcal{X}.
  \]

- **\( \Phi_\gamma,b(\mathcal{X}) \)**: the set of functions in \( \Phi_\gamma(\mathcal{X}) \) bounded from below.

**Definition**: duality for weak transport costs

One says that duality holds for the cost

\[
c : \mathcal{X} \times \mathcal{P}_\gamma(\mathcal{X}) \to [0, +\infty],
\]

if for all \( \mu, \nu \in \mathcal{P}_\gamma(\mathcal{X}) \), it holds

\[
T_c(\nu|\mu) := \inf_{\pi \in \Pi(\mu, \nu)} \int c(x, p_x) d\mu(x)
\]

\[
= \sup_{\varphi \in \Phi_\gamma,b(\mathcal{X})} \left\{ \int R_c \varphi d\mu - \int \varphi d\nu \right\},
\]

where for \( \varphi \in \Phi_\gamma,b(\mathcal{X}) \),

\[
R_c \varphi(x) = \inf_{p \in \mathcal{P}_\gamma(\mathcal{X})} \left\{ \int \varphi d\rho + c(x, p) \right\}, \quad x \in \mathcal{X}.
\]
Kantorovich duality for weak transport costs

\( \Phi_{\gamma}(\mathcal{X}) \) : the set of continuous functions \( \varphi : \mathcal{X} \to \mathbb{R} \) such that

\[
\left| \varphi(x) \right| \leq a + b \gamma(d(x, x_0)), \quad \forall x \in \mathcal{X}.
\]

\( \Phi_{\gamma, b}(\mathcal{X}) \) : the set of functions in \( \Phi_{\gamma}(\mathcal{X}) \) bounded from below.

**Definition : duality for weak transport costs**

One says that duality holds for the cost

\[
c : \mathcal{X} \times \mathcal{P}(\mathcal{X}) \to [0, +\infty],
\]

if for all \( \mu, \nu \in \mathcal{P}(\mathcal{X}) \), it holds

\[
\mathcal{T}_c(\nu|\mu) := \inf_{\pi \in \Pi(\mu, \nu)} \int c(x, p_x) \, d\mu(x)
\]

\[
= \sup_{\varphi \in \Phi_{\gamma, b}(\mathcal{X})} \left\{ \int R_c \varphi \, d\mu - \int \varphi \, d\nu \right\},
\]

where for \( \varphi \in \Phi_{\gamma, b}(\mathcal{X}) \),

\[
R_c \varphi(x) = \inf_{p \in \mathcal{P}(\mathcal{X})} \left\{ \int \varphi \, dp + c(x, p) \right\}, \quad x \in \mathcal{X}.
\]

Main assumptions for duality to hold:

- \( p \mapsto c(x, p) \) is convex,
- semi-continuity assumptions.
Kantorovich duality for weak transport costs

$\Phi_\gamma(\mathcal{X})$ : the set of continuous functions $\varphi : \mathcal{X} \to \mathbb{R}$ such that

$$|\varphi(x)| \leq a + b \gamma(d(x, x_0)), \quad \forall x \in \mathcal{X}.$$ 

$\Phi_\gamma, b(\mathcal{X})$ : the set of functions in $\Phi_\gamma(\mathcal{X})$ bounded from below.

**Definition : duality for weak transport costs**

One says that **duality holds** for the cost

$c : \mathcal{X} \times \mathcal{P}_\gamma(\mathcal{X}) \to [0, +\infty],$

if for all $\mu, \nu \in \mathcal{P}_\gamma(\mathcal{X})$, it holds

$$T_c(\nu|\mu) := \inf_{\pi \in \Pi(\mu, \nu)} \int c(x, p_x) d\mu(x)$$

$$= \sup_{\varphi \in \Phi_\gamma, b(\mathcal{X})} \left\{ \int R_c \varphi d\mu - \int \varphi d\nu \right\},$$

where for $\varphi \in \Phi_\gamma, b(\mathcal{X})$,

$$R_c \varphi(x) = \inf_{p \in \mathcal{P}_\gamma(\mathcal{X})} \left\{ \int \varphi dp + c(x, p) \right\}, \quad x \in \mathcal{X}.$$

Main assumptions for duality to hold :
- $p \mapsto c(x, p)$ is convex,
Kantorovich duality for weak transport costs

\[ \Phi_\gamma(\mathcal{X}) : \text{the set of continuous functions } \varphi : \mathcal{X} \to \mathbb{R} \text{ such that } \]
\[ |\varphi(x)| \leq a + b \gamma(d(x, x_0)), \quad \forall x \in \mathcal{X}. \]

\[ \Phi_\gamma, b(\mathcal{X}) : \text{the set of functions in } \Phi_\gamma(\mathcal{X}) \text{ bounded from below.} \]

**Definition : duality for weak transport costs**

One says that duality holds for the cost

\[ c : \mathcal{X} \times \mathcal{P}(\mathcal{X}) \to [0, +\infty], \]

if for all \( \mu, \nu \in \mathcal{P}(\mathcal{X}) \), it holds

\[ \mathcal{T}_c(\nu|\mu) := \inf_{\pi \in \Pi(\mu, \nu)} \int c(x, p_x) d\mu(x) \]
\[ = \sup_{\varphi \in \Phi_\gamma, b(\mathcal{X})} \left\{ \int R_c \varphi \ d\mu - \int \varphi \ d\nu \right\}, \]

where for \( \varphi \in \Phi_\gamma, b(\mathcal{X}) \),

\[ R_c \varphi(x) = \inf_{p \in \mathcal{P}(\mathcal{X})} \left\{ \int \varphi \ dp + c(x, p) \right\}, \quad x \in \mathcal{X}. \]

Main assumptions for duality to hold:
- \( p \mapsto c(x, p) \) is convex,
- semi-continuity assumptions.
Theorem [GRST ’15] : Examples of weak costs for which duality holds
Theorem [GRST ’15] : Examples of weak costs for which duality holds

Example 0 : For $c(x, p) = \int \omega(x, y) dp(y)$.

$$
\mathcal{T}_c(\nu|\mu) = \inf_{\pi \in \Pi(\mu, \nu)} \iint \omega(x, y) \pi(dx, dy) = \mathcal{T}_\omega(\mu, \nu)
$$

$$
= \sup_\varphi \left\{ \int Q_\omega \varphi d\mu - \int \varphi d\nu \right\}, \quad \mu, \nu \in \mathcal{P}_\gamma(\mathcal{X}),
$$
Theorem [GRST '15] : Examples of weak costs for which duality holds

Example 0 : For $c(x,p) = \int \omega(x,y) dp(y)$.

$$
T_c(\nu|\mu) = \inf_{\pi \in \Pi(\mu,\nu)} \int \int \omega(x,y) \pi(dx,dy) = T_\omega(\mu,\nu)
$$

$$
= \sup_{\varphi} \left\{ \int Q_\omega \varphi d\mu - \int \varphi d\nu \right\}, \quad \mu, \nu \in \mathcal{P}_\gamma(\mathcal{X}),
$$

with

$$
Q_\omega \varphi(x) = \inf_{p \in \mathcal{P}_\gamma(\mathcal{X})} \left\{ \int \varphi dp + \int \omega(x,y) dp(y) \right\} = \inf_{y \in \mathcal{X}} \{ \varphi(y) + \omega(x,y) \}. 
$$
Theorem [GRST ’15] : Examples of weak costs for which duality holds

Example 0 : For \( c(x, p) = \int \omega(x, y) dp(y) \),

\[
\mathcal{T}_c(\nu|\mu) = \inf_{\pi \in \Pi(\mu, \nu)} \iint \omega(x, y) \pi(dx, dy) = \mathcal{T}_\omega(\mu, \nu)
\]

\[
= \sup_\varphi \left\{ \int Q_\omega \varphi d\mu - \int \varphi d\nu \right\}, \quad \mu, \nu \in \mathcal{P}_\gamma(X),
\]

with

\[
Q_\omega \varphi(x) = \inf_{p \in \mathcal{P}_\gamma(X)} \left\{ \int \varphi dp + \int \omega(x, y) dp(y) \right\} = \inf_{y \in X} \{ \varphi(y) + \omega(x, y) \}.
\]
Theorem [GRST ’15] : Examples of weak costs for which duality holds

Example 0 : For \( c(x, p) = \int \omega(x, y) dp(y) \).

\[
\mathcal{T}_c(\nu|\mu) = \inf_{\pi \in \Pi(\mu, \nu)} \iint \omega(x, y) \pi(dx, dy) = \mathcal{T}_\omega(\mu, \nu)
\]

\[
= \sup_{\varphi} \left\{ \int Q_\omega \varphi d\mu - \int \varphi d\nu \right\}, \quad \mu, \nu \in \mathcal{P}_\gamma(\mathcal{X}),
\]

with

\[
Q_\omega \varphi(x) = \inf_{p \in \mathcal{P}_\gamma(\mathcal{X})} \left\{ \int \varphi dp + \int \omega(x, y) dp(y) \right\} = \inf_{y \in \mathcal{X}} \{ \varphi(y) + \omega(x, y) \}.
\]
Theorem [GRST ’15] : Examples of weak costs for which duality holds

Example 0 : For \( c(x, p) = \int \omega(x, y) dp(y) \).

\[
\mathcal{T}_c(\nu|\mu) = \inf_{\pi \in \Pi(\mu, \nu)} \iint \omega(x, y) \pi(dx, dy) = \mathcal{T}_\omega(\mu, \nu)
\]

\[
= \sup_{\varphi} \left\{ \int Q_\omega \varphi d\mu - \int \varphi d\nu \right\}, \quad \mu, \nu \in \mathcal{P}_\gamma(X),
\]

with

\[
Q_{\omega} \varphi(x) = \inf_{p \in \mathcal{P}_\gamma(X)} \left\{ \int \varphi dp + \int \omega(x, y) dp(y) \right\} = \inf_{y \in X} \{ \varphi(y) + \omega(x, y) \}.
\]

Example 1 : For \( c(x, p) = \alpha \left( \int \gamma(d(x, y)) dp(y) \right) \)
Theorem [GRST '15]: Examples of weak costs for which duality holds

**Example 0:** For \( c(x, p) = \int \omega(x, y) dp(y) \).

\[
\mathcal{T}_c (\nu | \mu) = \inf_{\pi \in \Pi(\mu, \nu)} \int \int \omega(x, y) \pi(dx, dy) = \mathcal{T}_\omega (\mu, \nu)
\]

\[
= \sup_{\varphi} \left\{ \int Q_\omega \varphi d\mu - \int \varphi d\nu \right\}, \quad \mu, \nu \in \mathcal{P}_\gamma (\mathcal{X}),
\]

with

\[
Q_\omega \varphi(x) = \inf_{p \in \mathcal{P}_\gamma (\mathcal{X})} \left\{ \int \varphi dp + \int \omega(x, y) dp(y) \right\} = \inf_{y \in \mathcal{X} } \{ \varphi(y) + \omega(x, y) \}.
\]

**Example 1:** For \( c(x, p) = \alpha \left( \int \gamma(d(x, y)) dp(y) \right) \)

with \( \alpha : \mathbb{R}^+ \to [0, +\infty] \) (lower semi-)continuous convex and \( \alpha(0) = 0 \).
Theorem [GRST ’15] : Examples of weak costs for which duality holds

Example 0 : For $c(x, p) = \int \omega(x, y)dp(y)$.

$$T_c(\nu|\mu) = \inf_{\pi \in \Pi(\mu, \nu)} \iint \omega(x, y)\pi(dx, dy) = T_\omega(\mu, \nu)$$

$$= \sup_\varphi \{ \int Q_\omega \varphi d\mu - \int \varphi d\nu \}, \quad \mu, \nu \in \mathcal{P}_\gamma(\mathcal{X}),$$

with $Q_\omega \varphi(x) = \inf_{p \in \mathcal{P}_\gamma(\mathcal{X})} \left\{ \int \varphi dp + \int \omega(x, y) dp(y) \right\} = \inf_{y \in \mathcal{X}} \{ \varphi(y) + \omega(x, y) \}.$

Example 1 : For $c(x, p) = \alpha \left( \int \gamma(d(x, y)) dp(y) \right)$

with $\alpha : \mathbb{R}^+ \to [0, +\infty]$ (lower semi-)continuous convex and $\alpha(0) = 0$.

$$\tilde{T}_\alpha(\nu|\mu) = \inf_{\pi \in \Pi(\mu, \nu)} \int \alpha \left( \int \gamma(d(x, y)) dp_x(y) \right) d\mu(x)$$

$$= \sup_\varphi \{ \int \tilde{Q}_\alpha \varphi d\mu - \int \varphi d\nu \}, \quad \mu, \nu \in \mathcal{P}_\gamma(\mathcal{X}),$$
Theorem [GRST ’15] : Examples of weak costs for which duality holds

Example 0 : For \( c(x, p) = \int \omega(x, y) dp(y) \).

\[
\mathcal{T}_c(\nu|\mu) = \inf_{\pi \in \Pi(\mu, \nu)} \int \int \omega(x, y) \pi(dx, dy) = \mathcal{T}_\omega(\mu, \nu)
\]

\[
= \sup_{\varphi} \left\{ \int Q_\omega \varphi \, d\mu - \int \varphi \, d\nu \right\}, \quad \mu, \nu \in \mathcal{P}_\gamma(\mathcal{X}),
\]

with \( Q_\omega \varphi(x) = \inf_{p \in \mathcal{P}_\gamma(\mathcal{X})} \left\{ \int \varphi \, dp + \int \omega(x, y) \, dp(y) \right\} = \inf_{y \in \mathcal{X}} \{ \varphi(y) + \omega(x, y) \} \).

Example 1 : For \( c(x, p) = \alpha \left( \int \gamma(d(x, y)) \, dp(y) \right) \)

with \( \alpha : \mathbb{R}^+ \to [0, +\infty] \) (lower semi-)continuous convex and \( \alpha(0) = 0 \).

\[
\mathcal{\tilde{T}}_\alpha(\nu|\mu) = \inf_{\pi \in \Pi(\mu, \nu)} \alpha \left( \int \gamma(d(x, y)) \, dp_x(y) \right) \, d\mu(x)
\]

\[
= \sup_{\varphi} \left\{ \int Q_\alpha \varphi \, d\mu - \int \varphi \, d\nu \right\}, \quad \mu, \nu \in \mathcal{P}_\gamma(\mathcal{X}),
\]

with \( Q_\alpha \varphi(x) = \inf_{p \in \mathcal{P}_\gamma(\mathcal{X})} \left\{ \int \varphi \, dp + \alpha \left( \int \gamma(d(x, y)) \, dp(y) \right) \right\} \).
Theorem [GRST ’15] : Examples of weak costs for which duality holds

Example 0 : For \( c(x, p) = \int \omega(x, y) dp(y) \).

\[
\mathcal{T}_c(\nu|\mu) = \inf_{\pi \in \Pi(\mu, \nu)} \iint \omega(x, y) \pi(dx, dy) = \mathcal{T}_\omega(\mu, \nu)
\]

\[
= \sup_{\varphi} \left\{ \int Q_\omega \varphi d\mu - \int \varphi d\nu \right\}, \quad \mu, \nu \in \mathcal{P}_\gamma(X),
\]

with

\[
Q_\omega \varphi(x) = \inf_{p \in \mathcal{P}_\gamma(X)} \left\{ \int \varphi dp + \int \omega(x, y) dp(y) \right\} = \inf_{y \in X} \{ \varphi(y) + \omega(x, y) \}.
\]

Example 1 : For \( c(x, p) = \alpha \left( \int \gamma(d(x, y)) dp(y) \right) \)

with \( \alpha : \mathbb{R}^+ \to [0, +\infty] \) (lower semi-)continuous convex and \( \alpha(0) = 0 \).

\[
\widetilde{\mathcal{T}}_\alpha(\nu|\mu) = \inf_{\pi \in \Pi(\mu, \nu)} \alpha \left( \int \gamma(d(x, y)) dp_x(y) \right) d\mu(x)
\]

\[
= \sup_{\varphi} \left\{ \int \widetilde{Q}_\alpha \varphi d\mu - \int \varphi d\nu \right\}, \quad \mu, \nu \in \mathcal{P}_\gamma(X),
\]

with

\[
\widetilde{Q}_\alpha \varphi(x) = \inf_{p \in \mathcal{P}_\gamma(X)} \left\{ \int \varphi dp + \alpha \left( \int \gamma(d(x, y)) dp(y) \right) \right\}.
\]
Theorem [GRST ’15]: Examples of weak costs for which duality holds

Example 2:
Theorem [GRST '15] : Examples of weak costs for which duality holds

Example 2 : Let $\mu_0$ denotes a reference probability measure on $\mathcal{X}$. 
Theorem [GRST ’15] : Examples of weak costs for which duality holds

Example 2 : Let $\mu_0$ denotes a reference probability measure on $\mathcal{X}$.

$$c(x, p) = \int \beta \left( \gamma(d(x, y)) \frac{dp}{d\mu_0}(y) \right) d\mu_0(y), \quad \text{if } p \ll \mu_0,$$

and $c(x, p) = +\infty$ otherwise, with $\beta : \mathbb{R}^+ \to [0, +\infty]$, convex and $\beta(0) = 0$. 

Particular case : a Talagrand’s cost for $\gamma_0 = 1$ $\mu = 0$, 

$$c(x, p) = \int \beta \left( \gamma(d(x, y)) \frac{dp}{d\mu_0}(y) \right) d\mu_0(y), \quad \text{if } p \ll \mu_0,$$

used by Talagrand (1996) as a main ingredient to reach deviation inequalities for supremum of empirical processes with Bernstein’s bounds, see also S. (2007).
Theorem [GRST '15] : Examples of weak costs for which duality holds

**Example 2** : Let $\mu_0$ denotes a reference probability measure on $\mathcal{X}$.

$$c(x, p) = \int \beta \left( \gamma(d(x, y)) \frac{dp}{d\mu_0}(y) \right) d\mu_0(y), \quad \text{if } p << \mu_0,$$

and $c(x, p) = +\infty$ otherwise, with $\beta : \mathbb{R}^+ \rightarrow [0, +\infty]$, convex and $\beta(0) = 0$.

$$\hat{T}_\beta(\nu|\mu) = \inf_{\pi \in \Pi(\mu, \nu)} \int \int \beta \left( \gamma(d(x, y)) \frac{dp_x}{d\mu_0}(y) \right) d\mu_0(y) d\mu(x) \geq \hat{T}_\beta(\nu|\mu).$$
Theorem [GRST ’15] : Examples of weak costs for which duality holds

Example 2 : Let $\mu_0$ denotes a reference probability measure on $\mathcal{X}$. 

$$c(x, p) = \int \beta \left( \gamma(d(x, y)) \frac{dp}{d\mu_0}(y) \right) d\mu_0(y), \quad \text{if } p \ll \mu_0,$$

and $c(x, p) = +\infty$ otherwise, with $\beta : \mathbb{R}^+ \to [0, +\infty]$, convex and $\beta(0) = 0$.

$$\widehat{T}_\beta(\nu|\mu) = \inf_{\pi \in \Pi(\mu, \nu)} \int \int \beta \left( \gamma(d(x, y)) \frac{dp_x}{d\mu_0}(y) \right) d\mu_0(y) d\mu(x)$$

$$\geq \widehat{T}_\beta(\nu|\mu)$$

$$\widehat{T}_\beta(\nu|\mu) = \sup_{\varphi} \left\{ \int \hat{Q}_\beta \varphi(x) d\mu(x) - \int \varphi(y) d\nu(y) \right\},$$

Particular case : a Talagrand’s cost for $\gamma = 1$, $\mu = \mu_0$, $\nu = \mu_0$, used by Talagrand (1996) as a main ingredient to reach deviation inequalities for supremum of empirical processes with Bernstein’s bounds, see also S. (2007).
Theorem [GRST ’15] : Examples of weak costs for which duality holds

Example 2 : Let $\mu_0$ denotes a reference probability measure on $\mathcal{X}$.

$$c(x, p) = \int \beta \left( \gamma(d(x, y)) \frac{dp}{d\mu_0}(y) \right) d\mu_0(y),$$

if $p << \mu_0$, and $c(x, p) = +\infty$ otherwise, with $\beta : \mathbb{R}^+ \to [0, +\infty]$, convex and $\beta(0) = 0$.

$$\widehat{T}_\beta(\nu | \mu) = \inf_{\pi \in \Pi(\mu, \nu)} \int \beta \left( \gamma(d(x, y)) \frac{dp_x}{d\mu_0}(y) \right) d\mu_0(y) d\mu(x)$$

$$\geq \widehat{T}_\beta(\nu | \mu)$$

$$\widehat{T}_\beta(\nu | \mu) = \sup_{\varphi} \left\{ \int \widetilde{Q}_\beta \varphi(x) d\mu(x) - \int \varphi(y) d\nu(y) \right\},$$

$$\widetilde{Q}_\beta \varphi(x) = \inf_{p \in \mathcal{P}_\gamma(X)} \left\{ \int \varphi(y) dp(y) + \int \beta \left( \gamma(d(x, y)) \frac{dp}{d\mu_0}(y) \right) d\mu_0(y) \right\}.$$
Theorem [GRST ’15] : Examples of weak costs for which duality holds

Example 2 : Let $\mu_0$ denotes a reference probability measure on $\mathcal{X}$.

$$c(x, p) = \int \beta \left( \gamma(d(x, y)) \frac{dp}{d\mu_0}(y) \right) d\mu_0(y), \quad \text{if } p << \mu_0,$$

and $c(x, p) = +\infty$ otherwise, with $\beta : \mathbb{R}^+ \to [0, +\infty]$, convex and $\beta(0) = 0$.

$$\widehat{T}_\beta(\nu|\mu) = \inf_{\pi \in \Pi(\mu, \nu)} \int \int \beta \left( \gamma(d(x, y)) \frac{dp_x}{d\mu_0}(y) \right) d\mu_0(y) d\mu(x)$$

$$\geq \widetilde{T}_\beta(\nu|\mu)$$

$$\widehat{T}_\beta(\nu|\mu) = \sup_\varphi \left\{ \int \hat{Q}_\beta \varphi(x) d\mu(x) - \int \varphi(y) d\nu(y) \right\},$$

$$\hat{Q}_\beta \varphi(x) = \inf_{p \in \mathcal{P}_\gamma(X)} \left\{ \int \varphi(y) dp(y) + \int \beta \left( \gamma(d(x, y)) \frac{dp}{d\mu_0}(y) \right) d\mu_0(y) \right\}.$$
Theorem [GRST ’15] : Examples of weak costs for which duality holds

Example 2 : Let $\mu_0$ denotes a reference probability measure on $\mathcal{X}$.

$$c(x, p) = \int \beta \left( \gamma(d(x, y)) \frac{dp}{d\mu_0}(y) \right) d\mu_0(y), \text{ if } p \ll \mu_0,$$

and $c(x, p) = +\infty$ otherwise, with $\beta : \mathbb{R}^+ \to [0, +\infty]$, convex and $\beta(0) = 0$.

$$\hat{T}_\beta (\nu | \mu) = \inf_{\pi \in \Pi(\mu, \nu)} \int \int \beta \left( \gamma(d(x, y)) \frac{dp_x}{d\mu_0}(y) \right) d\mu_0(y) d\mu(x)$$

$$\geq \hat{T}_\beta (\nu | \mu)$$

$$\hat{T}_\beta (\nu | \mu) = \sup_{\varphi} \left\{ \int \hat{Q}_\beta \varphi(x) d\mu(x) - \int \varphi(y) d\nu(y) \right\},$$

$$\hat{Q}_\beta \varphi(x) = \inf_{p \in \mathcal{P}_{\gamma}(X)} \left\{ \int \varphi(y) dp(y) + \int \beta \left( \gamma(d(x, y)) \frac{dp}{d\mu_0}(y) \right) d\mu_0(y) \right\}.$$
Theorem [GRST ’15] : Examples of weak costs for which duality holds

Example 2 : Let $\mu_0$ denotes a reference probability measure on $\mathcal{X}$.

$$c(x, p) = \int \beta \left( \gamma(d(x, y)) \frac{dp}{d\mu_0}(y) \right) d\mu_0(y), \quad \text{if } p \ll \mu_0,$$

and $c(x, p) = +\infty$ otherwise, with $\beta : \mathbb{R}^+ \to [0, +\infty]$, convex and $\beta(0) = 0$.

$$\hat{T}_\beta(\nu | \mu) = \inf_{\pi \in \Pi(\mu, \nu)} \int \int \beta \left( \gamma(d(x, y)) \frac{dp_x}{d\mu_0}(y) \right) d\mu_0(y) d\mu(x)$$

$$\geq \hat{T}_\beta(\nu | \mu)$$

$$\hat{T}_\beta(\nu | \mu) = \sup_{\varphi} \left\{ \int \hat{Q}_\beta \varphi(x) d\mu(x) - \int \varphi(y) d\nu(y) \right\},$$

$$\hat{Q}_\beta \varphi(x) = \inf_{p \in \mathcal{P}_\gamma(X)} \left\{ \int \varphi(y) dp(y) + \int \beta \left( \gamma(d(x, y)) \frac{dp}{d\mu_0}(y) \right) d\mu_0(y) \right\}.$$

Particular case : a Talagrand’s cost for $\gamma_0(u) = 1_{u \neq 0}$,

$$c(x, p) = \int \beta \left( 1_{x \neq y} \frac{dp}{d\mu_0}(y) \right) d\mu_0(y),$$

used by Talagrand (1996) as a main ingredient to reach deviation inequalities for supremum of empirical processes with Bernstein’s bounds, see also S. (2007).
Theorem [GRST ’15] : Examples of weak costs for which duality holds

Example 3 :
Theorem [GRST ’15]: Examples of weak costs for which duality holds

Example 3: Barycentric variant of Marton’s cost function when $\mathcal{X} \subset \mathbb{R}^m$. 
Theorem [GRST '15] : Examples of weak costs for which duality holds

Example 3 : Barycentric variant of Marton's cost function when $\mathcal{X} \subset \mathbb{R}^m$.

$$c(x, p) = \theta \left( x - \int y \, dp(y) \right), \quad p \in \mathcal{P}_1(\mathcal{X}),$$
Theorem [GRST '15]: Examples of weak costs for which duality holds

Example 3: Barycentric variant of Marton's cost function when $\mathcal{X} \subset \mathbb{R}^m$.

$$c(x, p) = \theta \left( x - \int y \, dp(y) \right), \quad p \in \mathcal{P}_1(\mathcal{X}),$$

with $\theta : \mathbb{R}^m \to [0, +\infty]$ (lower semi-)continuous convex and $\theta(0) = 0$. 

Remark: This cost has strong connections with convex functions.
Theorem [GRST ’15] : Examples of weak costs for which duality holds

**Example 3** : Barycentric variant of Marton’s cost function when $\mathcal{X} \subset \mathbb{R}^m$.

$$c(x, p) = \theta \left( x - \int y \, dp(y) \right), \quad p \in \mathcal{P}_1(\mathcal{X}),$$

with $\theta : \mathbb{R}^m \to [0, +\infty]$ (lower semi-)continuous convex and $\theta(0) = 0$.

$$\overline{T}_\theta(\nu | \mu) = \inf_{\pi \in \Pi(\mu, \nu)} \int \theta \left( \int x - \int y \, dp_x(y) \right) \, d\mu(x)$$

$$= \sup_{\varphi} \left\{ \int \overline{Q}_\theta \varphi \, d\mu - \int \varphi \, d\nu \right\}$$

Remark : This cost has strong connections with convex functions. Observe that

$$\overline{Q}_\theta \varphi = \inf_{z \in \mathbb{R}^m} \varphi(z) - \theta \left( \int x - \int y \, dp_x(y) \right)$$

The function $\varphi$ is convex. From this observation we get

$$\overline{T}_\theta(\nu | \mu) \leq \sup_{\varphi \text{ convex}} \left\{ \int \overline{Q}_\theta \varphi \, d\mu - \int \varphi \, d\nu \right\},$$

where the supremum runs over all convex Lipschitz functions $\varphi : \mathbb{R}^m \to \mathbb{R}$ bounded from below, and $\overline{Q}_\theta \varphi$ is the usual infimum-convolution operator.
Theorem [GRST '15] : Examples of weak costs for which duality holds

**Example 3** : Barycentric variant of Marton’s cost function when \( \mathcal{X} \subset \mathbb{R}^m \).

\[
c(x, p) = \theta \left( x - \int y \, dp(y) \right), \quad p \in \mathcal{P}_1(\mathcal{X}),
\]

with \( \theta : \mathbb{R}^m \rightarrow [0, +\infty] \) (lower semi-)continuous convex and \( \theta(0) = 0 \).

\[
\overline{T}_\theta(\nu|\mu) = \inf_{\pi \in \Pi(\mu, \nu)} \int \theta \left( \int x - \int y \, dp_x(y) \right) \, d\mu(x)
\]

\[= \sup_{\varphi} \left\{ \int \overline{Q}_\theta \varphi \, d\mu - \int \varphi \, d\nu \right\}
\]

with \( \overline{Q}_\theta \varphi(x) = \inf_{p \in \mathcal{P}_1(\mathcal{X})} \left\{ \int \varphi \, dp + \theta \left( x - \int y \, dp(y) \right) \right\} \).
Theorem [GRST '15] : Examples of weak costs for which duality holds

**Example 3** : Barycentric variant of Marton’s cost function when $\mathcal{X} \subset \mathbb{R}^m$.

$$ c(x, p) = \theta \left( x - \int y \, dp(y) \right), \quad p \in \mathcal{P}_1(\mathcal{X}), $$

with $\theta : \mathbb{R}^m \to [0, +\infty]$ (lower semi-)continuous convex and $\theta(0) = 0$.

$$ \mathcal{T}_\theta (\nu | \mu) = \inf_{\pi \in \Pi(\mu, \nu)} \int \theta \left( \int x - \int y \, dp_x(y) \right) \, d\mu(x) $$

$$ = \sup_{\varphi} \left\{ \int \overline{Q}_\theta \varphi \, d\mu - \int \varphi \, d\nu \right\} $$

with $\overline{Q}_\theta \varphi(x) = \inf_{p \in \mathcal{P}_1(\mathcal{X})} \left\{ \int \varphi \, dp + \theta \left( x - \int y \, dp(y) \right) \right\}$.

**Remark** : This cost has strong connections with convex functions.
**Theorem [GRST ’15] : Examples of weak costs for which duality holds**

**Example 3 :** Barycentric variant of Marton’s cost function when \( \mathcal{X} \subset \mathbb{R}^m \).

\[
c(x, p) = \theta \left( x - \int y \, dp(y) \right), \quad p \in \mathcal{P}_1(\mathcal{X}),
\]

with \( \theta : \mathbb{R}^m \to [0, +\infty] \) (lower semi-)continuous convex and \( \theta(0) = 0 \).

\[
\overline{T}_\theta(v|\mu) = \inf_{\pi \in \Pi(\mu, \nu)} \int \theta \left( \int x - \int y \, dp_x(y) \right) \, d\mu(x)
\]

\[
= \sup_\varphi \left\{ \int \overline{Q}_\theta \varphi \, d\mu - \int \varphi \, d\nu \right\}
\]

with \( \overline{Q}_\theta \varphi(x) = \inf_{p \in \mathcal{P}_1(\mathcal{X})} \left\{ \int \varphi \, dp + \theta \left( x - \int y \, dp(y) \right) \right\} \).

**Remark :** This cost has strong connections with convex functions. Observe that

\[
\overline{Q}_\theta \varphi(x) = \inf_{z \in \mathbb{R}^m} \left\{ \left( \inf_{p, \int y \, dp(y) = z} \int \varphi \, dp \right) + \theta \left( x - z \right) \right\}
\]

\[
:= \overline{\varphi}(z)
\]
**Theorem [GRST '15] : Examples of weak costs for which duality holds**

**Example 3 :** Barycentric variant of Marton’s cost function when \( X \subset \mathbb{R}^m \).

\[
c(x, p) = \theta \left( x - \int y \, dp(y) \right), \quad p \in \mathcal{P}_1(X),
\]

with \( \theta : \mathbb{R}^m \to [0, +\infty] \) (lower semi-)continuous convex and \( \theta(0) = 0 \).

\[
\overline{T}_\theta(\nu|\mu) = \inf_{\pi \in \Pi(\mu, \nu)} \int \theta \left( \int x - \int y \, dp_x(y) \right) \, d\mu(x)
\]

\[
= \sup_{\varphi} \left\{ \int \overline{Q}_{\theta} \varphi \, d\mu - \int \varphi \, d\nu \right\}
\]

with \( \overline{Q}_{\theta} \varphi(x) = \inf_{p \in \mathcal{P}_1(X)} \left\{ \int \varphi \, dp + \theta \left( x - \int y \, dp(y) \right) \right\} \).

**Remark :** This cost has strong connections with convex functions. Observe that

\[
\overline{Q}_{\theta} \varphi(x) = \inf_{z \in \mathbb{R}^m} \left\{ \left( \inf_{p, \int y \, dp(y) = z} \int \varphi \, dp \right) + \theta \left( x - z \right) \right\} = Q_{\theta} \varphi(x).
\]

\[
:= \overline{\varphi}(z)
\]
Theorem [GRST '15] : Examples of weak costs for which duality holds

Example 3 : Barycentric variant of Marton’s cost function when $\mathcal{X} \subset \mathbb{R}^m$.

$$c(x, p) = \theta \left( x - \int y \, dp(y) \right), \quad p \in \mathcal{P}_1(\mathcal{X}),$$

with $\theta : \mathbb{R}^m \to [0, +\infty]$ (lower semi-)continuous convex and $\theta(0) = 0$.

$$\overline{T}_\theta(\nu | \mu) = \inf_{\pi \in \Pi(\mu, \nu)} \int \theta \left( \int x - \int y \, dp_x(y) \right) \, d\mu(x)$$

$$= \sup_{\varphi} \left\{ \int \overline{Q}_\theta \varphi \, d\mu - \int \varphi \, d\nu \right\}$$

with $\overline{Q}_\theta \varphi(x) = \inf_{p \in \mathcal{P}_1(\mathcal{X})} \left\{ \int \varphi \, dp + \theta \left( x - \int y \, dp(y) \right) \right\}$.

Remark : This cost has strong connections with convex functions. Observe that

$$\overline{Q}_\theta \varphi(x) = \inf_{z \in \mathbb{R}^m} \left\{ \left( \inf_{p, \int y \, dp(y) = z} \int \varphi \, dp \right) + \theta \left( x - z \right) \right\} = Q_\theta \varphi(x).$$

The function $\varphi$ is convex.
Theorem [GRST '15]: Examples of weak costs for which duality holds

Example 3: Barycentric variant of Marton's cost function when $\mathcal{X} \subset \mathbb{R}^m$.

$$c(x, p) = \theta \left( x - \int y \, dp(y) \right), \quad p \in \mathcal{P}_1(\mathcal{X}),$$

with $\theta : \mathbb{R}^m \to [0, +\infty]$ (lower semi-)continuous convex and $\theta(0) = 0$.

$$\overline{T}_\theta(\nu|\mu) = \inf_{\pi \in \Pi(\mu, \nu)} \int \theta \left( \int x - \int y \, dp_x(y) \right) \, d\mu(x)$$

$$= \sup_{\varphi} \left\{ \int \overline{Q}_\theta \varphi \, d\mu - \int \varphi \, d\nu \right\}$$

with $\overline{Q}_\theta \varphi(x) = \inf_{p \in \mathcal{P}_1(\mathcal{X})} \left\{ \int \varphi \, dp + \theta \left( x - \int y \, dp(y) \right) \right\}$.

Remark: This cost has strong connections with convex functions. Observe that

$$\overline{Q}_\theta \varphi(x) = \inf_{z \in \mathbb{R}^m} \left\{ \left( \inf_{p, \int y \, dp(y) = z} \int \varphi \, dp \right) + \theta \left( x - z \right) \right\} = Q_\theta \overline{\varphi}(x).$$

The function $\overline{\varphi}$ is convex. From this observation we get

$$\overline{T}_\theta(\nu|\mu) = \sup_{\overline{\varphi}} \left\{ \int Q_\theta \overline{\varphi} \, d\mu - \int \overline{\varphi} \, d\nu \right\},$$
Theorem [GRST '15]: Examples of weak costs for which duality holds

Example 3: Barycentric variant of Marton's cost function when \( \mathcal{X} \subset \mathbb{R}^m \).

\[
c(x, p) = \theta \left( x - \int y \, dp(y) \right), \quad p \in \mathcal{P}_1(\mathcal{X}),
\]

with \( \theta : \mathbb{R}^m \to [0, +\infty] \) (lower semi-)continuous convex and \( \theta(0) = 0 \).

\[
\overline{T}_\theta(\nu | \mu) = \inf_{\pi \in \Pi(\mu, \nu)} \theta \left( \int x - \int y \, dp_x(y) \right) \, d\mu(x)
= \sup_{\varphi} \left\{ \int \overline{Q}_\theta \varphi \, d\mu - \int \varphi \, d\nu \right\}
\]

with \( \overline{Q}_\theta \varphi(x) = \inf_{p \in \mathcal{P}_1(\mathcal{X})} \left\{ \int \varphi \, dp + \theta \left( x - \int y \, dp(y) \right) \right\} \).

Remark: This cost has strong connections with convex functions. Observe that

\[
\overline{Q}_\theta \varphi(x) = \inf_{z \in \mathbb{R}^m} \left\{ \inf_{p, \int y \, dp(y) = z} \int \varphi \, dp \right\} + \theta \left( x - z \right) = Q_\theta \varphi(x).
\]

The function \( \varphi \) is convex. From this observation we get

\[
\overline{T}_\theta(\nu | \mu) = \sup_{\varphi \text{ convex}} \left\{ \int Q_\theta \varphi \, d\mu - \int \varphi \, d\nu \right\},
\]

where the supremum runs over all convex Lipschitz functions \( \varphi : \mathbb{R}^m \to \mathbb{R} \) bounded from below,
Theorem [GRST ’15] : Examples of weak costs for which duality holds

Example 3 : Barycentric variant of Marton’s cost function when \( \mathcal{X} \subset \mathbb{R}^m \).

\[
c(x, p) = \theta \left( x - \int y \, dp(y) \right), \quad p \in \mathcal{P}_1(\mathcal{X}),
\]

with \( \theta : \mathbb{R}^m \to [0, +\infty] \) (lower semi-)continuous convex and \( \theta(0) = 0 \).

\[
\mathcal{T}_\theta(\nu | \mu) = \inf_{\pi \in \Pi(\mu, \nu)} \int \theta \left( \int x - \int y \, dp(x) \right) \, d\mu(x)
\]

\[
= \sup_{\varphi} \left\{ \int \mathcal{Q}_\theta \varphi \, d\mu - \int \varphi \, d\nu \right\}
\]

with \( \mathcal{Q}_\theta \varphi(x) = \inf_{p \in \mathcal{P}_1(\mathcal{X})} \left\{ \int \varphi \, dp + \theta \left( x - \int y \, dp(y) \right) \right\} \).

Remark : This cost has strong connections with convex functions. Observe that

\[
\mathcal{Q}_\theta \varphi(x) = \inf_{z \in \mathbb{R}^m} \left\{ \left( \inf_{p, \int y \, dp(y) = z} \int \varphi \, dp \right) + \theta \left( x - z \right) \right\} = \mathcal{Q}_\theta \varphi(x).
\]

The function \( \varphi \) is convex. From this observation we get

\[
\mathcal{T}_\theta(\nu | \mu) = \sup_{\varphi \text{ convex}} \left\{ \int \mathcal{Q}_\theta \varphi \, d\mu - \int \varphi \, d\nu \right\},
\]

where the supremum runs over all convex Lipschitz functions \( \varphi : \mathbb{R}^m \to \mathbb{R} \) bounded from below, and \( \mathcal{Q}_\theta \varphi \) is the usual infimum-convolution operator.
A first use of barycentric cost for a Strassen result

\[
\overline{T}_\theta(\nu|\mu) = \sup_{\psi \text{ convex}} \left\{ \int Q_\theta \psi \, d\mu - \int \psi \, d\nu \right\},
\]

with

\[
Q_\theta \psi(x) = \inf_{z \in \mathbb{R}^m} \{\psi(z) + \theta(x - z)\}, \quad x \in \mathbb{R}^m
\]
A first use of barycentric cost for a Strassen result

\[ \overline{T}_\theta(\nu|\mu) = \sup_{\psi \text{ convex}} \left\{ \int Q_\theta \psi \, d\mu - \int \psi \, d\nu \right\}, \]

with

\[ Q_\theta \psi(x) = \inf_{z \in \mathbb{R}^m} \{ \psi(z) + \theta(x - z) \}, \quad x \in \mathbb{R}^m \]

Particular case:
A first use of barycentric cost for a Strassen result

\[ \overline{T}_\theta(\nu|\mu) = \sup_{\psi \text{ convex}} \left\{ \int Q_\theta \psi \, d\mu - \int \psi \, d\nu \right\}, \]

with

\[ Q_\theta \psi(x) = \inf_{z \in \mathbb{R}^m} \{ \psi(z) + \theta(x - z) \}, \quad x \in \mathbb{R}^m \]

**Particular case:** \( \theta(x - z) = |x - z|, \)
A first use of barycentric cost for a Strassen result

\[ \overline{T}_\theta(\nu|\mu) = \sup_{\psi \text{ convex}} \left\{ \int Q_\theta \psi \, d\mu - \int \psi \, d\nu \right\}, \]

with

\[ Q_\theta \psi(x) = \inf_{z \in \mathbb{R}^m} \{ \psi(z) + \theta(x - z) \}, \quad x \in \mathbb{R}^m \]

Particular case: \( \theta(x - z) = |x - z|, \quad \overline{T}_\theta = \overline{T}_1 \)
A first use of barycentric cost for a Strassen result

\[
\overline{T}_\theta(\nu|\mu) = \sup_{\psi \text{ convex}} \left\{ \int Q_\theta \psi \, d\mu - \int \psi \, d\nu \right\},
\]

with

\[
Q_\theta \psi(x) = \inf_{z \in \mathbb{R}^m} \{ \psi(z) + \theta(x - z) \}, \quad x \in \mathbb{R}^m
\]

Particular case: \(\theta(x - z) = |x - z|\), \(\overline{T}_\theta = \overline{T}_1\), \(Q_\theta \psi = Q_1 \psi\) is 1-Lipschitz,
A first use of barycentric cost for a Strassen result

\[ \mathcal{T}_\theta (\nu | \mu) = \sup_{\psi \text{ convex}} \left\{ \int Q_\theta \psi \, d\mu - \int \psi \, d\nu \right\}, \]

with

\[ Q_\theta \psi (x) = \inf_{z \in \mathbb{R}^m} \{ \psi (z) + \theta (x - z) \}, \quad x \in \mathbb{R}^m \]

Particular case: \( \theta (x - z) = |x - z| \), \( \mathcal{T}_\theta = \mathcal{T}_1 \), \( Q_\theta \psi = Q_1 \psi \) is 1-Lipschitz,

\[ Q_1 \psi = \psi. \]
A first use of barycentric cost for a Strassen result

$$\overline{T}_\theta(\nu|\mu) = \sup_{\psi \text{ convex}} \left\{ \int Q_\theta \psi \ d\mu - \int \psi \ d\nu \right\},$$

with

$$Q_\theta \psi(x) = \inf_{z \in \mathbb{R}^m} \{\psi(z) + \theta(x - z)\}, \quad x \in \mathbb{R}^m$$

**Particular case:** $\theta(x - z) = |x - z|$, $\overline{T}_\theta = \overline{T}_1$, $Q_\theta \psi = Q_1 \psi$ is 1-Lipschitz,

$$Q_1 \psi = \psi.$$

**Proposition. [GRST 2015]**

$$\overline{T}_1(\nu|\mu) = \inf_{(X,Y)} \mathbb{E}[|X - \mathbb{E}[Y|X]|]$$
A first use of barycentric cost for a Strassen result

\[ \overline{T}_\theta(\nu|\mu) = \sup_{\psi \text{ convex}} \left\{ \int Q_\theta \psi \, d\mu - \int \psi \, d\nu \right\}, \]

with

\[ Q_\theta \psi(x) = \inf_{z \in \mathbb{R}^m} \{ \psi(z) + \theta(x - z) \}, \quad x \in \mathbb{R}^m \]

**Particular case:** \( \theta(x - z) = |x - z|, \quad \overline{T}_\theta = \overline{T}_1, \quad Q_\theta \psi = Q_1 \psi \) is 1-Lipschitz,

\[ Q_1 \psi = \psi. \]

**Proposition. [GRST 2015]**

\[ \overline{T}_1(\nu|\mu) = \inf_{(X,Y)} \mathbb{E}[|X - \mathbb{E}[Y|X]|] = \sup_{\psi \text{ convex, 1-Lipschitz}} \left\{ \int \psi \, d\mu - \int \psi \, d\nu \right\}. \]
A first use of barycentric cost for a Strassen result

\[ \overline{T}_\theta(\nu|\mu) = \sup_{\psi \text{ convex}} \left\{ \int Q_\theta \psi \, d\mu - \int \psi \, d\nu \right\}, \]

with

\[ Q_\theta \psi(x) = \inf_{z \in \mathbb{R}^m} \{ \psi(z) + \theta(x - z) \}, \quad x \in \mathbb{R}^m \]

**Particular case:** \( \theta(x - z) = |x - z| \), \( \overline{T}_\theta = \overline{T}_1 \), \( Q_\theta \psi = Q_1 \psi \) is 1-Lipschitz,

\[ Q_1 \psi = \psi. \]

**Proposition.** [GRST 2015]

\[ \overline{T}_1(\nu|\mu) = \inf_{(X,Y)} \mathbb{E}[|X - \mathbb{E}[Y|X]|] = \sup_{\psi \text{ convex, 1-Lipschitz}} \left\{ \int \psi \, d\mu - \int \psi \, d\nu \right\}. \]

**Application:** A simple proof of a result by Strassen
A first use of barycentric cost for a Strassen result

\[ \overline{T}_\theta(\nu|\mu) = \sup_{\psi \text{ convex}} \left\{ \int Q_\theta \psi \, d\mu - \int \psi \, d\nu \right\}, \]

with

\[ Q_\theta \psi(x) = \inf_{z \in \mathbb{R}^m} \{ \psi(z) + \theta(x - z) \}, \quad x \in \mathbb{R}^m \]

**Particular case**: \( \theta(x - z) = |x - z| \), \( \overline{T}_\theta = \overline{T}_1 \), \( Q_\theta \psi = Q_1 \psi \) is 1-Lipschitz,

\[ Q_1 \psi = \psi. \]

**Proposition. [GRST 2015]**

\[ \overline{T}_1(\nu|\mu) = \inf_{(X,Y)} \mathbb{E}[|X - \mathbb{E}[Y|X]|] = \sup_{\psi \text{ convex, 1-Lipschitz}} \left\{ \int \psi \, d\mu - \int \psi \, d\nu \right\}. \]

**Application**: A simple proof of a result by Strassen

Let \( \mu, \nu \in \mathcal{P}_1(\mathbb{R}^m) \);
A first use of barycentric cost for a Strassen result

\[
\overline{T}_\theta (\nu | \mu) = \sup_{\psi \, \text{convex}} \left\{ \int Q_\theta \psi \, d\mu - \int \psi \, d\nu \right\},
\]

with

\[
Q_\theta \psi (x) = \inf_{z \in \mathbb{R}^m} \{ \psi (z) + \theta (x - z) \}, \quad x \in \mathbb{R}^m
\]

Particular case: \( \theta (x - z) = |x - z| \), \( \overline{T}_\theta = \overline{T}_1 \), \( Q_\theta \psi = Q_1 \psi \) is 1-Lipschitz, \( Q_1 \psi = \psi \).

**Proposition. [GRST 2015]**

\[
\overline{T}_1 (\nu | \mu) = \inf_{(X, Y)} \mathbb{E}[|X - \mathbb{E}[Y | X]|] = \sup_{\psi \, \text{convex, 1-Lipschitz}} \left\{ \int \psi \, d\mu - \int \psi \, d\nu \right\}.
\]

**Application:** A simple proof of a result by Strassen

Let \( \mu, \nu \in \mathcal{P}_1 (\mathbb{R}^m) \); one says that \( \mu \) is dominated by \( \nu \) in the convex order sense, \( \mu \preceq_C \nu \), if

\[
\int \psi \, d\mu \leq \int \psi \, d\nu,
\]

for all convex \( \psi : \mathbb{R}^m \to \mathbb{R} \).
A first use of barycentric cost for a Strassen result

\[
\overline{T}_\theta(\nu|\mu) = \sup_{\psi \text{ convex}} \left\{ \int Q_\theta \psi \,d\mu - \int \psi \,d\nu \right\},
\]

with

\[
Q_\theta \psi(x) = \inf_{z \in \mathbb{R}^m} \{ \psi(z) + \theta(x - z) \}, \quad x \in \mathbb{R}^m
\]

**Particular case:** \(\theta(x - z) = |x - z|\), \(\overline{T}_\theta = \overline{T}_1\), \(Q_\theta \psi = Q_1 \psi\) is 1-Lipschitz,

\[
Q_1 \psi = \psi.
\]

**Proposition. [GRST 2015]**

\[
\overline{T}_1(\nu|\mu) = \inf_{(X,Y)} \mathbb{E}[|X - \mathbb{E}[Y|X]|] = \sup_{\psi \text{ convex, 1-Lipschitz}} \left\{ \int \psi \,d\mu - \int \psi \,d\nu \right\}.
\]

**Application:** A simple proof of a result by Strassen

Let \(\mu, \nu \in \mathcal{P}_1(\mathbb{R}^m)\); one says that \(\mu\) is dominated by \(\nu\) in the convex order sense, \(\mu \leq_C \nu\), if

\[
\int \psi \,d\mu \leq \int \psi \,d\nu,
\]

for all convex \(\psi : \mathbb{R}^m \to \mathbb{R}\).

**Theorem. [Strassen 1965]**

Let \(\mu, \nu \in \mathcal{P}(\mathbb{R}^m)\).
A first use of barycentric cost for a Strassen result

\[ \mathcal{T}_\theta(\nu|\mu) = \sup_{\psi \text{ convex}} \left\{ \int Q_\theta \psi \, d\mu - \int \psi \, d\nu \right\}, \]

with

\[ Q_\theta \psi(x) = \inf_{z \in \mathbb{R}^m} \{ \psi(z) + \theta(x - z) \}, \quad x \in \mathbb{R}^m \]

Particular case: \( \theta(x - z) = |x - z|, \quad \mathcal{T}_\theta = \mathcal{T}_1, \quad Q_\theta \psi = Q_1 \psi \text{ is 1-Lipschitz}, \]

\[ Q_1 \psi = \psi. \]

**Proposition. [GRST 2015]**

\[ \mathcal{T}_1(\nu|\mu) = \inf_{(X,Y)} \mathbb{E}[|X - \mathbb{E}[Y|X]|] = \sup_{\psi \text{ convex, 1-Lipschitz}} \left\{ \int \psi \, d\mu - \int \psi \, d\nu \right\}. \]

**Application:** A simple proof of a result by Strassen

Let \( \mu, \nu \in \mathcal{P}_1(\mathbb{R}^m) \); one says that \( \mu \) is dominated by \( \nu \) in the convex order sense, \( \mu \preceq_C \nu \), if

\[ \int \psi \, d\mu \leq \int \psi \, d\nu, \]

for all convex \( \psi : \mathbb{R}^m \to \mathbb{R} \).

**Theorem. [Strassen 1965]**

Let \( \mu, \nu \in \mathcal{P}(\mathbb{R}^m) \). Then \( \mu \preceq_C \nu \) if and only if
A first use of barycentric cost for a Strassen result

\[
\overline{T}_\theta (\nu | \mu) = \sup_{\psi \text{ convex}} \left\{ \int Q_\theta \psi \, d\mu - \int \psi \, d\nu \right\},
\]

with

\[
Q_\theta \psi (x) = \inf_{z \in \mathbb{R}^m} \{ \psi (z) + \theta (x - z) \}, \quad x \in \mathbb{R}^m
\]

**Particular case:** \(\theta (x - z) = |x - z|\), \(\overline{T}_\theta = \overline{T}_1\), \(Q_\theta \psi = Q_1 \psi\) is 1-Lipschitz, \(Q_1 \psi = \psi\).

**Proposition.** [GRST 2015]

\[
\overline{T}_1 (\nu | \mu) = \inf_{(X,Y)} \mathbb{E}[|X - \mathbb{E}[Y|X]|] = \sup_{\psi \text{ convex, 1-Lipschitz}} \left\{ \int \psi \, d\mu - \int \psi \, d\nu \right\}.
\]

**Application:** A simple proof of a result by Strassen

Let \(\mu, \nu \in \mathcal{P}_1 (\mathbb{R}^m)\); one says that \(\mu\) is dominated by \(\nu\) in the convex order sense, \(\mu \leq_C \nu\), if

\[
\int \psi \, d\mu \leq \int \psi \, d\nu,
\]

for all convex \(\psi : \mathbb{R}^m \rightarrow \mathbb{R}\).

**Theorem.** [Strassen 1965]

Let \(\mu, \nu \in \mathcal{P}(\mathbb{R}^m)\). Then \(\mu \leq_C \nu\) if and only if there exists a martingale \((X, Y)\) \((\mathbb{E}[Y|X] = X)\), where \(X\) follows the law \(\mu\) and \(Y\) the law \(\nu\).
Examples of weak optimal transport costs for which duality holds

Example 4:

Let $\pi \in \Pi(p, q)$ such that $\mu \ll \nu$. According to Strassen Theorem,

$$\pi \in \Pi(p, q)$$

almost surely.

By definition, the martingale optimal cost associated to $\omega$:

$$\inf_{\pi \in \Pi(p, q)} \sum_{x \in \mathbb{R}} \omega(x, y) \pi(dx, dy)$$

This martingale cost can be expressed as a weak cost if:

$$\inf_{\pi \in \Pi(p, q)} \sum_{x \in \mathbb{R}} \omega(x, y) \pi(dx, dy)$$

Observe that the function $\omega$ is convex in $p$, and one has

$$\inf_{\pi \in \Pi(p, q)} \sum_{x \in \mathbb{R}} \omega(x, y) \pi(dx, dy)$$

with

$$\omega(x, y) = \int x \pi(dx, dy)$$

The cost $\omega$ is convex in $p$. The dual Kantorovich Theorem for weak cost applies and we recover the duality result by Beighböck-Henry-Labordère-Penker (2013).
Examples of weak optimal transport costs for which duality holds

Example 4: The martingale transport problem on the line.
Examples of weak optimal transport costs for which duality holds

**Example 4:** The martingale transport problem on the line.

Let \( \mu, \nu \in \mathcal{P}(\mathbb{R}) \) such that \( \mu \preceq_C \nu \). According to Strassen Theorem,

\[
\Pi^{\text{mart}}(\mu, \nu) := \left\{ \pi \in \Pi(\mu, \nu), \pi = \mu \otimes p, \int y dp_x(y) = x \mu \text{-almost surely} \right\} \neq \emptyset.
\]
Examples of weak optimal transport costs for which duality holds

**Example 4** : The martingale transport problem on the line.
Let $\mu, \nu \in \mathcal{P}(\mathbb{R})$ such that $\mu \preceq_C \nu$. According to Strassen Theorem,

$$\Pi^{mart}(\mu, \nu) := \left\{ \pi \in \Pi(\mu, \nu), \pi = \mu \otimes p, \int ydp_x(y) = x \mu \text{-almost surely} \right\} \neq \emptyset.\,$$

By definition, the martingale optimal cost associated to $\omega : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is

$$\mathcal{T}^{mart}_{\omega}(\nu | \mu) := \inf_{\pi \in \Pi^{mart}(\mu, \nu)} \iint \omega(x, y) \, d\pi(x, y).$$
Examples of weak optimal transport costs for which duality holds

**Example 4 :** The martingale transport problem on the line.
Let $\mu, \nu \in \mathcal{P}(\mathbb{R})$ such that $\mu \leq_{C} \nu$. According to Strassen Theorem,

$$\Pi^{\text{mart}}(\mu, \nu) := \left\{ \pi \in \Pi(\mu, \nu), \pi = \mu \otimes p, \int ydp_x(y) = x \mu\text{-almost surely} \right\} \neq \emptyset.$$

By definition, the martingale optimal cost associated to $\omega : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is

$$\mathcal{T}_{\omega}^{\text{mart}}(\nu|\mu) := \inf_{\pi \in \Pi^{\text{mart}}(\mu, \nu)} \iint \omega(x, y) d\pi(x, y).$$

How to express this martingale cost as a weak cost?
Examples of weak optimal transport costs for which duality holds

Example 4: The martingale transport problem on the line.
Let $\mu, \nu \in \mathcal{P}(\mathbb{R})$ such that $\mu \preceq \nu$. According to Strassen Theorem,

$$\Pi^{\text{mart}}(\mu, \nu) := \left\{ \pi \in \Pi(\mu, \nu), \pi = \mu \otimes p, \int y dp_x(y) = x \right\}_{\mu}\text{-almost surely} \neq \emptyset.$$

By definition, the martingale optimal cost associated to $\omega : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is

$$\mathcal{T}_{\omega}^{\text{mart}}(\nu|\mu) := \inf_{\pi \in \Pi^{\text{mart}}(\mu, \nu)} \iint \omega(x, y) d\pi(x, y).$$

How to express this martingale cost as a weak cost?

For $x \in \mathbb{R}$, $p \in \mathcal{P}_1(\mathbb{R})$, let

$$i(x, p) = \begin{cases} 0, & \text{if } \int y dp(y) = x, \\ +\infty, & \text{otherwise.} \end{cases}$$
Examples of weak optimal transport costs for which duality holds

Example 4: The martingale transport problem on the line.
Let $\mu, \nu \in \mathcal{P}(\mathbb{R})$ such that $\mu \preceq C \nu$. According to Strassen Theorem,

$$\Pi^{mart}(\mu, \nu) := \left\{ \pi \in \Pi(\mu, \nu), \pi = \mu \otimes p, \int y dp_x(y) = x \mu \text{-almost surely} \right\} \neq \emptyset.$$ 

By definition, the martingale optimal cost associated to $\omega : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is

$$\mathcal{T}^{mart}_{\omega}(\nu \mid \mu) := \inf_{\pi \in \Pi^{mart}(\mu, \nu)} \iint \omega(x, y) \, d\pi(x, y).$$

How to express this martingale cost as a weak cost?

For $x \in \mathbb{R}$, $p \in \mathcal{P}_1(\mathbb{R})$, let $i(x, p) = \left\{ \begin{array}{ll} 0, & \text{if } \int y \, dp(y) = x, \\ +\infty, & \text{otherwise}. \end{array} \right.$

Observe that the function $i$ is convex in $p$, 

...
Examples of weak optimal transport costs for which duality holds

Example 4: The martingale transport problem on the line.
Let $\mu, \nu \in \mathcal{P}(\mathbb{R})$ such that $\mu \preceq_{\mathcal{C}} \nu$. According to Strassen Theorem,

$$\Pi^{\text{mart}}(\mu, \nu) := \left\{ \pi \in \Pi(\mu, \nu), \pi = \mu \otimes p, \int ydp_x(y) = x \mu\text{-almost surely} \right\} \neq \emptyset.$$  

By definition, the martingale optimal cost associated to $\omega : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is

$$\mathcal{T}^{\text{mart}}(\nu|\mu) := \inf_{\pi \in \Pi^{\text{mart}}(\mu, \nu)} \iint \omega(x, y) \, d\pi(x, y).$$

How to express this martingale cost as a weak cost?

For $x \in \mathbb{R}$, $p \in \mathcal{P}_1(\mathbb{R})$, let

$$i(x, p) = \begin{cases} 0, & \text{if } \int y \, dp(y) = x, \\ +\infty, & \text{otherwise.} \end{cases}$$

Observe that the function $i$ is convex in $p$, and one has

$$\mathcal{T}^{\text{mart}}(\nu|\mu) = \inf_{\pi \in \Pi(\mu, \nu)} \left\{ \iint \omega(x, y) \, d\pi(x, y) + \int i(x, p_x) \, d\mu(x) \right\}$$

$$= \inf_{\pi \in \Pi(\mu, \nu)} \int c(x, p_x) \, d\mu(x),$$

where $c(x, p_x)$ is convex in $p$. The dual Kantorovich Theorem for weak cost applies and we recover the duality result by Beignébock-Henry-Labordère-Penker (2013).
Examples of weak optimal transport costs for which duality holds

Example 4: The martingale transport problem on the line.

Let $\mu, \nu \in \mathcal{P}(\mathbb{R})$ such that $\mu \leq_{c} \nu$. According to Strassen Theorem,

$$
\Pi_{\text{mart}}^{\mu}(\nu) := \left\{ \pi \in \Pi(\mu, \nu), \pi = \mu \otimes p, \int ydp_{x}(y) = x \mu\text{-almost surely} \right\} \neq \emptyset.
$$

By definition, the martingale optimal cost associated to $\omega : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is

$$
\mathcal{T}_{\omega}^{\text{mart}}(\nu | \mu) := \inf_{\pi \in \Pi_{\text{mart}}^{\mu}(\nu)} \iint \omega(x, y) d\pi(x, y).
$$

How to express this martingale cost as a weak cost?

For $x \in \mathbb{R}$, $p \in \mathcal{P}_{1}(\mathbb{R})$, let

$$
i(x, p) = \begin{cases} 
0, & \text{if } \int y dp(y) = x, \\
+\infty, & \text{otherwise}.
\end{cases}
$$

Observe that the function $i$ is convex in $p$, and one has

$$
\mathcal{T}_{\omega}^{\text{mart}}(\nu | \mu) = \inf_{\pi \in \Pi(\mu, \nu)} \left\{ \iint \omega(x, y) d\pi(x, y) + \int i(x, p_{x}) d\mu(x) \right\}
$$

$$
= \inf_{\pi \in \Pi(\mu, \nu)} \int c(x, p_{x}) d\mu(x),
$$

with $c(x, p) := \int \omega(x, y) dp(y) + i(x, p)$. 
Examples of weak optimal transport costs for which duality holds

**Example 4:** The martingale transport problem on the line.
Let $\mu, \nu \in \mathcal{P}(\mathbb{R})$ such that $\mu \preceq_c \nu$. According to Strassen Theorem,

$$\Pi_{\text{mart}}(\mu, \nu) := \left\{ \pi \in \Pi(\mu, \nu), \pi = \mu \otimes p, \int ydp_x(y) = x \mu\text{-almost surely} \right\} \neq \emptyset.$$ 

By definition, the martingale optimal cost associated to $\omega : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is

$$\mathcal{T}^\text{mart}_\omega (\nu | \mu) := \inf_{\pi \in \Pi_{\text{mart}}(\mu, \nu)} \iint \omega(x, y) \, d\pi(x, y).$$

**How to express this martingale cost as a weak cost?**

For $x \in \mathbb{R}$, $p \in \mathcal{P}_1(\mathbb{R})$, let

$$i(x, p) = \begin{cases} 0, & \text{if } \int y \, dp(y) = x, \\ +\infty, & \text{otherwise.} \end{cases}$$

Observe that the function $i$ is convex in $p$, and one has

$$\mathcal{T}^\text{mart}_\omega (\nu | \mu) = \inf_{\pi \in \Pi(\mu, \nu)} \left\{ \iint \omega(x, y) \, d\pi(x, y) + \int i(x, p_x) \, d\mu(x) \right\}$$

$$= \inf_{\pi \in \Pi(\mu, \nu)} \int c(x, p_x) \, d\mu(x),$$

with $c(x, p) := \int \omega(x, y) \, dp(y) + i(x, p)$. The cost $c$ is convex in $p$. 
Examples of weak optimal transport costs for which duality holds

**Example 4**: The martingale transport problem on the line.
Let $\mu, \nu \in \mathcal{P}(\mathbb{R})$ such that $\mu \leq_{\mathcal{C}} \nu$. According to Strassen Theorem,

$$
P^{\text{mart}}(\mu, \nu) := \left\{ \pi \in \Pi(\mu, \nu), \pi = \mu \otimes p, \int y dp_x(y) = x \mu\text{-almost surely} \right\} \neq \emptyset.
$$

By definition, the martingale optimal cost associated to $\omega : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is

$$
\mathcal{T}^{\text{mart}}_\omega(\nu | \mu) := \inf_{\pi \in P^{\text{mart}}(\mu, \nu)} \iint \omega(x, y) d\pi(x, y).
$$

How to express this martingale cost as a weak cost?

For $x \in \mathbb{R}, p \in \mathcal{P}_1(\mathbb{R})$, let $i(x, p) = \begin{cases} 0, & \text{if } \int y dp(y) = x, \\ +\infty, & \text{otherwise.} \end{cases}$

Observe that the function $i$ is convex in $p$, and one has

$$
\mathcal{T}^{\text{mart}}_\omega(\nu | \mu) = \inf_{\pi \in \Pi(\mu, \nu)} \left\{ \iint \omega(x, y) d\pi(x, y) + \int i(x, p_x) d\mu(x) \right\}
= \inf_{\pi \in \Pi(\mu, \nu)} \int c(x, p_x) d\mu(x),
$$

with $c(x, p) := \int \omega(x, y) dp(y) + i(x, p)$. The cost $c$ is convex in $p$.

The dual Kantorovich Theorem for weak cost applies.
Examples of weak optimal transport costs for which duality holds

**Example 4**: The martingale transport problem on the line.

Let $\mu, \nu \in \mathcal{P}(\mathbb{R})$ such that $\mu \preceq_{\mathcal{C}} \nu$. According to Strassen Theorem,

$$\Pi^{\text{mart}}_{\mu, \nu} := \left\{ \pi \in \Pi(\mu, \nu), \pi = \mu \otimes p, \int y dp_x(y) = x \mu\text{-almost surely} \right\} \neq \emptyset.$$

By definition, the martingale optimal cost associated to $\omega : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is

$$\mathcal{T}^{\text{mart}}_{\omega}(\nu|\mu) := \inf_{\pi \in \Pi^{\text{mart}}_{\mu, \nu}} \iint \omega(x, y) \, d\pi(x, y).$$

**How to express this martingale cost as a weak cost?**

For $x \in \mathbb{R}$, $p \in \mathcal{P}_1(\mathbb{R})$, let

$$i(x, p) = \begin{cases} 0, & \text{if } \int y \, dp(y) = x, \\ +\infty, & \text{otherwise}. \end{cases}$$

Observe that the function $i$ is convex in $p$, and one has

$$\mathcal{T}^{\text{mart}}_{\omega}(\nu|\mu) = \inf_{\pi \in \Pi(\mu, \nu)} \left\{ \iint \omega(x, y) \, d\pi(x, y) + \int i(x, p_x) \, d\mu(x) \right\}$$

$$= \inf_{\pi \in \Pi(\mu, \nu)} \int c(x, p_x) \, d\mu(x),$$

with $c(x, p) := \int \omega(x, y) \, dp(y) + i(x, p)$. The cost $c$ is convex in $p$.

The dual Kantorovich Theorem for weak cost applies and we recover the duality result by Beighböck-Henry-Labordère-Penker (2013).
Duality for martingale costs
Duality for martingale costs

Theorem: [B and al., 2013]

Let \( w : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) be a upper semi-continuous function, bounded from above.
Duality for martingale costs

**Theorem : [B and al., 2013]**

Let $w : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a upper semi-continuous function, bounded from above.

$$\sup_{\pi \in \Pi^{\text{mart}}(\mu, \nu)} \iint w \, d\pi = \inf_{f, g, h} \left\{ \int f \, d\mu + \int g \, d\nu \right\},$$
Duality for martingale costs

**Theorem** [B and al., 2013]

Let \( w : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) be a upper semi-continuous function, bounded from above. For all \( x, y \in \mathbb{R} \), \( \pi \) be a upper semi-continuous function, bounded from above.

\[
\sup_{\pi \in \Pi^{mart}(_{\mu, \nu})} \iint w \, d\pi = \inf_{f, g, h} \left\{ \int f \, d\mu + \int g \, d\nu \right\},
\]

where the infimum runs over all measurable bounded functions \( f, g, h \) such that for all \( x, y \in \mathbb{R} \), \( w(x, y) \leq f(x) + g(y) + h(x)(y - x) \).
Duality for martingale costs

**Theorem**: [B and al., 2013]

Let \( w : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) be a upper semi-continuous function, bounded from above.

\[
\sup_{\pi \in \Pi^{\text{mart}}(\mu, \nu)} \iint w \, d\pi = \inf_{f, g, h} \left\{ \int f \, d\mu + \int g \, d\nu \right\},
\]

where the infimum runs over all measurable bounded functions \( f, g, h \) such that for all \( x, y \in \mathbb{R} \), \( w(x, y) \leq f(x) + g(y) + h(x)(y - x) \).

**Idea of the proof**:
Duality for martingale costs

**Theorem**: [B and al., 2013]

Let \( w : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) be a upper semi-continuous function, bounded from above.

\[
\sup_{\pi \in \Pi^{mart}(\mu, \nu)} \iint w \, d\pi = \inf_{f, g, h} \left\{ \int f \, d\mu + \int g \, d\nu \right\},
\]

where the infimum runs over all measurable bounded functions \( f, g, h \) such that for all \( x, y \in \mathbb{R} \),

\[
w(x, y) \leq f(x) + g(y) + h(x)(y - x).
\]

**idea of the proof**: The inequality \( \leq \) is obvious since for all \( \pi \in \Pi^{mart}(\mu, \nu) \),

\[
\int h(x)(y - x) \, d\pi(x, y) = 0.
\]
Duality for martingale costs

**Theorem : [B and al., 2013]**

Let \( w : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) be a upper semi-continuous function, bounded from above.

\[
\sup_{\pi \in \Pi^{mart}(\mu, \nu)} \int \int w \, d\pi = \inf_{f, g, h} \left\{ \int f \, d\mu + \int g \, d\nu \right\},
\]

where the infimum runs over all measurable bounded functions \( f, g, h \) such that for all \( x, y \in \mathbb{R}, \quad w(x, y) \leq f(x) + g(y) + h(x)(y - x). \)

**Idea of the proof :** The inequality \( \leq \) is obvious since for all \( \pi \in \Pi^{mart}(\mu, \nu), \)

\[
\int h(x)(y - x) \, d\pi(x, y) = 0. \]

For the reverse inequality \( \geq : \) let \( \varepsilon > 0, \omega = -w. \)
Duality for martingale costs

**Theorem : [B and al., 2013]**

Let $w : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a upper semi-continuous function, bounded from above.

$$
\sup_{\pi \in \Pi^{mart}(\mu, \nu)} \iint w \, d\pi = \inf_{f, g, h} \left\{ \int f \, d\mu + \int g \, d\nu \right\},
$$

where the infimum runs over all measurable bounded functions $f, g, h$ such that for all $x, y \in \mathbb{R}$, $w(x, y) \leq f(x) + g(y) + h(x)(y - x)$.

**Idea of the proof:** The inequality $\leq$ is obvious since for all $\pi \in \Pi^{mart}(\mu, \nu)$,

$$
\int h(x)(y - x) \, d\pi(x, y) = 0.
$$

For the reverse inequality $\geq$: let $\varepsilon > 0$, $\omega = -w$.

$$
\sup_{\pi \in \Pi^{mart}(\mu, \nu)} \iint w \, d\pi = -\mathcal{W}^{mart}(\nu | \mu)
$$
Duality for martingale costs

**Theorem**: [B and al., 2013]

Let $w : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be a upper semi-continuous function, bounded from above.

$$\sup_{\pi \in \Pi^{mart}(\mu, \nu)} \iint w \, d\pi = \inf_{f, g, h} \left\{ \int f \, d\mu + \int g \, d\nu \right\},$$

where the infimum runs over all measurable bounded functions $f, g, h$ such that for all $x, y \in \mathbb{R}$, $w(x, y) \leq f(x) + g(y) + h(x)(y - x)$.

**idea of the proof**: The inequality $\leq$ is obvious since for all $\pi \in \Pi^{mart}(\mu, \nu)$,

$$\int h(x)(y - x) \, d\pi(x, y) = 0.$$ For the reverse inequality $\geq$: let $\varepsilon > 0$, $\omega = -w$.

$$\sup_{\pi \in \Pi^{mart}(\mu, \nu)} \iint w \, d\pi = -\mathcal{T}_{\omega}^{mart}(\nu | \mu) = \inf_{g} \left\{ \int (-Rc) \, d\nu + \int g \, d\mu \right\}$$
Duality for martingale costs

**Theorem : [B and al.,2013]**

Let \( w : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) be a upper semi-continuous function, bounded from above.

\[
\sup_{\pi \in \Pi^{mart}(\mu, \nu)} \int \int w \, d\pi = \inf_{f,g,h} \left\{ \int f \, d\mu + \int g \, d\nu \right\},
\]

where the infimum runs over all measurable bounded functions \( f, g, h \) such that for all \( x, y \in \mathbb{R} \), \( w(x, y) \leq f(x) + g(y) + h(x)(y - x) \).

**idea of the proof :** The inequality \( \leq \) is obvious since for all \( \pi \in \Pi^{mart}(\mu, \nu) \),

\[
\int h(x)(y - x) \, d\pi(x, y) = 0.
\]

For the reverse inequality \( \geq \) : let \( \varepsilon > 0 \), \( \omega = -w \).

\[
\sup_{\pi \in \Pi^{mart}(\mu, \nu)} \int \int w \, d\pi = -\mathcal{T}_\omega^{mart}(\nu | \mu) = \inf_g \left\{ \int (-R_c g) \, d\nu + \int g \, d\mu \right\}
\]

\[
\geq \int (-R_c g_0) \, d\nu + \int g_0 \, d\mu - \varepsilon.
\]
**Duality for martingale costs**

**Theorem : [B and al.,2013]**

Let \( w : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \) be a upper semi-continuous function, bounded from above.

\[
\sup_{\pi \in \Pi^{mart}(\mu, \nu)} \int \int w \, d\pi = \inf_{f,g,h} \left\{ \int f \, d\mu + \int g \, d\nu \right\},
\]

where the infimum runs over all measurable bounded functions \( f, g, h \) such that for all \( x, y \in \mathbb{R} \), \( w(x, y) \leq f(x) + g(y) + h(x)(y - x) \).

**idea of the proof**: The inequality \( \leq \) is obvious since for all \( \pi \in \Pi^{mart}(\mu, \nu) \),

\[
\int h(x)(y - x) \, d\pi(x, y) = 0.
\]

For the reverse inequality \( \geq \): let \( \varepsilon > 0 \), \( \omega = -w \).

\[
\sup_{\pi \in \Pi^{mart}(\mu, \nu)} \int \int w \, d\pi \geq \int (-Rc g_0) \, d\nu + \int g_0 \, d\mu - \varepsilon.
\]

Observe that \( i(x, p) = \sup_{\gamma \in \mathbb{R}} \gamma \cdot \left( \int y \, dp(y) - x \right) \),
Duality for martingale costs

**Theorem : [B and al.,2013]**

Let $w : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be a upper semi-continuous function, bounded from above.

$$
\sup_{\pi \in \Pi^{\text{mart}}(\mu, \nu)} \int \int w \ d\pi = \inf_{f, g, h} \left\{ \int f \ d\mu + \int g \ d\nu \right\},
$$

where the infimum runs over all measurable bounded functions $f, g, h$ such that for all $x, y \in \mathbb{R}$, $w(x, y) \leq f(x) + g(y) + h(x)(y - x)$.

**idea of the proof :** The inequality $\leq$ is obvious since for all $\pi \in \Pi^{\text{mart}}(\mu, \nu)$,

$$
\int h(x)(y - x) \ d\pi(x, y) = 0.
$$

For the reverse inequality $\geq$ : let $\varepsilon > 0$, $\omega = -w$.

$$
\sup_{\pi \in \Pi^{\text{mart}}(\mu, \nu)} \int \int w \ d\pi \geq \int \left( -R_c g_0 \right) \ d\nu + \int g_0 \ d\mu - \varepsilon.
$$

Observe that $i(x, p) = \sup_{\gamma} \gamma \cdot \left( \int y \ dp(y) - x \right)$, it follows that

$$
f_0(x) := -R_c g_0(x)
$$
Duality for martingale costs

**Theorem : [B and al.,2013]**

Let $w : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a upper semi-continuous function, bounded from above.

$$
\sup_{\pi \in \Pi^{mart}(\mu, \nu)} \iint w \ d\pi = \inf_{f, g, h} \left\{ \int f \ d\mu + \int g \ d\nu \right\},
$$

where the infimum runs over all measurable bounded functions $f, g, h$ such that for all $x, y \in \mathbb{R}$, $w(x, y) \leq f(x) + g(y) + h(x)(y - x)$.

**idea of the proof :** The inequality $\leq$ is obvious since for all $\pi \in \Pi^{mart}(\mu, \nu)$,

$$
\int h(x)(y - x) \ d\pi(x, y) = 0.
$$

For the reverse inequality $\geq$: let $\varepsilon > 0$, $\omega = -w$.

$$
\sup_{\pi \in \Pi^{mart}(\mu, \nu)} \iint w \ d\pi \geq \int ( - R_c g_0) \ d\nu + \int g_0 \ d\mu - \varepsilon.
$$

Observe that $i(x, p) = \sup_{\gamma \in \mathbb{R}} \gamma \cdot \left( \int y \ dp(y) - x \right)$, it follows that

$$
f_0(x) := -R_c g_0(x) = \sup_{p} \inf_{\gamma \in \mathbb{R}} \left\{ - \int g_0 \ dp + \int w(x, y) \ dp(y) - \int \gamma \cdot (y - x) \ dp(y) \right\}
$$
Duality for martingale costs

**Theorem : [B and al.,2013]**

Let \( w : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \) be a upper semi-continuous function, bounded from above.

\[
\sup_{\pi \in \Pi^{mart}(\mu, \nu)} \int \int w \, d\pi = \inf_{f,g,h} \left\{ \int f \, d\mu + \int g \, d\nu \right\},
\]

where the infimum runs over all measurable bounded functions \( f, g, h \) such that for all \( x, y \in \mathbb{R} \),

\[ w(x, y) \leq f(x) + g(y) + h(x)(y - x). \]

**idea of the proof :** The inequality \( \leq \) is obvious since for all \( \pi \in \Pi^{mart}(\mu, \nu) \),

\[ \int h(x)(y - x) \, d\pi(x, y) = 0. \]

For the reverse inequality \( \geq \) : let \( \varepsilon > 0, \omega = -w \).

\[
\sup_{\pi \in \Pi^{mart}(\mu, \nu)} \int \int w \, d\pi \geq \int (-R_c g_0) \, d\nu + \int g_0 \, d\mu - \varepsilon.
\]

Observe that

\[ i(x, \rho) = \sup_{\gamma \in \mathbb{R}} \gamma \cdot \left( \int y \, d\rho(y) - x \right) \]

it follows that

\[
f_0(x) := -R_c g_0(x) = \sup_{\rho} \inf_{\gamma \in \mathbb{R}} \left\{ -\int g_0 \, d\rho + \int w(x, y) \, d\rho(y) - \int \gamma \cdot (y - x) \, d\rho(y) \right\} \]

\[ = \inf_{\gamma \in \mathbb{R}} \sup_{y} \left\{ -g_0(y) + w(x, y) - \gamma \cdot (y - x) \right\} \]
Duality for martingale costs

**Theorem : [B and al.,2013]**

Let \( w : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) be a upper semi-continuous function, bounded from above.

\[
\sup_{\pi \in \Pi^{mart}(\mu, \nu)} \iint w \, d\pi = \inf_{f, g, h} \left\{ \int f \, d\mu + \int g \, d\nu \right\},
\]

where the infimum runs over all measurable bounded functions \( f, g, h \) such that for all \( x, y \in \mathbb{R} \), \( w(x, y) \leq f(x) + g(y) + h(x)(y - x) \).

**idea of the proof :** The inequality \( \leq \) is obvious since for all \( \pi \in \Pi^{mart}(\mu, \nu) \),

\[
\int h(x)(y - x) \, d\pi(x, y) = 0.
\]

For the reverse inequality \( \geq \) : let \( \varepsilon > 0, \omega = -w \).

\[
\sup_{\pi \in \Pi^{mart}(\mu, \nu)} \iint w \, d\pi \geq \int ( -R_c g_0 ) \, d\nu + \int g_0 \, d\mu - \varepsilon.
\]

Observe that \( i(x, p) = \sup_{\gamma \in \mathbb{R}} \gamma \cdot \left( \int y \, dp(y) - x \right) \), it follows that

\[
f_0(x) := -R_c g_0(x) = \sup_{\gamma \in \mathbb{R}} \inf_{p} \left\{ -\int g_0 \, dp + \int w(x, y) \, dp(y) - \int \gamma \cdot (y - x) \, dp(y) \right\}
\]

\[
= \inf_{\gamma \in \mathbb{R}} \sup_{y} \left\{ -g_0(y) + w(x, y) - \gamma \cdot (y - x) \right\}
\]

\[
\geq -g_0(y) + w(x, y) - \gamma(x) \cdot (y - x) - \varepsilon.
\]
Duality for martingale costs

**Theorem:** [B and al., 2013]

Let $w : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a upper semi-continuous function, bounded from above.

$$\sup_{\pi \in \Pi^{\text{mart}}(\mu, \nu)} \iint w \, d\pi = \inf_{f, g, h} \left\{ \int f \, d\mu + \int g \, d\nu \right\},$$

where the infimum runs over all measurable bounded functions $f, g, h$ such that for all $x, y \in \mathbb{R}$, $w(x, y) \leq f(x) + g(y) + h(x)(y - x)$.

**idea of the proof:** The inequality $\leq$ is obvious since for all $\pi \in \Pi^{\text{mart}}(\mu, \nu)$,

$$\int h(x)(y - x) \, d\pi(x, y) = 0.$$ For the reverse inequality $\geq$ : let $\varepsilon > 0$, $\omega = -w$.

$$\sup_{\pi \in \Pi^{\text{mart}}(\mu, \nu)} \iint w \, d\pi \geq \int ( - R_c g_0 ) \, d\nu + \int g_0 \, d\mu - \varepsilon.$$

Observe that $i(x, p) = \sup_{\gamma \in \mathbb{R}} \gamma \cdot \left( \int y \, dp(y) - x \right)$, it follows that

$$f_0(x) := -R_c g_0(x) = \sup_{p} \inf_{\gamma \in \mathbb{R}} \left\{ - \int g_0 \, dp + \int w(x, y) \, dp(y) - \int \gamma \cdot (y - x) \, dp(y) \right\}$$

$$= \inf_{\gamma \in \mathbb{R}} \sup_{y} \left\{ - g_0(y) + w(x, y) - \gamma \cdot (y - x) \right\}$$

$$\geq - g_0(y) + w(x, y) - \gamma(x) \cdot (y - x) - \varepsilon.$$
Duality for martingale costs

Theorem: [B and al., 2013]

Let $w : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a upper semi-continuous function, bounded from above.

$$\sup_{\pi \in \Pi^{\text{mart}}(\mu, \nu)} \iint w \, d\pi = \inf_{f, g, h} \left\{ \int f \, d\mu + \int g \, d\nu \right\},$$

where the infimum runs over all measurable bounded functions $f, g, h$ such that for all $x, y \in \mathbb{R}$, $w(x, y) \leq f(x) + g(y) + h(x)(y - x)$.

\textbf{Idea of the proof}: The inequality $\leq$ is obvious since for all $\pi \in \Pi^{\text{mart}}(\mu, \nu)$,

$$\int h(x)(y - x) \, d\pi(x, y) = 0.$$  

For the reverse inequality $\geq$ : let $\varepsilon > 0$, $\omega = -w$.

$$\sup_{\pi \in \Pi^{\text{mart}}(\mu, \nu)} \iint w \, d\pi \geq \int (-R_c g_0) \, d\nu + \int g_0 \, d\mu - \varepsilon.$$

Observe that $i(x, p) = \sup_{\gamma \in \mathbb{R}} \gamma \cdot \left( \int y \, dp(y) - x \right)$, it follows that

$$f_0(x) := -R_c g_0(x) = \sup_{p} \inf_{\gamma \in \mathbb{R}} \left\{ -\int g_0 \, dp + \int w(x, y) \, dp(y) - \int \gamma \cdot (y - x) \, dp(y) \right\}$$

$$= \inf_{\gamma} \sup_{y} \left\{ -g_0(y) + w(x, y) - \gamma \cdot (y - x) \right\}$$

$$\geq -g_0(y) + w(x, y) - \gamma(x) \cdot (y - x) - \varepsilon.$$

$$\sup_{\pi \in \Pi^{\text{mart}}(\mu, \nu)} \iint w \, d\pi \geq \inf_{f_0, g_0, \gamma} \left\{ \int f_0 \, d\mu + \int g_0 \, d\nu \right\} - \varepsilon,$$

over all $f_0, g_0, \gamma$, $f_0(x) + g_0(y) + \gamma(x) \cdot (y - x) + \varepsilon \geq w(x, y)$. 

\textbf{Transport inequality on the symmetric group}

\textbf{Universal transport inequalities}

\textbf{Barycentric transport inequalities}

\textbf{Examples in discrete}

\textbf{Weak transport costs.14}
Applications of duality to transport-entropy inequalities and concentration

Definition: Weak transport-entropy inequality $T_{c \alpha 1, \alpha 2 q}$

- The measure $\mu$ satisfies the transport-entropy inequality $T_{c \alpha 1, \alpha 2 q}$ if $\nu_1, \nu_2$.

Marton's inequality:

$T_{2 p \nu_2 | \nu_1 q d} (\alpha_1, \alpha_2 q \mathbb{P})$.

Proposition: Dual characterization for weak transport-entropy inequalities.

If the Kantorovich duality holds for the weak cost $T_{c \alpha 1, \alpha 2 q}$, then the following statements are equivalent:

1. $\mu$ satisfies $T_{c \alpha 1, \alpha 2 q}$
2. For all functions $\varphi \in \Phi$, $\mathbb{E}_{c \varphi \alpha_2 d \mu} \tilde{a}_2 \mathbb{E}_{c \varphi \alpha_1 d \mu} \tilde{a}_1 \leq 1$.

ii) is a generalisation of the so-called (convex) $\tau$-property introduced by Maurey (1990) to recover Talagrand's concentration results.
Applications of duality to transport-entropy inequalities and concentration

Definition: Weak transport-entropy inequality $T_c(a_1, a_2)$
### Applications of duality to transport-entropy inequalities and concentration

**Definition : Weak transport-entropy inequality** $T_c(a_1, a_2)$

The measure $\mu \in P_\gamma(X')$ satisfies the transport-entropy inequality $T_c(a_1, a_2)$, $a_1, a_2 > 0$, etc.
**Applications of duality to transport-entropy inequalities and concentration**

**Definition : Weak transport-entropy inequality** $T_c(a_1, a_2)$

The measure $\mu \in \mathcal{P}_\gamma(X)$ satisfies the transport-entropy inequality $T_c(a_1, a_2)$, $a_1, a_2 > 0$,

$$T_c(\nu_1|\nu_2) \leq a_1 H(\nu_1|\mu) + a_2 H(\nu_2|\mu) \quad \nu_1, \nu_2 \in \mathcal{P}_\gamma(X).$$
Applications of duality to transport-entropy inequalities and concentration

**Definition : Weak transport-entropy inequality** $T_c(a_1, a_2)$

The measure $\mu \in \mathcal{P}_\gamma(\mathcal{X})$ satisfies the transport-entropy inequality $T_c(a_1, a_2)$, $a_1, a_2 > 0$,

$$T_c(\nu_1|\nu_2) \leq a_1 H(\nu_1|\mu) + a_2 H(\nu_2|\mu) \quad \nu_1, \nu_2 \in \mathcal{P}_\gamma(\mathcal{X}).$$
Applications of duality to transport-entropy inequalities and concentration

**Definition : Weak transport-entropy inequality** $T_c(a_1, a_2)$

The measure $\mu \in \mathcal{P}_\gamma(\mathcal{X})$ satisfies the transport-entropy inequality $T_c(a_1, a_2)$, $a_1, a_2 > 0$,

$$T_c(\nu_1|\nu_2) \leq a_1 H(\nu_1|\mu) + a_2 H(\nu_2|\mu) \quad \nu_1, \nu_2 \in \mathcal{P}_\gamma(\mathcal{X}).$$

**Marton’s inequality** : $\tilde{T}_2(\nu_2|\nu_1) \leq \frac{2}{s} H(\nu_1|\mu^n) + \frac{2}{1-s} H(\nu_2|\mu^n), \forall s \in (0, 1).$
Applications of duality to transport-entropy inequalities and concentration

**Definition : Weak transport-entropy inequality** $T_c(a_1, a_2)$

The measure $\mu \in \mathcal{P}_\gamma(X)$ satisfies the transport-entropy inequality $T_c(a_1, a_2)$, $a_1, a_2 > 0$,

$$T_c(\nu_1|\nu_2) \leq a_1 H(\nu_1|\mu) + a_2 H(\nu_2|\mu) \quad \nu_1, \nu_2 \in \mathcal{P}_\gamma(X).$$

**Marton’s inequality :** $\tilde{T}_2(\nu_2|\nu_1) \leq \frac{2}{s} H(\nu_1|\mu^n) + \frac{2}{1-s} H(\nu_2|\mu^n), \ \forall s \in (0, 1)$.

**Proposition : Dual characterization for weak transport-entropy inequalities.**
Applications of duality to transport-entropy inequalities and concentration

**Definition : Weak transport-entropy inequality $T_c(a_1, a_2)$**

The measure $\mu \in \mathcal{P}_\gamma(X)$ satisfies the transport-entropy inequality $T_c(a_1, a_2)$, $a_1, a_2 > 0$,

$$T_c(\nu_1|\nu_2) \leq a_1 H(\nu_1|\mu) + a_2 H(\nu_2|\mu) \quad \nu_1, \nu_2 \in \mathcal{P}_\gamma(X).$$

**Marton’s inequality :** $\tilde{T}_2(\nu_2|\nu_1) \leq \frac{2}{s} H(\nu_1|\mu^n) + \frac{2}{1-s} H(\nu_2|\mu^n), \forall s \in (0, 1)$.

**Proposition : Dual characterization for weak transport-entropy inequalities.**

If the Kantorovich duality holds for the weak cost $T_c$, then the following statements are equivalents :

- **i)** $\mu$ satisfies $T_c(a_1, a_2)$,
- **ii)** For all functions $\phi \in \Phi_{\gamma, b}$,

$$\int_X \phi d\mu \leq \text{inf} \left\{ \int_X \phi d\nu : \nu \in \mathcal{P}_\gamma(X), \right\}$$

**ii)** is a generalisation of the so-called (convex) $\tau$-property introduced by Maurey (1990) to recover Talagrand’s concentration results.
Applications of duality to transport-entropy inequalities and concentration

**Definition : Weak transport-entropy inequality $T_c(a_1, a_2)$**

The measure $\mu \in \mathcal{P}_\gamma(X)$ satisfies the transport-entropy inequality $T_c(a_1, a_2)$, $a_1, a_2 > 0$,

$$T_c(\nu_1 | \nu_2) \leq a_1 H(\nu_1 | \mu) + a_2 H(\nu_2 | \mu) \quad \nu_1, \nu_2 \in \mathcal{P}_\gamma(X).$$

**Marton’s inequality :** $\tilde{T}_2(\nu_2 | \nu_1) \leq 2s H(\nu_1 | \mu^n) + \frac{2}{1-s} H(\nu_2 | \mu^n), \forall s \in (0, 1).$

**Proposition : Dual characterization for weak transport-entropy inequalities.**

If the Kantorovich duality holds for the weak cost $T_c$, then the following statements are equivalents :

- i) $\mu$ satisfies $T_c(a_1, a_2)$ ($a_1, a_2 > 0$)
Applications of duality to transport-entropy inequalities and concentration

**Definition : Weak transport-entropy inequality** $T_c(a_1, a_2)$

The measure $\mu \in \mathcal{P}(\mathcal{X})$ satisfies the transport-entropy inequality $T_c(a_1, a_2)$, $a_1, a_2 > 0$,

$$T_c(\nu_1|\nu_2) \leq a_1 H(\nu_1|\mu) + a_2 H(\nu_2|\mu) \quad \nu_1, \nu_2 \in \mathcal{P}(\mathcal{X}).$$

**Marton’s inequality** : $\tilde{T}_2(\nu_2|\nu_1) \leq \frac{2}{s} H(\nu_1|\mu^n) + \frac{2}{1-s} H(\nu_2|\mu^n), \forall s \in (0, 1)$.

**Proposition : Dual characterization for weak transport-entropy inequalities.**

If the Kantorovich duality holds for the weak cost $T_c$, then the following statements are equivalents :

i) $\mu$ satisfies $T_c(a_1, a_2)$ ($a_1, a_2 > 0$)

ii) For all functions $\varphi \in \Phi_{\gamma,b}(\mathcal{X})$,

$$\left( \int e^{\frac{R_c\varphi}{a_2}} d\mu \right)^{a_2} \left( \int e^{-\frac{\varphi}{a_1}} d\mu \right)^{a_1} \leq 1$$

$$R_c\varphi(x) = \inf_{p \in \mathcal{P}(\mathcal{X})} \left\{ \int \varphi(y) dp(y) + c(x, p) \right\}, \quad x \in \mathcal{X}.$$
Applications of duality to transport-entropy inequalities and concentration

**Definition : Weak transport-entropy inequality** $T_c(a_1, a_2)$

The measure $\mu \in P_\gamma(\mathcal{X})$ satisfies the transport-entropy inequality $T_c(a_1, a_2)$, $a_1, a_2 > 0$,

$$T_c(\nu_1 | \nu_2) \leq a_1 H(\nu_1 | \mu) + a_2 H(\nu_2 | \mu), \quad \nu_1, \nu_2 \in P_\gamma(\mathcal{X}).$$

**Marton’s inequality** : $\tilde{T}_2(\nu_2 | \nu_1) \leq \frac{2}{s} H(\nu_1 | \mu^n) + \frac{2}{1-s} H(\nu_2 | \mu^n), \quad \forall s \in (0, 1)$.

**Proposition : Dual characterization for weak transport-entropy inequalities.**

If the Kantorovich duality holds for the weak cost $T_c$, then the following statements are equivalents:

i) $\mu$ satisfies $T_c(a_1, a_2)$ ($a_1, a_2 > 0$)

ii) For all functions $\varphi \in \Phi_{\gamma,b}(\mathcal{X})$,

$$\left( \int e^{\varphi \mu} \frac{a_2}{a_1} d\mu \right)^{a_2} \left( \int e^{-\varphi \mu} \frac{a_1}{a_2} d\mu \right)^{a_1} \leq 1$$

$$R_c \varphi(x) = \inf_{p \in P_\gamma(\mathcal{X})} \left\{ \int \varphi(y) dp(y) + c(x, p) \right\}, \quad x \in \mathcal{X}.$$

ii) is a generalisation of the so-called (convex) $\tau$-property introduced by Maurey (1990) to recover Talagrand’s concentration results.
Idea of the proof (Bobkov-Götze 1999)
Idea of the proof (Bobkov-Götze 1999)

Assume that for all $\nu_1, \nu_2 \in \mathcal{P}_\gamma(X)$,

$$T_c(\nu_1|\nu_2) \leq a_1 H(\nu_1|\mu) + a_2 H(\nu_2|\mu)$$
Idea of the proof (Bobkov-Götze 1999)

Assume that for all $\nu_1, \nu_2 \in \mathcal{P}_\gamma(X)$,

$$\mathcal{T}_c(\nu_1 | \nu_2) = \sup_{\varphi \in \Phi_{\gamma,b}(X)} \left\{ \int R_c \varphi \, d\nu_2 - \int \varphi \, d\nu_1 \right\} \leq a_1 H(\nu_1 | \mu) + a_2 H(\nu_2 | \mu),$$
Idea of the proof (Bobkov-Götze 1999)

Assume that for all \( \nu_1, \nu_2 \in \mathcal{P}_\gamma(\mathcal{X}) \),

\[
\mathcal{T}_c(\nu_1 | \nu_2) = \sup_{\varphi \in \Phi_{\gamma,b}(\mathcal{X})} \left\{ \int R_c \varphi \, d\nu_2 - \int \varphi \, d\nu_1 \right\} \leq a_1 H(\nu_1 | \mu) + a_2 H(\nu_2 | \mu),
\]

Therefore, for all \( \varphi \in \Phi_{\gamma,b}(\mathcal{X}) \), and all \( \nu_1, \nu_2 \in \mathcal{P}_\gamma(\mathcal{X}) \)
Idea of the proof (Bobkov-Götze 1999)

Assume that for all $\nu_1, \nu_2 \in \mathcal{P}_\gamma(\mathcal{X})$,

$$\mathcal{T}_c(\nu_1 | \nu_2) = \sup_{\varphi \in \Phi_{\gamma,b}(\mathcal{X})} \left\{ \int R_c \varphi \, d\nu_2 - \int \varphi \, d\nu_1 \right\} \leq a_1 H(\nu_1 | \mu) + a_2 H(\nu_2 | \mu),$$

Therefore, for all $\varphi \in \Phi_{\gamma,b}(\mathcal{X})$, and all $\nu_1, \nu_2 \in \mathcal{P}_\gamma(\mathcal{X})$

$$a_2 \left( \int \frac{R_c \varphi}{a_2} \, d\nu_2 - H(\nu_2 | \mu) \right) + a_1 \left( \int -\frac{\varphi}{a_1} \, d\nu_1 - H(\nu_1 | \mu) \right) \leq 0.$$
Idea of the proof (Bobkov-Götze 1999)

Assume that for all $\nu_1, \nu_2 \in \mathcal{P}_\gamma(\mathcal{X})$,

$$\mathcal{T}_c(\nu_1 | \nu_2) = \sup_{\varphi \in \Phi_{\gamma,b}(\mathcal{X})} \left\{ \int R_c \varphi \, d\nu_2 - \int \varphi \, d\nu_1 \right\} \leq a_1 H(\nu_1 | \mu) + a_2 H(\nu_2 | \mu),$$

Therefore, for all $\varphi \in \Phi_{\gamma,b}(\mathcal{X})$, and all $\nu_1, \nu_2 \in \mathcal{P}_\gamma(\mathcal{X})$

$$a_2 \left( \int \frac{R_c \varphi}{a_2} \, d\nu_2 - H(\nu_2 | \mu) \right) + a_1 \left( \int -\frac{\varphi}{a_1} \, d\nu_1 - H(\nu_1 | \mu) \right) \leq 0.$$
Idea of the proof (Bobkov-Götze 1999)

Assume that for all $\nu_1, \nu_2 \in \mathcal{P}_\gamma(X)$,

$$T_c(\nu_1|\nu_2) = \sup_{\varphi \in \Phi_{\gamma,b}(X)} \left\{ \int R_c \varphi \, d\nu_2 - \int \varphi \, d\nu_1 \right\} \leq a_1 H(\nu_1|\mu) + a_2 H(\nu_2|\mu),$$

Therefore, for all $\varphi \in \Phi_{\gamma,b}(X)$, and all $\nu_1, \nu_2 \in \mathcal{P}_\gamma(X)$

$$a_2 \left( \int \frac{R_c \varphi}{a_2} \, d\nu_2 - H(\nu_2|\mu) \right) + a_1 \left( \int -\frac{\varphi}{a_1} \, d\nu_1 - H(\nu_1|\mu) \right) \leq 0.$$
Idea of the proof (Bobkov-Götze 1999)

Assume that for all $\nu_1, \nu_2 \in \mathcal{P}_\gamma(\mathcal{X})$,

$$
\mathcal{T}_c(\nu_1 | \nu_2) = \sup_{\varphi \in \Phi_{\gamma,b}(\mathcal{X})} \left\{ \int R_c \varphi \, d\nu_2 - \int \varphi \, d\nu_1 \right\} \leq a_1 H(\nu_1 | \mu) + a_2 H(\nu_2 | \mu),
$$

Therefore, for all $\varphi \in \Phi_{\gamma,b}(\mathcal{X})$, and all $\nu_1, \nu_2 \in \mathcal{P}_\gamma(\mathcal{X})$

$$
a_2 \left( \int \frac{R_c \varphi}{a_2} \, d\nu_2 - H(\nu_2 | \mu) \right) + a_1 \left( \int \frac{\varphi}{a_1} \, d\nu_1 - H(\nu_1 | \mu) \right) \leq 0.
$$

By optimizing over all $\nu_1, \nu_2$ we get
Idea of the proof (Bobkov-Götze 1999)

Assume that for all $\nu_1, \nu_2 \in \mathcal{P}_\gamma(X)$,

$$T_c(\nu_1|\nu_2) = \sup_{\varphi \in \Phi_{\gamma,b}(X)} \left\{ \int R_c \varphi \, d\nu_2 - \int \varphi \, d\nu_1 \right\} \leq a_1 H(\nu_1|\mu) + a_2 H(\nu_2|\mu),$$

Therefore, for all $\varphi \in \Phi_{\gamma,b}(X)$,

$$a_2 \sup_{\nu_2} \left\{ \int \frac{R_c \varphi}{a_2} \, d\nu_2 - H(\nu_2|\mu) \right\} + a_1 \sup_{\nu_1} \left\{ \int -\frac{\varphi}{a_1} \, d\nu_1 - H(\nu_1|\mu) \right\} \leq 0.$$
Idea of the proof (Bobkov-Götze 1999)

Assume that for all \( \nu_1, \nu_2 \in \mathcal{P}_\gamma(X) \),

\[
\mathcal{T}_c(\nu_1 | \nu_2) = \sup_{\varphi \in \Phi_{\gamma,b}(X)} \left\{ \int R_c \varphi \, d\nu_2 - \int \varphi \, d\nu_1 \right\} \leq a_1 H(\nu_1 | \mu) + a_2 H(\nu_2 | \mu),
\]

Therefore, for all \( \varphi \in \Phi_{\gamma,b}(X) \),

\[
a_2 \sup_{\nu_2} \left\{ \int \frac{R_c \varphi}{a_2} \, d\nu_2 - H(\nu_2 | \mu) \right\} + a_1 \sup_{\nu_1} \left\{ \int -\frac{\varphi}{a_1} \, d\nu_1 - H(\nu_1 | \mu) \right\} \leq 0.
\]

Since \( \sup_{\nu \in \mathcal{P}_\gamma(X)} \left\{ \int \psi \, d\nu - H(\nu | \mu) \right\} = \log \int e^\psi \, d\mu, \quad \forall \psi \in \Phi_{\gamma,b}(X) \),

Therefore, for all \( \varphi \in \Phi_{\gamma,b}(X) \),

\[
a_2 \sup_{\nu_2} \left\{ \int \frac{R_c \varphi}{a_2} \, d\nu_2 - H(\nu_2 | \mu) \right\} + a_1 \sup_{\nu_1} \left\{ \int -\frac{\varphi}{a_1} \, d\nu_1 - H(\nu_1 | \mu) \right\} \leq 0.
\]

or equivalently

\[
\hat{\varphi} e^{\int \frac{R_c \varphi}{a_2} \, d\nu_2} \hat{\varphi} - a_2 H(\nu_2 | \mu) \leq 0.
\]

Since \( \sup_{\nu \in \mathcal{P}_\gamma(X)} \left\{ \int \psi \, d\nu - H(\nu | \mu) \right\} = \log \int e^\psi \, d\mu, \quad \forall \psi \in \Phi_{\gamma,b}(X) \),
Idea of the proof (Bobkov-Götze 1999)

Assume that for all $\nu_1, \nu_2 \in \mathcal{P}_\gamma(X)$,

$$
\mathcal{T}_c(\nu_1|\nu_2) = \sup_{\varphi \in \Phi_{\gamma,b}(X)} \left\{ \int R_c \varphi\, d\nu_2 - \int \varphi\, d\nu_1 \right\} \leq a_1 H(\nu_1|\mu) + a_2 H(\nu_2|\mu),
$$

Therefore, for all $\varphi \in \Phi_{\gamma,b}(X)$,

$$
a_2 \sup_{\nu_2} \left\{ \int \frac{R_c \varphi}{a_2}\, d\nu_2 - H(\nu_2|\mu) \right\} + a_1 \sup_{\nu_1} \left\{ \int -\frac{\varphi}{a_1}\, d\nu_1 - H(\nu_1|\mu) \right\} \leq 0.
$$

Since $\sup_{\nu \in \mathcal{P}_\gamma(X)} \left\{ \int \psi\, d\nu - H(\nu|\mu) \right\} = \log \int e^\psi\, d\mu$, $\forall \psi \in \Phi_{\gamma,b}(X)$,

it follows that

$$
a_2 \log \int e^{R_c \varphi/a_2}\, d\mu + a_1 \log \int e^{-\varphi/a_1}\, d\mu \leq 0.
$$
Idea of the proof (Bobkov-Götze 1999)

Assume that for all $\nu_1, \nu_2 \in \mathcal{P}_\gamma(X)$,

$$
\mathcal{T}_c(\nu_1 | \nu_2) = \sup_{\varphi \in \Phi_{\gamma,b}(X)} \left\{ \int R_c \varphi \, d\nu_2 - \int \varphi \, d\nu_1 \right\} \leq a_1 H(\nu_1 | \mu) + a_2 H(\nu_2 | \mu),
$$

Therefore, for all $\varphi \in \Phi_{\gamma,b}(X)$,

$$
a_2 \sup_{\nu_2} \left\{ \int \frac{R_c \varphi}{a_2} \, d\nu_2 - H(\nu_2 | \mu) \right\} + a_1 \sup_{\nu_1} \left\{ \int -\frac{\varphi}{a_1} \, d\nu_1 - H(\nu_1 | \mu) \right\} \leq 0.
$$

Since

$$
\sup_{\nu \in \mathcal{P}_\gamma(X)} \left\{ \int \psi \, d\nu - H(\nu | \mu) \right\} = \log \int e^{\psi} \, d\mu, \quad \forall \psi \in \Phi_{\gamma,b}(X),
$$

it follows that

$$
a_2 \log \int e^{R_c \varphi / a_2} \, d\mu + a_1 \log \int e^{-\varphi / a_1} \, d\mu \leq 0.
$$

or equivalently

$$
\left( \int e^{R_c \varphi / a_2} \, d\mu \right)^{a_2} \left( \int e^{-\varphi / a_1} \, d\mu \right)^{a_1} \leq 1.
$$
From dual characterization of transport-entropy inequality to concentration
From dual characterization of transport-entropy inequality to concentration

We assume that for all measurable functions $\varphi : \mathcal{X} \to \mathbb{R} \cup \{+\infty\}$ bounded from below

\[
\left( \int e^{\frac{Rc\varphi}{a_2}} \, d\mu \right)^{a_2} \left( \int e^{\frac{-\varphi}{a_1}} \, d\mu \right)^{a_1} \leq 1
\]

where $Rc\varphi(x) = \inf_{p \in \mathcal{P}_\gamma(\mathcal{X})} \left\{ \int \varphi \, dp + c(x, p) \right\}, \quad x \in \mathcal{X}.$
From dual characterization of transport-entropy inequality to concentration

We assume that for all measurable functions $\varphi : \mathcal{X} \to \mathbb{R} \cup \{+\infty\}$ bounded from below

$$\left(\int e^{\frac{R_c \varphi}{a_2}} d\mu\right)^{a_2} \left(\int e^{-\frac{\varphi}{a_1}} d\mu\right)^{a_1} \leq 1$$

where $R_c \varphi(x) = \inf_{p \in \mathcal{P}_\gamma(\mathcal{X})} \left\{ \int \varphi \, dp + c(x, p) \right\}$, $x \in \mathcal{X}$.

Let $A \subset \mathcal{X}$.
From dual characterization of transport-entropy inequality to concentration

We assume that for all measurable functions \( \varphi : \mathcal{X} \to \mathbb{R} \cup \{ +\infty \} \) bounded from below

\[
\left( \int e^{\frac{R_c \varphi}{a_2}} d\mu \right)^{a_2} \left( \int e^{-\frac{\varphi}{a_1}} d\mu \right)^{a_1} \leq 1
\]

where \( R_c \varphi(x) = \inf_{p \in \mathcal{P}_c(\mathcal{X})} \left\{ \int \varphi \, dp + c(x, p) \right\} \), \( x \in \mathcal{X} \).

Let \( A \subset \mathcal{X} \). Applying this inequality to the function

\[
\varphi(x) = i_A(x) := \begin{cases} 
0 & \text{if } x \in A, \\
+\infty & \text{otherwise},
\end{cases}
\]
From dual characterization of transport-entropy inequality to concentration

We assume that for all measurable functions $\varphi : X \to \mathbb{R} \cup \{+\infty\}$ bounded from below

$$\left( \int e^{\frac{R_{c} \varphi_{X}}{a_2}} d\mu \right)^{a_2} \left( \int e^{-\frac{\varphi_{X}}{a_1}} d\mu \right)^{a_1} \leq 1$$

where

$$R_{c} \varphi_{X}(x) = \inf_{p \in \mathcal{P}_{X}} \left\{ \int \varphi dp + c(x, p) \right\}, \quad x \in X.$$

Let $A \subset X$. Applying this inequality to the function

$$\varphi(x) = i_{A}(x) := \begin{cases} 0 & \text{if } x \in A, \\ +\infty & \text{otherwise}, \end{cases}$$

since

$$\int e^{-\frac{i_{A}}{a_1}} d\mu = \mu(A),$$
From dual characterization of transport-entropy inequality to concentration

We assume that for all measurable functions \( \varphi : X \to \mathbb{R} \cup \{+\infty\} \) bounded from below

\[
\left( \int e^{\frac{R_c \varphi}{a_2}} d\mu \right)^{a_2} \left( \int e^{-\frac{\varphi}{a_1}} d\mu \right)^{a_1} \leq 1
\]

where \( R_c \varphi(x) = \inf_{p \in \mathcal{P}(X)} \left\{ \int \varphi \, dp + c(x, p) \right\} \), \( x \in X \).

Let \( A \subset X \). Applying this inequality to the function

\[
\varphi(x) = i_A(x) := \begin{cases} 
0 & \text{if } x \in A, \\
+\infty & \text{otherwise},
\end{cases}
\]

since \( \int e^{-\frac{i_A}{a_1}} d\mu = \mu(A) \),

and \( R_c i_A(x) = \inf_{p \in \mathcal{P}(X)} \left\{ \int i_A \, dp + c(x, p) \right\} \).
From dual characterization of transport-entropy inequality to concentration

We assume that for all measurable functions \( \varphi : \mathcal{X} \to \mathbb{R} \cup \{+\infty\} \) bounded from below

\[
\left( \int e^{-\frac{R_c \varphi}{a_2}} d\mu \right)^{a_2} \left( \int e^{-\frac{\varphi}{a_1}} d\mu \right)^{a_1} \leq 1
\]

where \( R_c \varphi(x) = \inf_{p \in \mathcal{P}(\mathcal{X})} \left\{ \int \varphi \, dp + c(x, p) \right\} \), \( x \in \mathcal{X} \).

Let \( A \subset \mathcal{X} \). Applying this inequality to the function

\[
\varphi(x) = i_A(x) := \begin{cases} 0 & \text{if } x \in A, \\ +\infty & \text{otherwise}, \end{cases}
\]

since \( \int e^{-\frac{i_A}{a_1}} d\mu = \mu(A) \),

and \( R_c i_A(x) = \inf_{p \in \mathcal{P}(\mathcal{X})} \left\{ \int i_A \, dp + c(x, p) \right\} = \inf_{p, \mu(A)=1} c(x, p) \).
From dual characterization of transport-entropy inequality to concentration

We assume that for all measurable functions \( \varphi : \mathcal{X} \to \mathbb{R} \cup \{+\infty\} \) bounded from below

\[
\left( \int e^{\frac{R_c \varphi}{a_2}} \, d\mu \right)^{a_2} \left( \int e^{-\frac{\varphi}{a_1}} \, d\mu \right)^{a_1} \leq 1
\]

where \( R_c \varphi(x) = \inf_{p \in \mathcal{P}(\mathcal{X})} \left\{ \int \varphi \, dp + c(x, p) \right\} \), \( x \in \mathcal{X} \).

Let \( A \subset \mathcal{X} \). Applying this inequality to the function

\[
\varphi(x) = i_A(x) := \begin{cases} 0 & \text{if } x \in A, \\ +\infty & \text{otherwise}, \end{cases}
\]

since \( \int e^{-\frac{i_A}{a_1}} \, d\mu = \mu(A) \),

and \( R_c i_A(x) = \inf_{p \in \mathcal{P}(\mathcal{X})} \left\{ \int i_A \, dp + c(x, p) \right\} = \inf_{p, p(A) = 1} c(x, p) := c(x, A) \),
From dual characterization of transport-entropy inequality to concentration

We assume that for all measurable functions \( \varphi : \mathcal{X} \to \mathbb{R} \cup \{+\infty\} \) bounded from below

\[
\left( \int e^{\frac{R_c \varphi}{a_2}} d\mu \right)^{a_2} \left( \int e^{\frac{-\varphi}{a_1}} d\mu \right)^{a_1} \leq 1
\]

where \( R_c \varphi(x) = \inf_{p \in \mathcal{P}_\gamma(\mathcal{X})} \left\{ \int \varphi \, dp + c(x, p) \right\}, \quad x \in \mathcal{X}. \)

Let \( A \subset \mathcal{X} \). Applying this inequality to the function

\[
\varphi(x) = i_A(x) := \begin{cases} 0 & \text{if } x \in A, \\ +\infty & \text{otherwise}, \end{cases}
\]

since \( \int e^{-\frac{i_A}{a_1}} d\mu = \mu(A) \),

and \( R_c i_A(x) = \inf_{p \in \mathcal{P}(\mathcal{X})} \left\{ \int i_A dp + c(x, p) \right\} = \inf_{p, p(A)=1} c(x, p) := c(x, A) \),

we get the following type of Talagrand’s concentration result

\[
\left( \int e^{\frac{c(x, A)}{a_2}} d\mu(x) \right)^{a_2} \mu(A)^{a_1} \leq 1.
\]
From dual characterization of transport-entropy inequality to concentration

We assume that for all measurable functions $\varphi : \mathcal{X} \to \mathbb{R} \cup \{+\infty\}$ bounded from below

$$\left( \int e^{\frac{R_c \varphi}{a_2}} \, d\mu \right)^{a_2} \left( \int e^{-\frac{\varphi}{a_1}} \, d\mu \right)^{a_1} \leq 1$$

where $R_c \varphi(x) = \inf_{p \in \mathcal{P}_c(x)} \left\{ \int \varphi \, dp + c(x, p) \right\}$, $x \in \mathcal{X}$.

Let $A \subset \mathcal{X}$. Applying this inequality to the function

$$\varphi(x) = i_A(x) := \begin{cases} 0 & \text{if } x \in A, \\ +\infty & \text{otherwise,} \end{cases}$$

since $\int e^{-\frac{i_A}{a_1}} \, d\mu = \mu(A)$,

and $R_c i_A(x) = \inf_{p \in \mathcal{P}(x)} \left\{ \int i_A \, dp + c(x, p) \right\} = \inf_{p, p(A)=1} c(x, p) := c(x, A)$,

we get the following type of Talagrand’s concentration result

$$\left( \int e^{\frac{c(x, A)}{a_2}} \, d\mu(x) \right)^{a_2} \mu(A)^{a_1} \leq 1.$$  

By Markov inequality,

$$\mu(\mathcal{X} \setminus A_t) = \mu(\{x \in \mathcal{X}, c(x, A) > t\})$$
From dual characterization of transport-entropy inequality to concentration

We assume that for all measurable functions \( \varphi : \mathcal{X} \to \mathbb{R} \cup \{+\infty\} \) bounded from below

\[
\left( \int e^{-\frac{R_c \varphi}{a_2}} \, d\mu \right)^{a_2} \left( \int e^{-\frac{\varphi}{a_1}} \, d\mu \right)^{a_1} \leq 1
\]

where \( R_c \varphi(x) = \inf_{p \in \mathcal{P}_{\gamma}(\mathcal{X})} \left\{ \int \varphi \, dp + c(x, p) \right\} \), \( x \in \mathcal{X} \).

Let \( A \subset \mathcal{X} \). Applying this inequality to the function

\[
\varphi(x) = i_A(x) := \begin{cases} 
0 & \text{if } x \in A, \\
+\infty & \text{otherwise,}
\end{cases}
\]

since

\[
\int e^{-\frac{i_A}{a_1}} \, d\mu = \mu(A),
\]

and

\[
R_c i_A(x) = \inf_{p \in \mathcal{P}(\mathcal{X})} \left\{ \int i_A \, dp + c(x, p) \right\} = \inf_{p, p(A)=1} c(x, p) := c(x, A),
\]

we get the following type of Talagrand’s concentration result

\[
\left( \int e^{-\frac{c(x, A)}{a_2}} \, d\mu(x) \right)^{a_2} \mu(A)^{a_1} \leq 1.
\]

By Markov inequality,

\[
\mu(\mathcal{X} \setminus A_t) = \mu(\{x \in \mathcal{X}, c(x, A) > t\}) \leq e^{-t/a_2} \int e^{-\frac{c(x, A)}{a_2}} \, d\mu(x).
\]
From dual characterization of transport-entropy inequality to concentration

We assume that for all measurable functions $\varphi : \mathcal{X} \to \mathbb{R} \cup \{+\infty\}$ bounded from below

\[ \left( \int e^{\frac{R_c \varphi}{a_2}} \, d\mu \right)^{a_2} \left( \int e^{-\frac{\varphi}{a_1}} \, d\mu \right)^{a_1} \leq 1 \]

where

\[ R_c \varphi(x) = \inf_{p \in \mathcal{P}_Y(\mathcal{X})} \left\{ \int \varphi \, dp + c(x, p) \right\}, \quad x \in \mathcal{X}. \]

Let $A \subset \mathcal{X}$. Applying this inequality to the function

\[ \varphi(x) = i_A(x) := \begin{cases} 0 & \text{if } x \in A, \\ +\infty & \text{otherwise}, \end{cases} \]

since

\[ \int e^{-\frac{i_A}{a_1}} \, d\mu = \mu(A), \]

and

\[ R_c i_A(x) = \inf_{p \in \mathcal{P}(\mathcal{X})} \left\{ \int i_A \, dp + c(x, p) \right\} = \inf_{\rho, \rho(A)=1} c(x, p) := c(x, A), \]

we get the following type of Talagrand’s concentration result

\[ \left( \int e^{\frac{c(x, A)}{a_2}} \, d\mu(x) \right)^{a_2} \mu(A)^{a_1} \leq 1. \]

By Markov inequality,

\[ \mu(\mathcal{X}\setminus A_t) = \mu(\{x \in \mathcal{X}, c(x, A) > t\}) \leq e^{-t/a_2} \int e^{\frac{c(x, A)}{a_2}} \, d\mu(x). \]

It follows that

\[ \mu(\mathcal{X}\setminus A_t)^{a_2} \mu(A)^{a_1} \leq e^{-t}, \quad \forall t > 0. \]
From dual characterization of transport-entropy inequality to concentration

We assume that for all measurable functions $\varphi : \mathcal{X} \to \mathbb{R} \cup \{+\infty\}$ bounded from below

$$\left( \int e^{\frac{R_c \varphi}{a_2}} d\mu \right)^{a_2} \left( \int e^{-\frac{\varphi}{a_1}} d\mu \right)^{a_1} \leq 1$$

where $R_c \varphi(x) = \inf_{p \in \mathcal{P}(\mathcal{X})} \left\{ \int \varphi \, dp + c(x, p) \right\}$, $x \in \mathcal{X}$.

Let $A \subset \mathcal{X}$. Applying this inequality to the function

$$\varphi(x) = i_A(x) := \begin{cases} 0 & \text{if } x \in A, \\ +\infty & \text{otherwise}, \end{cases}$$

since $\int e^{-\frac{i_A}{a_1}} d\mu = \mu(A)$,

and $R_c i_A(x) = \inf_{p \in \mathcal{P}(\mathcal{X})} \left\{ \int i_A \, dp + c(x, p) \right\} = \inf_{p, p(A)=1} c(x, p) := c(x, A)$,

we get the following type of Talagrand’s concentration result

$$\left( \int e^{\frac{c(x, A)}{a_2}} d\mu(x) \right)^{a_2} \mu(A)^{a_1} \leq 1.$$  

By Markov inequality,

$$\mu(\mathcal{X} \setminus A_t) = \mu(\{x \in \mathcal{X}, c(x, A) > t\}) \leq e^{-t/a_2} \int e^{\frac{c(x, A)}{a_2}} d\mu(x).$$

It follows that $\mu(\mathcal{X} \setminus A_t)^{a_2} \mu(A)^{a_1} \leq e^{-t}$, $\forall t > 0$. 


Examples of weak transport inequality in product spaces

Universal transport inequalities

Theorem [Dembo 1996] : A universal weak transport entropy inequality

Let $s \in \mathbb{P}_{p,1}$, $\mu \in \mathbb{P}_p(X,q)$ satisfies $r T c_{p,1} \{ \mu \}$, where $c_{p,1} : \mathbb{R} \to \mathbb{R}$ is an (optimal) convex function.

Theorem [S. 2007] : Another universal weak transport entropy inequality

Let $s \in \mathbb{P}_{p,1}$, $\mu \in \mathbb{P}_p(X,q)$ satisfies $p T c_{p,1} \{ \mu \}$, where $c_{p,1} : \mathbb{R} \to \mathbb{R}$ is an (optimal) convex function.

As for Marton's transport inequality, weak transport inequalities tensorize with $c_n : \mathbb{P}_p \times \mathbb{P}_q \to \mathbb{R}$ for $p, q \in [1,\infty]$. Any product probability measure $\mu_n$ satisfies $r T c_n \{ \mu_n \}$ and $p T c_n \{ \mu_n \}$.

• Any product probability measure $\mu_n$ satisfies $r T c_n \{ \mu_n \}$.

• Any product probability measure $\mu_n$ has a Dim-Free Concentration property. Improves some Talagrand's results (1996) for product measures (convex hull method).

• Gives Bernstein deviation's bounds for suprema of empirical processes (S. 07) as an alternative method to the Ledoux entropy method (Herbst argument).

• Extended to non-product measures, with mixing conditions (S. 2000, Marton 2003), Dobrushing conditions (Paulin 2014).
Examples of weak transport inequality in product spaces

Universal transport inequalities

Theorem [Dembo 1996]: A universal weak transport entropy inequality

Theorem [S. 2007]: Another universal weak transport entropy inequality

As for Marton's transport inequality, weak transport inequalities tensorize with

\[
\begin{align*}
\text{any product probability measure } \mu_n \text{ satisfies } & \\
\text{weak transport inequalities for product measures } & \\
\text{Dim-Free Concentration property} & \\
\text{Improves some Talagrand's results (1996) for product measures} (\text{convex hull method).} & \\
\text{Gives Bernstein deviation's bounds for suprema of empirical processes (S. 07)} & \\
\text{an alternative method to the Ledoux entropy method (Herbst argument).} & \\
\text{Extented to non-product measures, with mixing conditions (S. 2000, } & \\
\text{Marton 2003), Dobrushing contitions (Paulin 2014).} & \\
\end{align*}
\]
Examples of weak transport inequality in product spaces

Universal transport inequalities

**Theorem [Dembo 1996] : A universal weak transport entropy inequality**

Let \( s \in (0, 1) \). Any measure \( \mu \in \mathcal{P}(\mathcal{X}) \) satisfies \( \tilde{T}_c(1/(1 - s), 1/s) \), where
Examples of weak transport inequality in product spaces

Universal transport inequalities

**Theorem [Dembo 1996] : A universal weak transport entropy inequality**

Let $s \in (0, 1)$. Any measure $\mu \in \mathcal{P}(\mathcal{X})$ satisfies $\bar{T}_c(1/(1 - s), 1/s)$, where

$$c(x, \rho) = \alpha_s \left( \int 1_{x \neq y} \rho(y) \right), \quad x \in \mathcal{X}, \rho \in \mathcal{P}(\mathcal{X}),$$

and

As for Marton's transport inequality, weak transport inequalities tensorize with $c_{n,p} = \sum_{i=1}^{n} c(x_i, \rho_i)$, $x \in \mathcal{X}^n$, $\rho \in \mathcal{P}^n(\mathcal{X})$. Any product probability measure $\mu$ satisfies $\bar{T}_c(1/(1 - s), 1/s)$ and $\mu^\otimes n$.

Universal transport inequalities

**Weak transport inequalities**

Dual characterization to concentration

**Barycentric transport inequalities**

examples

characterization on $\mathbb{R}$

**Transport inequality on the symmetric group**

introduction

Ewens distribution

deviation inequalities

**The Schrödinger minimization problem**

definition

curvature in discrete spaces

functional inequalities

Examples in discrete

Weak transport costs
Examples of weak transport inequality in product spaces

Universal transport inequalities

Theorem [Dembo 1996]: A universal weak transport entropy inequality

Let $s \in (0, 1)$. Any measure $\mu \in \mathcal{P}(\mathcal{X})$ satisfies $\mathbb{T}_c(1/(1 - s), 1/s)$, where

$$c(x, \rho) = \alpha_s \left( \int 1_{x \neq y} d\rho(y) \right), \quad x \in \mathcal{X}, \rho \in \mathcal{P}(\mathcal{X}),$$

and $\alpha_s : \mathbb{R}^+ \to \mathbb{R}^+ \cup \{+\infty\}$ is an (optimal) convex function ($\alpha_s(h) \geq h^2/2$).
Examples of weak transport inequality in product spaces

Universal transport inequalities

**Theorem [Dembo 1996] :** A universal weak transport entropy inequality

Let \( s \in (0, 1) \). Any measure \( \mu \in \mathcal{P}(\mathcal{X}) \) satisfies \( \tilde{T}_c(1/(1 - s), 1/s) \), where

\[
c(x, p) = \alpha_s \left( \int 1_{x \neq y} dp(y) \right), \quad x \in \mathcal{X}, p \in \mathcal{P}(\mathcal{X}),
\]

and \( \alpha_s : \mathbb{R}^+ \to \mathbb{R}^+ \cup \{+\infty\} \) is an (optimal) convex function \( \alpha_s(h) \geq h^2/2 \).

**Theorem [S. 2007] :** Another universal weak transport entropy inequality

---

**introduction**
- Marton’s inequality
- Talagrand’s concentration

**Kantorovich duality**
- for classical costs
- for weak costs

**Examples of weak cost**
- Marton’s type of cost
- Barycentric cost
- Strassen’s result
- Martingale costs

**Weak transport inequalities**
- Dual characterization to concentration

**Universal transport inequalities**

**Barycentric transport inequalities**
- examples
- characterisation on \( \mathbb{R} \)

**Transport inequality on the symmetric group**
- introduction
- Ewens distribution
- deviation inequalities

**The Schrödinger minimization problem**
- definition
- curvature in discrete spaces
- functional inequalities
- Examples in discrete

**Weak transport costs.18**
### Examples of weak transport inequality in product spaces

**Universal transport inequalities**

**Theorem [Dembo 1996] : A universal weak transport entropy inequality**

Let \( s \in (0, 1) \). Any measure \( \mu \in \mathcal{P}(\mathcal{X}) \) satisfies \( \tilde{T}_c(1/(1 - s), 1/s) \), where

\[
c(x, p) = \alpha_s \left( \int 1_{x \neq y} \, dp(y) \right), \quad x \in \mathcal{X}, \, p \in \mathcal{P}(\mathcal{X}),
\]

and \( \alpha_s : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \cup \{ +\infty \} \) is an (optimal) convex function (\( \alpha_s(h) \geq h^2/2 \)).

**Theorem [S. 2007] : Another universal weak transport entropy inequality**

Let \( s \in (0, 1) \). Any measure \( \mu \in \mathcal{P}(\mathcal{X}) \) satisfies \( \tilde{T}_c(1/(1 - s), 1/s) \), where
Examples of weak transport inequality in product spaces

Universal transport inequalities

Theorem [Dembo 1996] : A universal weak transport entropy inequality

Let $s \in (0, 1)$. Any measure $\mu \in \mathcal{P}(\mathcal{X})$ satisfies $\tilde{T}_c(1/(1 - s), 1/s)$, where

$c(x, p) = \alpha_s \left( \int 1_{x \neq y} dp(y) \right), \quad x \in \mathcal{X}, p \in \mathcal{P}(\mathcal{X}),$

and $\alpha_s : \mathbb{R}^+ \to \mathbb{R}^+ \cup \{+\infty\}$ is an (optimal) convex function ($\alpha_s(h) \geq h^2/2$).

Theorem [S. 2007] : Another universal weak transport entropy inequality

Let $s \in (0, 1)$. Any measure $\mu \in \mathcal{P}(\mathcal{X})$ satisfies $\hat{T}_c(1/(1 - s), 1/s)$, where

$c(x, p) = \int \beta_s \left( \int 1_{x \neq y} \frac{d\rho}{d\mu} (y) \right) d\mu(y), \quad x \in \mathcal{X}, p \ll \mu,$
Examples of weak transport inequality in product spaces

Universal transport inequalities

Theorem [Dembo 1996] : A universal weak transport entropy inequality

Let $s \in (0, 1)$. Any measure $\mu \in \mathcal{P}(\mathcal{X})$ satisfies $\tilde{T}_c(1/(1 - s), 1/s)$, where

$$c(x, p) = \alpha_s \left( \int 1_{x \neq y} d\rho(y) \right), \quad x \in \mathcal{X}, p \in \mathcal{P}(\mathcal{X}),$$

and $\alpha_s : \mathbb{R}^+ \to \mathbb{R}^+ \cup \{+\infty\}$ is an (optimal) convex function ($\alpha_s(h) \geq h^2/2$).

Theorem [S. 2007] : Another universal weak transport entropy inequality

Let $s \in (0, 1)$. Any measure $\mu \in \mathcal{P}(\mathcal{X})$ satisfies $\tilde{T}_c(1/(1 - s), 1/s)$, where

$$c(x, p) = \int \beta_s \left( \int 1_{x \neq y} \frac{d\rho}{d\mu}(y) \right) d\mu(y), \quad x \in \mathcal{X}, p \ll \mu,$$

and $\beta_s : \mathbb{R}^+ \to \mathbb{R}^+ \cup \{+\infty\}$ is an (optimal) convex function.
Universal transport inequalities

Theorem [Dembo 1996]: A universal weak transport entropy inequality

Let \( s \in (0, 1) \). Any measure \( \mu \in \mathcal{P}(\mathcal{X}) \) satisfies \( \tilde{T}_c(1/(1 - s), 1/s) \), where

\[
c(x, p) = \alpha_s \left( \int 1_{x \neq y} dp(y) \right), \quad x \in \mathcal{X}, p \in \mathcal{P}(\mathcal{X}),
\]

and \( \alpha_s : \mathbb{R}^+ \to \mathbb{R}^+ \cup \{+\infty\} \) is an (optimal) convex function (\( \alpha_s(h) \geq h^2/2 \)).

Theorem [S. 2007]: Another universal weak transport entropy inequality

Let \( s \in (0, 1) \). Any measure \( \mu \in \mathcal{P}(\mathcal{X}) \) satisfies \( \hat{T}_c(1/(1 - s), 1/s) \), where

\[
c(x, p) = \int \beta_s \left( \int 1_{x \neq y} \frac{dp}{d\mu}(y) \right) d\mu(y), \quad x \in \mathcal{X}, p \ll \mu,
\]

and \( \beta_s : \mathbb{R}^+ \to \mathbb{R}^+ \cup \{+\infty\} \) is an (optimal) convex function.
Examples of weak transport inequality in product spaces

Universal transport inequalities

Theorem [Dembo 1996]: A universal weak transport entropy inequality

Let \( s \in (0, 1) \). Any measure \( \mu \in \mathcal{P}(\mathcal{X}) \) satisfies \( \hat{T}_c(1/(1-s), 1/s) \), where

\[
c(x, p) = \alpha_s \left( \int 1_{x \neq y} dp(y) \right), \quad x \in \mathcal{X}, p \in \mathcal{P}(\mathcal{X}),
\]

and \( \alpha_s : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \cup \{+\infty\} \) is an (optimal) convex function (\( \alpha_s(h) \geq h^2/2 \)).

Theorem [S. 2007]: Another universal weak transport entropy inequality

Let \( s \in (0, 1) \). Any measure \( \mu \in \mathcal{P}(\mathcal{X}) \) satisfies \( \hat{T}_c(1/(1-s), 1/s) \), where

\[
c(x, p) = \int \beta_s \left( \int 1_{x \neq y} \frac{dp}{d\mu}(y) \right) d\mu(y), \quad x \in \mathcal{X}, p \ll \mu,
\]

and \( \beta_s : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \cup \{+\infty\} \) is an (optimal) convex function.

As for Marton’s transport inequality, weak transport inequalities tensorize with

\[
c^n(x, p) = \sum_{i=1}^n c(x_i, p_i), \quad x = (x_1, \ldots, x_n).
\]
Examples of weak transport inequality in product spaces

Universal transport inequalities

Theorem [Dembo 1996]: A universal weak transport entropy inequality

Let \( s \in (0, 1) \). Any measure \( \mu \in \mathcal{P}(\mathcal{X}) \) satisfies \( \mathcal{T}_c^{1/(1 - s), 1/s}(\mu) \), where

\[
c(x, \rho) = \alpha_s \left( \int 1_{x \neq y} d\rho(y) \right), \quad x \in \mathcal{X}, \rho \in \mathcal{P}(\mathcal{X}),
\]

and \( \alpha_s : \mathbb{R}^+ \to \mathbb{R}^+ \cup \{+\infty\} \) is an (optimal) convex function \( (\alpha_s(h) \geq h^2/2) \).

Theorem [S. 2007]: Another universal weak transport entropy inequality

Let \( s \in (0, 1) \). Any measure \( \mu \in \mathcal{P}(\mathcal{X}) \) satisfies \( \mathcal{T}_c^{1/(1 - s), 1/s}(\mu) \), where

\[
c(x, \rho) = \int \beta_s \left( \int 1_{x \neq y} \frac{d\rho}{d\mu}(y) \right) d\mu(y), \quad x \in \mathcal{X}, \rho \ll \mu,
\]

and \( \beta_s : \mathbb{R}^+ \to \mathbb{R}^+ \cup \{+\infty\} \) is an (optimal) convex function.

As for Marton’s transport inequality, weak transport inequalities tensorize with

\[
c_n(x, \rho) = \sum_{i=1}^n c(x_i, \rho_i), \quad x = (x_1, \ldots, x_n).
\]

- Any product probability measure \( \mu^n \) satisfies \( \mathcal{T}_c^{1/(1 - s), 1/s}(\mu^n) \) and \( \mathcal{T}_c^{1/(1 - s), 1/s}(\mu^n) \).

Examples of weak transport costs.18
Examples of weak transport inequality in product spaces

Universal transport inequalities

**Theorem [Dembo 1996] : A universal weak transport entropy inequality**

Let $s \in (0, 1)$. Any measure $\mu \in \mathcal{P}(\mathcal{X})$ satisfies $\tilde{T}_c(1/(1 - s), 1/s)$, where

$$c(x, p) = \alpha_s \left( \int 1_{x \neq y} dp(y) \right), \quad x \in \mathcal{X}, p \in \mathcal{P}(\mathcal{X}),$$

and $\alpha_s : \mathbb{R}^+ \to \mathbb{R}^+ \cup \{+\infty\}$ is an (optimal) convex function ($\alpha_s(h) \geq h^2/2$).

**Theorem [S. 2007] : Another universal weak transport entropy inequality**

Let $s \in (0, 1)$. Any measure $\mu \in \mathcal{P}(\mathcal{X})$ satisfies $\hat{T}_c(1/(1 - s), 1/s)$, where

$$c(x, p) = \int \beta_s \left( \int 1_{x \neq y} \frac{dp}{d\mu}(y) \right) d\mu(y), \quad x \in \mathcal{X}, p \ll \mu,$$

and $\beta_s : \mathbb{R}^+ \to \mathbb{R}^+ \cup \{+\infty\}$ is an (optimal) convex function.

As for Marton’s transport inequality, weak transport inequalities tensorize with

$$c^n(x, p) = \sum_{i=1}^n c(x_i, p_i), \quad x = (x_1, \ldots, x_n).$$

- Any product probability measure $\mu^n$ satisfies $\tilde{T}_{c^n}(1/(1 - s), 1/s)$ and $\hat{T}_{c^n}(1/(1 - s), 1/s)$.
- Any product probability measure $\mu^n$ a Dim-Free Concentration property.
Examples of weak transport inequality in product spaces

**Universal transport inequalities**

**Theorem [Dembo 1996] : A universal weak transport entropy inequality**

Let \( s \in (0, 1) \). Any measure \( \mu \in \mathcal{P}(\mathcal{X}) \) satisfies \( \hat{T}_c(1/(1 - s), 1/s) \), where
\[
c(x, p) = \alpha_s \left( \int 1_{x \neq y} d\rho(y) \right), \quad x \in \mathcal{X}, p \in \mathcal{P}(\mathcal{X}),
\]
and \( \alpha_s : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \cup \{ +\infty \} \) is an (optimal) convex function (\( \alpha_s(h) \geq h^2/2 \)).

**Theorem [S. 2007] : Another universal weak transport entropy inequality**

Let \( s \in (0, 1) \). Any measure \( \mu \in \mathcal{P}(\mathcal{X}) \) satisfies \( \hat{T}_c(1/(1 - s), 1/s) \), where
\[
c(x, p) = \int \beta_s \left( \int 1_{x \neq y} \frac{d\rho}{d\mu}(y) \right) d\mu(y), \quad x \in \mathcal{X}, p << \mu,
\]
and \( \beta_s : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \cup \{ +\infty \} \) is an (optimal) convex function.

As for Marton’s transport inequality, weak transport inequalities tensorize with
\[
c^n(x, p) = \sum_{i=1}^n c(x_i, p_i), \quad x = (x_1, \ldots, x_n).
\]
- Any product probability measure \( \mu^n \) satisfies \( \hat{T}_{cn}(1/(1 - s), 1/s) \) and \( \hat{T}_{cn}(1/(1 - s), 1/s) \).
- Any product probability measure \( \mu^n \) a Dim-Free Concentration property
  → Improves some Talagrand’s results (1996) for product measures
  (convex hull method).
Examples of weak transport inequality in product spaces

Universal transport inequalities

**Theorem [Dembo 1996] : A universal weak transport entropy inequality**

Let \( s \in (0, 1) \). Any measure \( \mu \in \mathcal{P}(\mathcal{X}) \) satisfies \( \hat{T}_c(1/(1 - s), 1/s) \), where

\[
c(x, \rho) = \alpha_s \left( \int 1_{x \neq y} \, d\rho(y) \right), \quad x \in \mathcal{X}, \rho \in \mathcal{P}(\mathcal{X}),
\]

and \( \alpha_s : \mathbb{R}^+ \to \mathbb{R}^+ \cup \{+\infty\} \) is an (optimal) convex function \( (\alpha_s(h) \geq h^2/2) \).

**Theorem [S. 2007] : Another universal weak transport entropy inequality**

Let \( s \in (0, 1) \). Any measure \( \mu \in \mathcal{P}(\mathcal{X}) \) satisfies \( \hat{T}_c(1/(1 - s), 1/s) \), where

\[
c(x, \rho) = \int \beta_s \left( \int 1_{x \neq y} \frac{d\rho}{d\mu}(y) \right) \, d\mu(y), \quad x \in \mathcal{X}, \rho \ll \mu,
\]

and \( \beta_s : \mathbb{R}^+ \to \mathbb{R}^+ \cup \{+\infty\} \) is an (optimal) convex function.

As for Marton’s transport inequality, weak transport inequalities tensorize with

\[
c^n(x, \rho) = \sum_{i=1}^n c(x_i, \rho_i), \quad x = (x_1, \ldots, x_n).
\]

- Any product probability measure \( \mu^n \) satisfies \( \hat{T}_c^n(1/(1 - s), 1/s) \) and \( \hat{T}_{c^n}(1/(1 - s), 1/s) \).
- Any product probability measure \( \mu^n \) a Dim-Free Concentration property
- Improves some Talagrand’s results (1996) for product measures (convex hull method).
- Gives Bernstein deviation’s bounds for suprema of empirical processes (S. 07)
Examples of weak transport inequality in product spaces

Universal transport inequalities

**Theorem [Dembo 1996] : A universal weak transport entropy inequality**

Let $s \in (0, 1)$. Any measure $\mu \in \mathcal{P}(\mathcal{X})$ satisfies $\tilde{T}_c(1/(1 - s), 1/s)$, where

$$c(x, p) = \alpha_s \left( \int 1_{x \neq y} dp(y) \right), \quad x \in \mathcal{X}, p \in \mathcal{P}(\mathcal{X}),$$

and $\alpha_s : \mathbb{R}^+ \to \mathbb{R}^+ \cup \{+\infty\}$ is an (optimal) convex function ($\alpha_s(h) \geq h^2/2$).

**Theorem [S. 2007] : Another universal weak transport entropy inequality**

Let $s \in (0, 1)$. Any measure $\mu \in \mathcal{P}(\mathcal{X})$ satisfies $\tilde{T}_c(1/(1 - s), 1/s)$, where

$$c(x, p) = \int \beta_s \left( \int 1_{x \neq y} \frac{dp(y)}{d\mu(y)} \right) d\mu(y), \quad x \in \mathcal{X}, p \ll \mu,$$

and $\beta_s : \mathbb{R}^+ \to \mathbb{R}^+ \cup \{+\infty\}$ is an (optimal) convex function.

As for Marton’s transport inequality, weak transport inequalities tensorize with

$$c^n(x, p) = \sum_{i} c(x_i, p_i), \quad x = (x_1, \ldots, x_n).$$

- Any product probability measure $\mu^n$ satisfies $\tilde{T}_{c_n}(1/(1 - s), 1/s)$ and $\tilde{T}_{c^n}(1/(1 - s), 1/s)$.
- Any product probability measure $\mu^n$ a Dim-Free Concentration property
  → Improves some Talagrand’s results (1996) for product measures
  (convex hull method).
  - Gives Bernstein deviation’s bounds for suprema of empirical processes (S. 07)
  → an alternative method to the Ledoux entropy method (Herbst argument).
Examples of weak transport inequality in product spaces

Universal transport inequalities

**Theorem [Dembo 1996] : A universal weak transport entropy inequality**

Let $s \in (0, 1)$. Any measure $\mu \in \mathcal{P}(\mathcal{X})$ satisfies $\tilde{T}_c(1/(1 - s), 1/s)$, where

$$c(x, p) = \alpha_s \left( \int 1_{x \neq y} dp(y) \right), \quad x \in \mathcal{X}, p \in \mathcal{P}(\mathcal{X}),$$

and $\alpha_s : \mathbb{R}^+ \to \mathbb{R}^+ \cup \{+\infty\}$ is an (optimal) convex function ($\alpha_s(h) \geq h^2/2$).

**Theorem [S. 2007] : Another universal weak transport entropy inequality**

Let $s \in (0, 1)$. Any measure $\mu \in \mathcal{P}(\mathcal{X})$ satisfies $\tilde{T}_c(1/(1 - s), 1/s)$, where

$$c(x, p) = \int \beta_s \left( \int 1_{x \neq y} \frac{dp}{d\mu}(y) \right) d\mu(y), \quad x \in \mathcal{X}, p \ll \mu,$$

and $\beta_s : \mathbb{R}^+ \to \mathbb{R}^+ \cup \{+\infty\}$ is an (optimal) convex function.

As for Marton’s transport inequality, weak transport inequalities tensorize with

$$c^n(x, p) = \sum_{i=1}^n c(x_i, p_i), \quad x = (x_1, \ldots, x_n).$$

- Any product probability measure $\mu^n$ satisfies $\tilde{T}_c^n(1/(1 - s), 1/s)$ and $\tilde{T}_c^n(1/(1 - s), 1/s)$.
- Any product probability measure $\mu^n$ a Dim-Free Concentration property
  → Improves some Talagrand’s results (1996) for product measures (convex hull method).
  - Gives Bernstein deviation’s bounds for suprema of empirical processes (S. 07)
    → an alternative method to the Ledoux entropy method (Herbst argument).
  - Extented to non-product measures, with mixing conditions (S. 2000, Marton 2003), Dobrushing contions (Paulin 2014).
Examples of weak transport inequality in product spaces

Universal transport inequalities

**Theorem [Dembo 1996] : A universal weak transport entropy inequality**

Let $s \in (0, 1)$. Any measure $\mu \in \mathcal{P}(\mathcal{X})$ satisfies $\tilde{T}_c(1/(1 - s), 1/s)$, where

$$c(x, p) = \alpha_s \left( \int 1_{x \neq y} dp(y) \right), \quad x \in \mathcal{X}, p \in \mathcal{P}(\mathcal{X}),$$

and $\alpha_s : \mathbb{R}^+ \to \mathbb{R}^+ \cup \{+\infty\}$ is an (optimal) convex function ($\alpha_s(h) \geq h^2/2$).

**Theorem [S. 2007] : Another universal weak transport entropy inequality**

Let $s \in (0, 1)$. Any measure $\mu \in \mathcal{P}(\mathcal{X})$ satisfies $\hat{T}_c(1/(1 - s), 1/s)$, where

$$c(x, p) = \int \beta_s \left( \int 1_{x \neq y} \frac{dp}{d\mu}(y) \right) d\mu(y), \quad x \in \mathcal{X}, p \ll \mu,$$

and $\beta_s : \mathbb{R}^+ \to \mathbb{R}^+ \cup \{+\infty\}$ is an (optimal) convex function.

As for Marton’s transport inequality, weak transport inequalities tensorize with

$$c^n(x, p) = \sum_{i=1}^n c(x_i, p_i), \quad x = (x_1, \ldots, x_n).$$

- Any product probability measure $\mu^n$ satisfies $\tilde{T}_{c^n}(1/(1 - s), 1/s)$ and $\hat{T}_{c^n}(1/(1 - s), 1/s)$.

- Any product probability measure $\mu^n$ a Dim-Free Concentration property

  → Improves some Talagrand’s results (1996) for product measures (convex hull method).

  – Gives Bernstein deviation’s bounds for suprema of empirical processes (S. 07)

    → an alternative method to the Ledouix entropy method (Herbst argument).

- Extented to non-product measures, with mixing conditions (S. 2000, Marton 2003), Dobrushing contitions (Paulin 2014).
Examples of weak transport-entropy inequalities in discrete spaces
Barycentric transport inequalities
Examples of weak transport-entropy inequalities in discrete spaces

Barycentric transport inequalities

Proposition [S. 2003] : Weak transport inequalities for the Bernoulli measure

Proposition [GRST 2015] : Weak transport inequalities for the binomial law

Choose $q^\lambda$ and use the weak convergence as $n \to \infty$ of the binomial law $\mu^\lambda$ to the Poisson measure $\mu^\lambda_k$.
Examples of weak transport-entropy inequalities in discrete spaces

Barycentric transport inequalities

**Proposition [S. 2003] : Weak transport inequalities for the Bernoulli measure**

The Bernoulli measure $\mu_q$ on $\mathcal{X} = \{0, 1\}$ with parameter $q = \mu_q(1)$ satisfies $\overline{T}_{cs}(1/(1 - s), 1/s), s \in (0, 1)$
**Examples of weak transport-entropy inequalities in discrete spaces**

**Barycentric transport inequalities**

**Proposition [S. 2003]: Weak transport inequalities for the Bernoulli measure**

The Bernoulli measure $\mu_q$ on $\mathcal{X} = \{0, 1\}$ with parameter $q = \mu_q(1)$ satisfies $\overline{T}_s(1/(1 - s), 1/s)$, $s \in (0, 1)$ where

$$c_s(x, p) = \theta_s \left( x - \int y dp(y) \right), \quad x \in \{0, 1\}, \quad p \in \mathcal{P}(\{0, 1\})$$
Examples of weak transport-entropy inequalities in discrete spaces

Barycentric transport inequalities

Proposition [S. 2003] : Weak transport inequalities for the Bernoulli measure

The Bernoulli measure $\mu_q$ on $\mathcal{X} = \{0, 1\}$ with parameter $q = \mu_q(1)$ satisfies $\overline{T}_{cs}(1/(1 - s), 1/s)$, $s \in (0, 1)$ where

$$c_s(x, p) = \theta_s \left( x - \int ydp(y) \right), \quad x \in \{0, 1\}, \ p \in \mathcal{P}(\{0, 1\})$$

with $\theta_s(h) \sim_0 + \frac{h^2}{2(1-q)}$, and $\theta_s(h) \sim_0 - \frac{h^2}{2q}$. 

weak transport\-entropies
Examples of weak transport-entropy inequalities in discrete spaces

Barycentric transport inequalities

Proposition [S. 2003] : Weak transport inequalities for the Bernoulli measure

The Bernoulli measure $\mu_q$ on $\mathcal{X} = \{0, 1\}$ with parameter $q = \mu_q(1)$ satisfies $\overline{T}_{cs}(1/(1 - s), 1/s), s \in (0, 1)$ where

$$c_s(x, p) = \theta_s \left( x - \int ydp(y) \right), \quad x \in \{0, 1\}, p \in \mathcal{P}(\{0, 1\})$$

with $\theta_s(h) \sim 0 + \frac{h^2}{2(1-q)}$, and $\theta_s(h) \sim 0 - \frac{h^2}{2q}$.

As a consequence, the product measure $\mu_q^n$ on $\{0, 1\}^n$ satisfies $\overline{T}_{cs}(1/(1 - s), 1/s)$,
Examples of weak transport-entropy inequalities in discrete spaces

Barycentric transport inequalities

**Proposition [S. 2003] : Weak transport inequalities for the Bernoulli measure**

The Bernoulli measure $\mu_q$ on $\mathcal{X} = \{0, 1\}$ with parameter $q = \mu_q(1)$ satisfies

$$\bar{T}_{c_s}(1/(1 - s), 1/s), \ s \in (0, 1)$$

where

$$c_s(x, p) = \theta_s \left( x - \int ydp(y) \right), \quad x \in \{0, 1\}, \ p \in \mathcal{P}\{\{0, 1\}\}$$

with $\theta_s(h) \sim 0^+ \frac{h^2}{2(1-q)}$, and $\theta_s(h) \sim 0^- \frac{h^2}{2q}$.

As a consequence, the product measure $\mu^n_q$ on $\{0, 1\}^n$ satisfies

$$\bar{T}_{c^n_s}(1/(1 - s), 1/s), \text{ and by projection arguments } ((x_1, \ldots, x_n) \mapsto \sum_{i=1}^n x_i),$$

**Proposition [GRST 2015] : Weak transport inequalities for the binomial law**

The Binomial law $\mu_{q, n}$ on $\mathcal{X} = \{0, 1, \ldots, n\}$ satisfies

$$\bar{T}_{c^n_s}(1/(1 - s), 1/s), \text{ and by projection arguments } ((x_1, \ldots, x_n) \mapsto \sum_{i=1}^n x_i),$$

where $\theta_s(h) \sim 0^+ \frac{h^2}{2(1-q)}$, and $\theta_s(h) \sim 0^- \frac{h^2}{2q}$. 

As a consequence, the product measure $\mu^n_{q, n}$ on $\{0, 1, \ldots, n\}^n$ satisfies

$$\bar{T}_{c^n_s}(1/(1 - s), 1/s), \text{ and by projection arguments } ((x_1, \ldots, x_n) \mapsto \sum_{i=1}^n x_i),$$
Examples of weak transport-entropy inequalities in discrete spaces

Barycentric transport inequalities

**Proposition [S. 2003]: Weak transport inequalities for the Bernoulli measure**

The Bernoulli measure $\mu_q$ on $\mathcal{X} = \{0, 1\}$ with parameter $q = \mu_q(1)$ satisfies $\mathcal{T}_{cs}(1/(1 - s), 1/s)$, $s \in (0, 1)$ where

$$c_s(x, p) = \theta_s \left( x - \int ydp(y) \right), \quad x \in \{0, 1\}, \; p \in \mathcal{P}\{0, 1\}$$

with $\theta_s(h) \sim 0^+ \frac{h^2}{2(1-q)}$, and $\theta_s(h) \sim 0^- \frac{h^2}{2q}$.

As a consequence, the product measure $\mu_q^n$ on $\{0, 1\}^n$ satisfies $\mathcal{T}_{cs}(1/(1 - s), 1/s)$, and by projection arguments ($(x_1, \ldots, x_n) \mapsto \sum_{i=1}^n x_i$),

**Proposition [GRST 2015]: Weak transport inequalities for the binomial law**

The Binomial law $\mu_{q,n}$ on $\{0, 1, \ldots, n\}$ satisfies $\mathcal{T}_{cs,n}(1/(1 - s), 1/s)$.
Examples of weak transport-entropy inequalities in discrete spaces

Barycentric transport inequalities

**Proposition [S. 2003] : Weak transport inequalities for the Bernoulli measure**

The Bernoulli measure $\mu_q$ on $\mathcal{X} = \{0, 1\}$ with parameter $q = \mu_q(1)$ satisfies $\overline{T}_{cs}(1/(1 - s), 1/s)$, $s \in (0, 1)$ where

$$c_s(x, p) = \theta_s \left( x - \int y dp(y) \right), \quad x \in \{0, 1\}, p \in \mathcal{P}(\{0, 1\})$$

with $\theta_s(h) \sim_0 + \frac{h^2}{2(1 - q)}$, and $\theta_s(h) \sim_0 - \frac{h^2}{2q}$.

As a consequence, the product measure $\mu_q^n$ on $\{0, 1\}^n$ satisfies $\overline{T}_{cs}^n(1/(1 - s), 1/s)$, and by projection arguments ($(x_1, \ldots, x_n) \mapsto \sum_{i=1}^n x_i$).

**Proposition [GRST 2015] : Weak transport inequalities for the binomial law**

The Binomial law $\mu_{q,n}$ on $\{0, 1, \ldots, n\}$ satisfies $\overline{T}_{cs,n}(1/(1 - s), 1/s)$ with

$$c_{s,n}(x, p) = n \theta_s \left( \frac{1}{n} \left( x - \int y dp(y) \right) \right), \quad x \in \{0, 1, \ldots, n\}.$$
Examples of weak transport-entropy inequalities in discrete spaces

**Barycentric transport inequalities**

**Proposition [S. 2003] : Weak transport inequalities for the Bernoulli measure**

The Bernoulli measure $\mu_q$ on $\mathcal{X} = \{0, 1\}$ with parameter $q = \mu_q(1)$ satisfies $\overline{T}_{cs_q}(1/(1 - s), 1/s)$, $s \in (0, 1)$ where

$$c_s(x, p) = \theta_s \left( x - \int ydp(y) \right), \quad x \in \{0, 1\}, \ p \in \mathcal{P}(\{0, 1\})$$

with $\theta_s(h) \sim 0^+ \frac{h^2}{2(1 - q)}$, and $\theta_s(h) \sim 0^- \frac{h^2}{2q}$.

As a consequence, the product measure $\mu_q^n$ on $\{0, 1\}^n$ satisfies $\overline{T}_{c_s^n}(1/(1 - s), 1/s)$, and by projection arguments $((x_1, \ldots, x_n) \mapsto \sum_{i=1}^n x_i)$,

**Proposition [GRST 2015] : Weak transport inequalities for the binomial law**

The Binomial law $\mu_{q,n}$ on $\{0, 1, \ldots, n\}$ satisfies $\overline{T}_{c_{s,n}}(1/(1 - s), 1/s)$ with

$$c_{s,n}(x, p) = n \theta_s \left( \frac{1}{n} \left( x - \int ydp(y) \right) \right), \quad x \in \{0, 1, \ldots, n\}.$$ 

$\theta_s$ is the same cost function as for the Bernoulli measure.
Examples of weak transport-entropy inequalities in discrete spaces

Barycentric transport inequalities

**Proposition [S. 2003] : Weak transport inequalities for the Bernoulli measure**

The Bernoulli measure \( \mu_q \) on \( \mathcal{X} = \{0, 1\} \) with parameter \( q = \mu_q(1) \) satisfies \( \mathcal{T}_{c_s}(1/(1 - s), 1/s), s \in (0, 1) \) where

\[
c_s(x, p) = \theta_s \left( x - \int y dp(y) \right), \quad x \in \{0, 1\}, p \in \mathcal{P}(\{0, 1\})
\]

with \( \theta_s(h) \sim 0^+ \frac{h^2}{2(1 - q)} \), and \( \theta_s(h) \sim 0^- \frac{h^2}{2q} \).

As a consequence, the product measure \( \mu_q^n \) on \( \{0, 1\}^n \) satisfies \( \mathcal{T}_{c^n_s}(1/(1 - s), 1/s) \), and by projection arguments \( \left((x_1, \ldots, x_n) \mapsto \sum_{i=1}^n x_i\right) \),

**Proposition [GRST 2015] : Weak transport inequalities for the binomial law**

The Binomial law \( \mu_{q,n} \) on \( \{0, 1, \ldots, n\} \) satisfies \( \mathcal{T}_{c_{s,n}}(1/(1 - s), 1/s) \) with

\[
c_{s,n}(x, p) = n \theta_s \left( \frac{1}{n} \left( x - \int y dp(y) \right) \right), \quad x \in \{0, 1, \ldots, n\}.
\]

\( \theta_s \) is the same cost function as for the Bernoulli measure.

**Proposition [GRST 2015] : Weak transport inequalities for the Poisson measure**

Choose \( q = \lambda \) \( \lambda \in (0, \infty) \), and use the weak convergence as \( n \to \infty \) of the binomial law \( \mu_{\lambda,n} \) to the Poisson measure \( \nu_{\lambda} \) with

\[
\nu_{\lambda} = \frac{e^{-\lambda}}{\lambda^k} \frac{\lambda^k}{k!}.
\]
Examples of weak transport-entropy inequalities in discrete spaces

Barycentric transport inequalities

**Proposition [S. 2003] : Weak transport inequalities for the Bernoulli measure**

The Bernoulli measure $\mu_q$ on $\mathcal{X} = \{0, 1\}$ with parameter $q = \mu_q(1)$ satisfies $\overline{T}_{cs}(1/(1 - s), 1/s)$, $s \in (0, 1)$ where

$$c_s(x, p) = \theta_s \left( x - \int y dp(y) \right), \quad x \in \{0, 1\}, p \in \mathcal{P}(\{0, 1\})$$

with $\theta_s(h) \sim_0 + \frac{h^2}{2(1 - q)}$, and $\theta_s(h) \sim_0 - \frac{h^2}{2q}$.

As a consequence, the product measure $\mu_q^n$ on $\{0, 1\}^n$ satisfies $\overline{T}_{cs^n}(1/(1 - s), 1/s)$, and by projection arguments ($(x_1, \ldots, x_n) \mapsto \sum_{i=1}^n x_i$).

**Proposition [GRST 2015] : Weak transport inequalities for the binomial law**

The Binomial law $\mu_{q, n}$ on $\{0, 1, \ldots, n\}$ satisfies $\overline{T}_{cs,n}(1/(1 - s), 1/s)$ with

$$c_{s,n}(x, p) = n \theta_s \left( \frac{1}{n} \left( x - \int y dp(y) \right) \right), \quad x \in \{0, 1, \ldots, n\}.$$  

$\theta_s$ is the same cost function as for the Bernoulli measure.

**Proposition [GRST 2015] : Weak transport inequalities for the Poisson measure**

Choose $q = \lambda/n$, $\lambda > 0$, and use the weak convergence as $n \rightarrow +\infty$ of the binomial law $\mu_{\lambda/n, n}$ to the Poisson measure $p_{\lambda}(k) = \frac{\lambda^k}{k!} e^{-\lambda}$, $k \in \mathbb{N}$.
Characterization of probability measures on \( \mathbb{R} \) satisfying a barycentric transport-entropy inequality
Characterization of probability measures on \( \mathbb{R} \) satisfying a barycentric transport-entropy inequality

Let \( \theta : \mathbb{R} \rightarrow \mathbb{R}^+ \) be a symmetric convex cost function satisfying

\[
\theta(t) = t^2, \quad \forall t \leq t_0, \quad \text{for some } t_0 > 0.
\]
Characterization of probability measures on $\mathbb{R}$ satisfying a barycentric transport-entropy inequality

Let $\theta : \mathbb{R} \to \mathbb{R}^+$ be a symmetric convex cost function satisfying

$$\theta(t) = t^2, \quad \forall t \leq t_0, \text{ for some } t_0 > 0.$$  

For $a > 0$, let $\theta_a(t) = \theta(at), \ t \in \mathbb{R}.$
Characterization of probability measures on $\mathbb{R}$ satisfying a barycentric transport-entropy inequality

Let $\theta : \mathbb{R} \rightarrow \mathbb{R}^+$ be a symmetric convex cost function satisfying

$$\theta(t) = t^2, \quad \forall t \leq t_0, \quad \text{for some } t_0 > 0.$$  

For $a > 0$, let $\theta_a(t) = \theta(at), \ t \in \mathbb{R}$. For any $\mu, \nu \in \mathcal{P}(\mathbb{R})$, we consider the barycentric transport cost

$$\overline{T}_{\theta_a}(\nu|\mu) = \inf_{\pi \in \Pi(\mu, \nu)} \int \theta_a \left( \int x - \int y \, d\rho_x(y) \right) \, d\mu(x).$$
Characterization of probability measures on $\mathbb{R}$ satisfying a barycentric transport-entropy inequality

Let $\theta : \mathbb{R} \to \mathbb{R}^+$ be a symmetric convex cost function satisfying

$$\theta(t) = t^2, \quad \forall t \leq t_0, \quad \text{for some } t_0 > 0.$$ 

For $a > 0$, let $\theta_a(t) = \theta(at)$, $t \in \mathbb{R}$.

For any $\mu, \nu \in \mathcal{P}(\mathbb{R})$, we consider the barycentric transport cost

$$\overline{T}_{\theta_a}(\nu|\mu) = \inf_{\pi \in \Pi(\mu, \nu)} \int \theta_a \left( \int x - \int y \, d\rho_x(y) \right) \, d\mu(x).$$

**Theorem:** [Gozlan-Roberto-S.-Shu-Tetali 2017]
Characterization of probability measures on $\mathbb{R}$ satisfying a barycentric transport-entropy inequality

Let $\theta : \mathbb{R} \to \mathbb{R}^+$ be a symmetric convex cost function satisfying

$$\theta(t) = t^2, \quad \forall t \leq t_0, \text{ for some } t_0 > 0.$$ 

For $a > 0$, let $\theta_a(t) = \theta(at)$, $t \in \mathbb{R}$.

For any $\mu, \nu \in \mathcal{P}(\mathbb{R})$, we consider the barycentric transport cost

$$\bar{T}_{\theta_a}(\nu | \mu) = \inf_{\pi \in \Pi(\mu, \nu)} \int \theta_a \left( \int x - \int y \, d\rho_x(y) \right) \, d\mu(x).$$

**Theorem : [Gozlan-Roberto-S.-Shu-Tetali 2017]**

Let $\mu \in \mathcal{P}(\mathbb{R})$. 

---

**introduction**
Marton's inequality
Talagrand's concentration
Kantorovich duality
for classical costs
for weak costs
Examples of weak cost
Marton's type of cost
Barycentric cost
Strassen's result
Martingale costs
Weak transport inequalities
Dual characterization to concentration
Universal transport inequalities
Barycentric transport inequalities
dual characterisation
examples
characterisation on $\mathbb{R}$
Transport inequality on the symmetric group
introduction
Ewens distribution
deviation inequalities
The Schrödinger minimization problem
definition
curvature in discrete spaces
functional inequalities
Examples in discrete
Weak transport costs
Characterization of probability measures on $\mathbb{R}$ satisfying a barycentric transport-entropy inequality

Let $\theta : \mathbb{R} \to \mathbb{R}^+$ be a symmetric convex cost function satisfying

$$\theta(t) = t^2, \quad \forall t \leq t_0, \text{ for some } t_0 > 0.$$ 

For $a > 0$, let $\theta_a(t) = \theta(at)$, $t \in \mathbb{R}$.

For any $\mu, \nu \in \mathcal{P}(\mathbb{R})$, we consider the barycentric transport cost

$$\overline{T}_{\theta_a}(\nu | \mu) = \inf_{\pi \in \Pi(\mu, \nu)} \int \theta_a \left( \int x - \int y \, dp_x(y) \right) d\mu(x).$$

**Theorem:** [Gozlan-Roberto-S.-Shu-Tetali 2017]

Let $\mu \in \mathcal{P}(\mathbb{R})$. The following propositions are equivalent:

1. There exists $a > 0$ such that for all $\nu \in \mathcal{P}(\mathbb{R})$,
   $$\overline{T}_{\theta_a}(\nu | \mu) \leq H(\nu | \mu),$$
   and
   $$\overline{T}_{\theta_a}(\mu | \nu) \leq H(\mu | \nu),$$

2. There exists $b > 0$ such that for all $u > 0$,
   $$\sup_{x \in U} \left| \mu^{-\frac{1}{2}} \left( x - \int y \, dp_x(y) \right) \right| \leq b \theta^{-\frac{1}{2}} \left( \frac{u}{t_2} \right)^{t_2}$$
   where $U = \{ x \in \mathbb{R} : \mu^{-\frac{1}{2}} \left| x \right| \leq u \}$,
   $\mu^{-\frac{1}{2}} = \begin{cases} \int |x| \, d\mu, & \text{if } x \geq 0, \\ \int |x| \, d\mu, & \text{if } x \leq 0. \end{cases}$

Used by Strzelecka-Strzelecki-Tkocz (2017) to show that any symmetric probability measure with log-concave tails satisfies a barycentric transport inequality with optimal cost, up to a universal constant.
Characterization of probability measures on $\mathbb{R}$ satisfying a barycentric transport-entropy inequality

Let $\theta : \mathbb{R} \to \mathbb{R}^+$ be a symmetric convex cost function satisfying

$$\theta(t) = t^2, \quad \forall t \leq t_0, \quad \text{for some } t_0 > 0.$$  

For $a > 0$, let $\theta_a(t) = \theta(at)$, $t \in \mathbb{R}$.

For any $\mu, \nu \in \mathcal{P}(\mathbb{R})$, we consider the barycentric transport cost

$$\overline{T}_{\theta_a}(\nu | \mu) = \inf_{\pi \in \Pi(\mu, \nu)} \int \theta_a \left( \int x - \int y \, dp_x(y) \right) \, d\mu(x).$$

**Theorem : [Gozlan-Roberto-S.-Shu-Tetali 2017]**

Let $\mu \in \mathcal{P}(\mathbb{R})$. The following propositions are equivalent:

1. There exists $a > 0$ such that for all $\nu \in \mathcal{P}(\mathbb{R})$,

   $$\overline{T}_{\theta_a}(\nu | \mu) \leq H(\nu | \mu), \quad \text{and} \quad \overline{T}_{\theta_a}(\mu | \nu) \leq H(\nu | \mu).$$
Characterization of probability measures on $\mathbb{R}$ satisfying a barycentric transport-entropy inequality

Let $\theta : \mathbb{R} \to \mathbb{R}^+$ be a symmetric convex cost function satisfying

$$\theta(t) = t^2, \quad \forall t \leq t_0, \quad \text{for some } t_0 > 0.$$ 

For $a > 0$, let $\theta_a(t) = \theta(at)$, $t \in \mathbb{R}$.

For any $\mu, \nu \in \mathcal{P}(\mathbb{R})$, we consider the barycentric transport cost

$$\overline{T}_{\theta_a}(\nu \| \mu) = \inf_{\pi \in \Pi(\mu, \nu)} \int \theta_a \left( \int x - \int y \, dp_x(y) \right) d\mu(x).$$

**Theorem :** [Gozlan-Roberto-S.-Shu-Tetali 2017]

Let $\mu \in \mathcal{P}(\mathbb{R})$. The following propositions are equivalent:

i) There exists $a > 0$ such that for all $\nu \in \mathcal{P}(\mathbb{R})$,

$$\overline{T}_{\theta_a}(\nu \| \mu) \leq H(\nu \| \mu), \quad \text{and} \quad \overline{T}_{\theta_a}(\mu \| \nu) \leq H(\nu \| \mu).$$

ii) There exists $b > 0$ such that for all $u > 0$,

$$\sup_x (U_\mu(x + u) - U_\mu(x)) \leq \frac{1}{b} \theta^{-1}(u + t_0^2),$$
Characterization of probability measures on \( \mathbb{R} \) satisfying a barycentric transport-entropy inequality

Let \( \theta : \mathbb{R} \to \mathbb{R}^+ \) be a symmetric convex cost function satisfying
\[
\theta(t) = t^2, \quad \forall t \leq t_0, \quad \text{for some } t_0 > 0.
\]

For \( a > 0 \), let \( \theta_a(t) = \theta(at) \), \( t \in \mathbb{R} \).
For any \( \mu, \nu \in \mathcal{P}(\mathbb{R}) \), we consider the barycentric transport cost
\[
\overline{T}_{\theta_a}(\nu|\mu) = \inf_{\pi \in \Pi(\mu,\nu)} \int \theta_a \left( \int x - \int y \, d\pi_x(y) \right) \, d\mu(x).
\]

**Theorem :** [Gozlan-Roberto-S.-Shu-Tetali 2017]

Let \( \mu \in \mathcal{P}(\mathbb{R}) \). The following propositions are equivalent:

i) There exists \( a > 0 \) such that for all \( \nu \in \mathcal{P}(\mathbb{R}) \),
\[
\overline{T}_{\theta_a}(\nu|\mu) \leq H(\nu|\mu), \quad \text{and} \quad \overline{T}_{\theta_a}(\mu|\nu) \leq H(\nu|\mu).
\]

ii) There exists \( b > 0 \) such that for all \( u > 0 \),
\[
\sup_x (U_\mu(x + u) - U_\mu(x)) \leq \frac{1}{b} \theta^{-1}(u + t_0^2),
\]
where
\[
U_\mu(x) := \begin{cases} 
F_\mu^{-1} \left( 1 - \frac{1}{2} e^{-|x|} \right), & \text{if } x \geq 0, \\
F_\mu^{-1} \left( e^{-|x|} \right), & \text{if } x \leq 0.
\end{cases}
\]
Characterization of probability measures on $\mathbb{R}$ satisfying a barycentric transport-entropy inequality

Let $\theta : \mathbb{R} \to \mathbb{R}^+$ be a symmetric convex cost function satisfying

$$\theta(t) = t^2, \quad \forall t \leq t_0,$$

for some $t_0 > 0$.

For $a > 0$, let $\theta_a(t) = \theta(at), \quad t \in \mathbb{R}$.

For any $\mu, \nu \in \mathcal{P}(\mathbb{R})$, we consider the barycentric transport cost

$$\overline{T}_{\theta_a}(\nu|\mu) = \inf_{\pi \in \Pi(\mu, \nu)} \int \theta_a \left( \int x - \int y \, dp_x(y) \right) \, d\mu(x).$$

Theorem: [Gozlan-Roberto-S.-Shu-Tetali 2017]

Let $\mu \in \mathcal{P}(\mathbb{R})$. The following propositions are equivalent:

i) There exists $a > 0$ such that for all $\nu \in \mathcal{P}(\mathbb{R})$,

$$\overline{T}_{\theta_a}(\nu|\mu) \leq H(\nu|\mu), \quad \text{and} \quad \overline{T}_{\theta_a}(\mu|\nu) \leq H(\nu|\mu).$$

ii) There exists $b > 0$ such that for all $u > 0$,

$$\sup_x (U_{\mu}(x + u) - U_{\mu}(x)) \leq \frac{1}{b} \theta^{-1}(u + t_o^2),$$

where

$$U_{\mu}(x) := \begin{cases} 
F_{\mu}^{-1} \left( 1 - \frac{1}{2} e^{-x} \right), & \text{if } x \geq 0, \\
F_{\mu}^{-1} (e^{-|x|}), & \text{if } x \leq 0.
\end{cases}$$

Used by Strzelecka-Strzelecki-Tkocz (2017) to show that any symmetric probability measure with log-concave tails satisfies a barycentric transport inequality with optimal cost, up to a universal constant.
Characterization of probability measures on $\mathbb{R}$ satisfying a barycentric transport-entropy inequality

Let $\theta : \mathbb{R} \to \mathbb{R}^+$ be a symmetric convex cost function satisfying

$$\theta(t) = t^2, \quad \forall t \leq t_0, \quad \text{for some } t_0 > 0.$$

For $a > 0$, let $\theta_a(t) = \theta(at), \quad t \in \mathbb{R}$. For any $\mu, \nu \in \mathcal{P}(\mathbb{R})$, we consider the barycentric transport cost

$$\overline{T}_{\theta_a}(\nu|\mu) = \inf_{\pi \in \Pi(\mu, \nu)} \int \theta_a \left( \int x - \int y \, dp_x(y) \right) \, d\mu(x).$$

**Theorem : [Gozlan-Roberto-S.-Shu-Tetali 2017]**

Let $\mu \in \mathcal{P}(\mathbb{R})$. The following propositions are equivalent :

i) There exists $a > 0$ such that for all $\nu \in \mathcal{P}(\mathbb{R})$,

$$\overline{T}_{\theta_a}(\nu|\mu) \leq H(\nu|\mu), \quad \text{and} \quad \overline{T}_{\theta_a}(\mu|\nu) \leq H(\nu|\mu).$$

ii) There exists $b > 0$ such that for all $u > 0$,

$$\sup_x (U_{\mu}(x + u) - U_{\mu}(x)) \leq \frac{1}{b} \theta^{-1}(u + t_0^2),$$

where

$$U_{\mu}(x) := \begin{cases} 
F^{-1}_\mu \left( 1 - \frac{1}{2} e^{-|x|} \right), & \text{if } x \geq 0, \\
F^{-1}_\mu \left( e^{-|x|} \right), & \text{if } x \leq 0.
\end{cases}$$

Used by Strzelecka-Strzelecki-Tkocz (2017) to show that any symmetric probability measure with log-concave tails satisfies a barycentric transport inequality with optimal cost, up to a universal constant.

→ comparison results for weak and strong moments for random vectors of independent coordinates with log-concave tails.
Weak transport-entropy inequalities on the symmetric group
Weak transport-entropy inequalities on the symmetric group

\( S_n \) : the group of permutations from \( \{1, \ldots, n\} \) to \( \{1, \ldots, n\} \).
Weak transport-entropy inequalities on the symmetric group

\( S_n \) : the group of permutations from \( \{1, \ldots, n\} \) to \( \{1, \ldots, n\} \).

\( d_H \) : the Hamming distance on \( S_n \)
Weak transport-entropy inequalities on the symmetric group

$S_n$ : the group of permutations from $\{1, \ldots, n\}$ to $\{1, \ldots, n\}$.

d_H : the Hamming distance on $S_n$

$$d_H(\sigma, \tau) := \sum_{i=1}^{n} \mathbb{1}_{\sigma(i) \neq \tau(i)}, \quad \sigma, \tau \in S_n.$$
Weak transport-entropy inequalities on the symmetric group

$S_n$ : the group of permutations from $\{1, \ldots, n\}$ to $\{1, \ldots, n\}$.

$d_H$ : the Hamming distance on $S_n$

$$d_H(\sigma, \tau) := \sum_{i=1}^{n} \mathbf{1}_{\sigma(i) \neq \tau(i)}, \quad \sigma, \tau \in S_n.$$  

$\mu_o$ : the uniform distribution on $S_n$,
Weak transport-entropy inequalities on the symmetric group

$S_n$: the group of permutations from $\{1, \ldots, n\}$ to $\{1, \ldots, n\}$.

$d_H$: the Hamming distance on $S_n$

$$d_H(\sigma, \tau) := \sum_{i=1}^{n} 1_{\sigma(i) \neq \tau(i)}, \quad \sigma, \tau \in S_n.$$

$\mu_o$: the uniform distribution on $S_n$, $\mu_o(\sigma) = \frac{1}{n!}$. 

Proof based on Hoeffding's inequality - martingale method.
Weak transport-entropy inequalities on the symmetric group

\( S_n \): the group of permutations from \( \{1, \ldots, n\} \) to \( \{1, \ldots, n\} \).

\( d_H \): the Hamming distance on \( S_n \)

\[ d_H(\sigma, \tau) := \sum_{i=1}^{n} 1_{\sigma(i) \neq \tau(i)}, \quad \sigma, \tau \in S_n. \]

\( \mu_o \): the uniform distribution on \( S_n \), \( \mu_o(\sigma) = \frac{1}{n!} \).

Theorem: [Maurey 1979]
Weak transport-entropy inequalities on the symmetric group

$S_n$ : the group of permutations from $\{1, \ldots, n\}$ to $\{1, \ldots, n\}$.

$d_H$ : the Hamming distance on $S_n$

$$d_H(\sigma, \tau) := \sum_{i=1}^{n} 1_{\sigma(i) \neq \tau(i)}, \quad \sigma, \tau \in S_n.$$ 

$\mu_o$ : the uniform distribution on $S_n$, $\mu_o(\sigma) = \frac{1}{n!}$.

**Theorem : [Maurey 1979]**

For any subset $A \subset S_n$ such that $\mu_o(A) \geq 1/2$,

$$\mu_o(A_t) \geq 1 - 2e^{-\frac{t^2}{64n}}, \quad \forall t \geq 0,$$

Introduction
- Marton's inequality
- Talagrand's concentration

Kantorovich duality
- for classical costs
- for weak costs

Examples of weak cost
- Marton's type of cost
- Barycentric cost
- Strassen's result
- Martingale costs

Weak transport inequalities
- Dual characterization to concentration

Universal transport inequalities

Barycentric transport inequalities
- examples
- characterisation on $\mathbb{R}$

Transport inequality on the symmetric group

Introduction
- Ewens distribution
- deviation inequalities

The Schrödinger minimization problem
- definition
- curvature in discrete spaces
- functional inequalities
- Examples in discrete

Weak transport costs.21
Weak transport-entropy inequalities on the symmetric group

\[ S_n : \text{the group of permutations from } \{1, \ldots, n\} \text{ to } \{1, \ldots, n\}. \]
\[ d_H : \text{the Hamming distance on } S_n \]

\[ d_H(\sigma, \tau) := \sum_{i=1}^{n} 1_{\sigma(i) \neq \tau(i)}, \quad \sigma, \tau \in S_n. \]

\[ \mu_o : \text{the uniform distribution on } S_n, \quad \mu_o(\sigma) = \frac{1}{n!}. \]

**Theorem : [Maurey 1979]**

For any subset \( A \subset S_n \) such that \( \mu_o(A) \geq 1/2, \)

\[ \mu_o(A_t) \geq 1 - 2e^{-\frac{t^2}{64n}}, \quad \forall t \geq 0, \]

where \( A_t := \{ \sigma \in S_n, d_H(\sigma, A) \leq t \}, \quad d_H(\sigma, A) = \inf_{\tau \in A} d_H(\sigma, \tau). \)
Weak transport-entropy inequalities on the symmetric group

\( S_n \) : the group of permutations from \( \{1, \ldots, n\} \) to \( \{1, \ldots, n\} \).

\( d_H \) : the Hamming distance on \( S_n \)

\[
d_H(\sigma, \tau) := \sum_{i=1}^{n} 1_{\sigma(i) \neq \tau(i)}, \quad \sigma, \tau \in S_n.
\]

\( \mu_o \) : the uniform distribution on \( S_n \), \( \mu_o(\sigma) = \frac{1}{n!} \).

**Theorem :** [Maurey 1979]

For any subset \( A \subset S_n \) such that \( \mu_o(A) \geq 1/2 \),

\[
\mu_o(A_t) \geq 1 - 2e^{-\frac{t^2}{64n}}, \quad \forall t \geq 0,
\]

where \( A_t := \{ \sigma \in S_n, d_H(\sigma, A) \leq t \} \), \( d_H(\sigma, A) = \inf_{\tau \in A} d_H(\sigma, \tau) \).

Proof based on Hoeffding’s inequality - martingale method.
Weak transport-entropy inequalities on the symmetric group

$S_n$ : the group of permutations from $\{1, \ldots, n\}$ to $\{1, \ldots, n\}$.

$d_H$ : the Hamming distance on $S_n$

$$d_H(\sigma, \tau) := \sum_{i=1}^{n} 1_{\sigma(i) \neq \tau(i)}, \quad \sigma, \tau \in S_n.$$ 

$\mu_o$ : the uniform distribution on $S_n$, $\mu_o(\sigma) = \frac{1}{n!}$.

**Theorem : [Maurey 1979]**

For any subset $A \subset S_n$ such that $\mu_o(A) \geq 1/2$, 

$$\mu_o(A_t) \geq 1 - 2e^{-\frac{t^2}{64n}}, \quad \forall t \geq 0,$$

where $A_t := \{\sigma \in S_n, d_H(\sigma, A) \leq t\}, \quad d_H(\sigma, A) = \inf_{\tau \in A} d_H(\sigma, \tau)$.

Proof based on Hoeffding’s inequality - martingale method.
Improved concentration result by Talagrand for $\mu_o$
Convex-hull method on $S_n$:
Improved concentration result by Talagrand for $\mu_0$
Convex-hull method on $S_n$: Let $A \subseteq S_n$ and $\sigma \in S_n$,

$$d_H(\sigma, A) := \inf_{\tau \in A} \sum_{i=1}^{n} 1_{\sigma(i) \neq \tau(i)}$$
Improved concentration result by Talagrand for $\mu_0$

Convex-hull method on $S_n$: Let $A \subset S_n$ and $\sigma \in S_n,$

$$d_H(\sigma, A) := \inf_{\tau \in A} \sum_{i=1}^{n} 1_{\sigma(i) \neq \tau(i)} = \inf_{\mathcal{P}(A)} \sum_{i=1}^{n} \int 1_{\sigma(i) \neq \tau(i)} \, dp(\tau).$$
Improved concentration result by Talagrand for $\mu_0$
Convex-hull method on $S_n$: Let $A \subset S_n$ and $\sigma \in S_n$,

$$d_H(\sigma, A) := \inf_{\tau \in A} \sum_{i=1}^{n} 1_{\sigma(i) \neq \tau(i)} = \inf_{p \in \mathcal{P}(A)} \sum_{i=1}^{n} \int 1_{\sigma(i) \neq \tau(i)} \, dp(\tau).$$
Improved concentration result by Talagrand for $\mu_o$
Convex-hull method on $S_n$: Let $A \subset S_n$ and $\sigma \in S_n$,

$$d_H(\sigma, A) := \inf_{\tau \in A} \sum_{i=1}^{n} 1_{\sigma(i) \neq \tau(i)} = \inf_{p \in \mathcal{P}(A)} \sum_{i=1}^{n} \int 1_{\sigma(i) \neq \tau(i)} \ dp(\tau).$$

$$\rightarrow \ c(\sigma, A) := \inf_{p \in \mathcal{P}(A)} c(\sigma, p) = \inf_{p \in \mathcal{P}(A)} \sum_{i=1}^{n} \left( \int 1_{\sigma(i) \neq \tau(i)} \ dp(\tau) \right)^2.$$
Improved concentration result by Talagrand for $\mu_0$

Convex-hull method on $S_n$: Let $A \subset S_n$ and $\sigma \in S_n$,

$$d_H(\sigma, A) := \inf_{\tau \in A} \sum_{i=1}^{n} 1_{\sigma(i) \neq \tau(i)} = \inf_{p \in \mathcal{P}(A)} \sum_{i=1}^{n} \int 1_{\sigma(i) \neq \tau(i)} \, dp(\tau).$$

$$\rightarrow$$

$$c(\sigma, A) := \inf_{p \in \mathcal{P}(A)} c(\sigma, p) = \inf_{p \in \mathcal{P}(A)} \sum_{i=1}^{n} \left( \int 1_{\sigma(i) \neq \tau(i)} \, dp(\tau) \right)^2.$$

By Cauchy-Schwarz inequality, $c(\sigma, A) \geq \frac{1}{n} d_H^2(\sigma, A).$
Improved concentration result by Talagrand for $\mu_0$

Convex-hull method on $S_n$: Let $A \subset S_n$ and $\sigma \in S_n$,

$$d_H(\sigma, A) := \inf_{\tau \in A} \sum_{i=1}^{n} \mathbb{1}_{\sigma(i) \neq \tau(i)} = \inf_{p \in \mathcal{P}(A)} \sum_{i=1}^{n} \int \mathbb{1}_{\sigma(i) \neq \tau(i)} \, dp(\tau).$$

$$\rightarrow \quad c(\sigma, A) := \inf_{p \in \mathcal{P}(A)} c(\sigma, p) = \inf_{p \in \mathcal{P}(A)} \sum_{i=1}^{n} \left( \int \mathbb{1}_{\sigma(i) \neq \tau(i)} \, dp(\tau) \right)^2.$$

By Cauchy-Schwarz inequality, $c(\sigma, A) \geq \frac{1}{n} d_H^2(\sigma, A)$.

**Theorem. [Talagrand 1995]**
Improved concentration result by Talagrand for $\mu_0$

Convex-hull method on $S_n$: Let $A \subset S_n$ and $\sigma \in S_n,$

$$d_H(\sigma, A) := \inf_{\tau \in A} \sum_{i=1}^{n} \mathbb{1}_{\sigma(i) \neq \tau(i)} = \inf_{p \in \mathcal{P}(A)} \sum_{i=1}^{n} \int \mathbb{1}_{\sigma(i) \neq \tau(i)} \, dp(\tau).$$

$$\rightarrow \quad c(\sigma, A) := \inf_{p \in \mathcal{P}(A)} c(\sigma, p) = \inf_{p \in \mathcal{P}(A)} \sum_{i=1}^{n} \left( \int \mathbb{1}_{\sigma(i) \neq \tau(i)} \, dp(\tau) \right)^2.$$

By Cauchy-Schwarz inequality, $c(\sigma, A) \geq \frac{1}{n} d_H^2(\sigma, A)$.

**Theorem.** [Talagrand 1995]

For any subset $A \subset S_n,$

$$\int_{S_n} e^{c(\sigma, A)/16} d\mu_0(\sigma) \leq \frac{1}{\mu_0(A)}.$$
Improved concentration result by Talagrand for $\mu_0$

Convex-hull method on $S_n$ : Let $A \subset S_n$ and $\sigma \in S_n$,

$$d_H(\sigma, A) := \inf_{\tau \in A} \sum_{i=1}^{n} 1_{\sigma(i) \neq \tau(i)} = \inf_{p \in \mathcal{P}(A)} \sum_{i=1}^{n} \int 1_{\sigma(i) \neq \tau(i)} \, dp(\tau).$$

$$\longrightarrow \quad c(\sigma, A) := \inf_{p \in \mathcal{P}(A)} c(\sigma, p) = \inf_{p \in \mathcal{P}(A)} \sum_{i=1}^{n} \left( \int 1_{\sigma(i) \neq \tau(i)} \, dp(\tau) \right)^2.$$

By Cauchy-Schwarz inequality, $c(\sigma, A) \geq \frac{1}{n} d_H^2(\sigma, A)$.

**Theorem. [Talagrand 1995]**

For any subset $A \subset S_n$,

$$\int_{S_n} e^{c(\sigma, A)/16} \, d\mu_0(\sigma) \leq \frac{1}{\mu_0(A)}.$$

By Markov’s inequality, if $\mu_0(A) \geq 1/2$, then

$$\frac{1}{\mu_0(A)} \leq \frac{1}{\mu_0(A)}.$$
Convex-hull method on $S_n$: Let $A \subset S_n$ and $\sigma \in S_n$,

$$d_H(\sigma, A) := \inf_{\tau \in A} \sum_{i=1}^{n} 1_{\sigma(i) \neq \tau(i)} = \inf_{p \in \mathcal{P}(A)} \sum_{i=1}^{n} \int 1_{\sigma(i) \neq \tau(i)} \, dp(\tau).$$

"convex-hull" $c(\sigma, A) := \inf_{p \in \mathcal{P}(A)} c(\sigma, p) = \inf_{p \in \mathcal{P}(A)} \sum_{i=1}^{n} \left( \int 1_{\sigma(i) \neq \tau(i)} \, dp(\tau) \right)^2.$

By Cauchy-Schwarz inequality, $c(\sigma, A) \geq \frac{1}{n} d_H^2(\sigma, A)$.

**Theorem. [Talagrand 1995]**

For any subset $A \subset S_n$,

$$\int_{S_n} e^{c(\sigma, A)/16} \, d\mu_0(\sigma) \leq \frac{1}{\mu_0(A)}.$$

By Markov’s inequality, if $\mu_0(A) \geq 1/2$, then

$$\mu_0(A_{c,r}) \geq 1 - 2e^{-r/16}, \quad u \geq 0,$$
Improved concentration result by Talagrand for $\mu_o$

Convex-hull method on $S_n$: Let $A \subset S_n$ and $\sigma \in S_n$,

$$d_H(\sigma, A) := \inf_{\tau \in A} \sum_{i=1}^{n} 1_{\sigma(i) \neq \tau(i)} = \inf_{\rho \in \mathcal{P}(A)} \sum_{i=1}^{n} \int 1_{\sigma(i) \neq \tau(i)} \, dp(\tau).$$

$$\rightarrow \quad c(\sigma, A) := \inf_{\rho \in \mathcal{P}(A)} c(\sigma, p) = \inf_{\rho \in \mathcal{P}(A)} \sum_{i=1}^{n} \left( \int 1_{\sigma(i) \neq \tau(i)} \, dp(\tau) \right)^2.$$

By Cauchy-Schwarz inequality, $c(\sigma, A) \geq \frac{1}{n} d_H^2(\sigma, A)$.

**Theorem. [Talagrand 1995]**

For any subset $A \subset S_n$,

$$\int_{S_n} e^{c(\sigma, A)} / 16 \, d\mu_o(\sigma) \leq \frac{1}{\mu_o(A)}.$$

By Markov’s inequality, if $\mu_o(A) \geq 1/2$, then

$$\mu_o(A_{c,r}) \geq 1 - 2e^{-r/16}, \quad u \geq 0,$$

where $A_{c,r} = \{\sigma \in S_n, c(\sigma, A) \leq r\}$.
Improved concentration result by Talagrand for $\mu_o$
Convex-hull method on $S_n$: Let $A \subset S_n$ and $\sigma \in S_n,$

$$d_H(\sigma, A) := \inf_{\tau \in A} \sum_{i=1}^{n} 1(\sigma(i) \neq \tau(i)) = \inf_{p \in \mathcal{P}(A)} \sum_{i=1}^{n} \int 1(\sigma(i) \neq \tau(i)) \, dp(\tau).$$

$$\longrightarrow \quad c(\sigma, A) := \inf_{p \in \mathcal{P}(A)} c(\sigma, p) = \inf_{p \in \mathcal{P}(A)} \left( \sum_{i=1}^{n} \int 1(\sigma(i) \neq \tau(i)) \, dp(\tau) \right)^2.$$

By Cauchy-Schwarz inequality, $$c(\sigma, A) \geq \frac{1}{n} d_H^2(\sigma, A).$$

**Theorem. [Talagrand 1995]**
For any subset $A \subset S_n,$

$$\int_{S_n} e^{c(\sigma, A)/16} \, d\mu_o(\sigma) \leq \frac{1}{\mu_o(A)}.$$

By Markov’s inequality, if $\mu_o(A) \geq 1/2,$ then

$$\mu_o(A_{c,r}) \geq 1 - 2e^{-r/16}, \quad u \geq 0,$$

where $A_{c,r} = \{\sigma \in S_n, c(\sigma, A) \leq r\}.$

$$A_{c,r} \subset A_{\sqrt{n}r}.$$
Improved concentration result by Talagrand for $\mu_o$
Convex-hull method on $S_n$ : Let $A \subset S_n$ and $\sigma \in S_n$,

$$ d_H(\sigma, A) := \inf_{\tau \in A} \sum_{i=1}^{n} \mathbb{1}_{\sigma(i) \neq \tau(i)} = \inf_{p \in \mathcal{P}(A)} \sum_{i=1}^{n} \int \mathbb{1}_{\sigma(i) \neq \tau(i)} \, dp(\tau). $$

$$ \rightarrow \quad c(\sigma, A) := \inf_{p \in \mathcal{P}(A)} c(\sigma, p) = \inf_{p \in \mathcal{P}(A)} \sum_{i=1}^{n} \left( \int \mathbb{1}_{\sigma(i) \neq \tau(i)} \, dp(\tau) \right)^2. $$

By Cauchy-Schwarz inequality, $c(\sigma, A) \geq \frac{1}{n} d_H^2(\sigma, A)$.

**Theorem.** [Talagrand 1995]
For any subset $A \subset S_n$,

$$ \int_{S_n} e^{c(\sigma, A)/16} \, d\mu_o(\sigma) \leq \frac{1}{\mu_o(A)}. $$

By Markov’s inequality, if $\mu_o(A) \geq 1/2$, then

$$ \mu_o(A_t) \geq \mu_o(A_{c,r}) \geq 1 - 2e^{-r/16}, \quad u \geq 0, $$

where $A_{c,r} = \{ \sigma \in S_n, c(\sigma, A) \leq r \}$.

$A_{c,r} \subset A_{\sqrt{n}r}$, setting $t = \sqrt{n}r$ we recover Maurey’s concentration inequality.
Extension to subgroups of $S_n$, and for non-uniform probability measures

introduction
Marton's inequality
Talagrand's concentration

Kantorovich duality
for classical costs
for weak costs

Examples of weak cost
Marton's type of cost
Barycentric cost
Strassen's result
Martingale costs

Weak transport inequalities
Dual characterization to concentration

Universal transport inequalities

Barycentric transport inequalities
examples
characterisation on $\mathbb{R}$

Transport inequality on the symmetric group

introduction
Ewens distribution
deviation inequalities

The Schrödinger minimization problem
definition
curvature in discrete spaces
functional inequalities
Examples in discrete

Weak transport costs.23
2002 C. McDiarmid extends Talagrand's results to uniform law on product of symmetric groups.
Extension to subgroups of $S_n$, and for non-uniform probability measures

- 2002 C. McDiarmid extends Talagrand's results to uniform law on product of symmetric groups.
- 2003 M.J. Luczak and McDiarmid $\leadsto$ uniform law on locally acting groups of permutations.
Extension to subgroups of $S_n$, and for non-uniform probability measures

- 2002 C. McDiarmid extends Talagrand's results to uniform law on product of symmetric groups.
- 2003 M.J. Luczak and McDiarmid ⇝ uniform law on locally acting groups of permutations.
- 2017 S. Extensions of Luczak-McDiarmid-Talagrand's results:

$\mu$ P M
Transport-entropy inequalities
Concentration properties for $\mu$.
Extension to subgroups of $S_n$, and for non-uniform probability measures

- 2002 C. McDiarmid extends Talagrand's results to uniform law on product of symmetric groups.

- 2003 M.J. Luczak and McDiarmid ↛ uniform law on locally acting groups of permutations.

- 2017 S. Extensions of Luczak-McDiarmid-Talagrand's results:
  
  1. We consider a larger class of measures $\mathcal{M}$, defined on subgroups $G$ of $S_n$. 
Extension to subgroups of $S_n$, and for non-uniform probability measures

- 2002 C. McDiarmid extends Talagrand's results to uniform law on product of symmetric groups.

- 2003 M.J. Luczak and McDiarmid → uniform law on locally acting groups of permutations.

- 2017 S. Extensions of Luczak-McDiarmid-Talagrand's results:
  1. We consider a larger class of measures $\mathcal{M}$, defined on subgroups $G$ of $S_n$.
  2. We prove some “weak” transport-entropy inequalities for $\mu \in \mathcal{M}$,
Extension to subgroups of $S_n$, and for non-uniform probability measures

- 2002 C. McDiarmid extends Talagrand’s results to uniform law on product of symmetric groups.

- 2003 M.J. Luczak and McDiarmid $\rightsquigarrow$ uniform law on locally acting groups of permutations.

- 2017 S. Extensions of Luczak-McDiarmid-Talagrand’s results:
  1. We consider a larger class of measures $\mathcal{M}$, defined on subgroups $G$ of $S_n$.
  2. We prove some “weak” transport-entropy inequalities for $\mu \in \mathcal{M}$,

  Transport-entropy inequalities for $\mu \implies$ Concentration properties for $\mu$. 

---

Examples of weak cost

- Marton’s type of cost
- Barycentric cost
- Strassen’s result
- Martingale costs

Weak transport inequalities

- Dual characterization to concentration

Universal transport inequalities

- Example characterisation on $\mathbb{R}$

Barycentric transport inequalities

- Examples
- Characterisation on $\mathbb{R}$
Specific example: the Ewens distribution on the symmetric group
Specific example: the Ewens distribution on the symmetric group

\[ \mu^\theta \] : the Ewens distribution of parameter \( \theta > 0 \) on the symmetric group \( S_n \).
Specific example: the Ewens distribution on the symmetric group

\[ \mu^\theta : \text{the Ewens distribution of parameter } \theta > 0 \text{ on the symmetric group } S_n, \]

\[ \mu^\theta (\sigma) = \frac{\theta |\sigma|}{\theta(n)}, \sigma \in S_n, \]
Specific example: the Ewens distribution on the symmetric group

\(\mu^\theta\) : the Ewens distribution of parameter \(\theta > 0\) on the symmetric group \(S_n\),

\[
\mu^\theta(\sigma) = \frac{\theta|\sigma|}{\theta(n)}, \sigma \in S_n,
\]

where

- \(|\sigma|\) is the number of cycles in the cycle decomposition of \(\sigma\),
Specific example: the Ewens distribution on the symmetric group

\( \mu^\theta \): the Ewens distribution of parameter \( \theta > 0 \) on the symmetric group \( S_n \),

\[
\mu^\theta(\sigma) = \frac{\theta^{|\sigma|}}{\theta(n)}, \sigma \in S_n,
\]

where

- \(|\sigma|\) is the number of cycles in the cycle decomposition of \( \sigma \),
- \( \theta(n) \) is the Pochhammer symbol defined by \( \theta(n) = \frac{\Gamma(\theta + n)}{\Gamma(\theta)} \),

Specific example: the Ewens distribution on the symmetric group

\( \mu^\theta \): the Ewens distribution of parameter \( \theta > 0 \) on the symmetric group \( S_n \),

\[
\mu^\theta(\sigma) = \frac{\theta^{|\sigma|}}{\theta(n)}, \sigma \in S_n,
\]

where

- \(|\sigma|\) is the number of cycles in the cycle decomposition of \( \sigma \),
- \( \theta(n) \) is the Pochhammer symbol defined by \( \theta(n) = \frac{\Gamma(\theta + n)}{\Gamma(\theta)} \),

Specific example: the Ewens distribution on the symmetric group

\( \mu^\theta \): the Ewens distribution of parameter \( \theta > 0 \) on the symmetric group \( S_n \),

\[
\mu^\theta(\sigma) = \frac{\theta^{|\sigma|}}{\theta(n)}, \sigma \in S_n,
\]

where

- \(|\sigma|\) is the number of cycles in the cycle decomposition of \( \sigma \),
- \( \theta(n) \) is the Pochhammer symbol defined by \( \theta(n) = \frac{\Gamma(\theta + n)}{\Gamma(\theta)} \),

Specific example: the Ewens distribution on the symmetric group

\( \mu^\theta \) : the Ewens distribution of parameter \( \theta > 0 \) on the symmetric group \( S_n \),

\[
\mu^\theta(\sigma) = \frac{\theta^{|\sigma|}}{\theta(n)}, \sigma \in S_n,
\]

where

- \( |\sigma| \) is the number of cycles in the cycle decomposition of \( \sigma \),
- \( \theta(n) \) is the Pochhammer symbol defined by \( \theta(n) = \frac{\Gamma(\theta + n)}{\Gamma(\theta)} \),

\[
\Gamma(\theta) = \int_0^{+\infty} s^{\theta-1} e^{-s} \, ds.
\]
Specific example: the Ewens distribution on the symmetric group

\( \mu^\theta \) : the Ewens distribution of parameter \( \theta > 0 \) on the symmetric group \( S_n \),

\[
\mu^\theta(\sigma) = \frac{\theta^{\mid \sigma \mid}}{\theta(n)}, \sigma \in S_n,
\]

where

- \( \mid \sigma \mid \) is the number of cycles in the cycle decomposition of \( \sigma \),
- \( \theta(n) \) is the Pochhammer symbol defined by \( \theta(n) = \frac{\Gamma(\theta + n)}{\Gamma(\theta)} \),

\[
\Gamma(\theta) = \int_0^{+\infty} s^{\theta-1} e^{-s} ds.
\]

Result: The Chinese restaurant process. \( \mu^\theta \) is the law of the product of transpositions

\[
(n, U_n)(n-1, U_{n-1}) \cdots (2, U_2),
\]
Specific example: the Ewens distribution on the symmetric group

\[ \mu^\theta : \text{the Ewens distribution of parameter } \theta > 0 \text{ on the symmetric group } S_n, \]

\[ \mu^\theta (\sigma) = \frac{\theta |\sigma|}{\theta(n)}, \sigma \in S_n, \]

where

- \(|\sigma|\) is the number of cycles in the cycle decomposition of \(\sigma\),
- \(\theta(n)\) is the Pochhammer symbol defined by \(\theta(n) = \frac{\Gamma(\theta + n)}{\Gamma(\theta)}\),

\[ \Gamma(\theta) = \int_0^{+\infty} s^{\theta-1} e^{-s} \, ds. \]

Result: The Chinese restaurant process. \(\mu^\theta\) is the law of the product of transpositions

\[ (n, U_n)(n-1, U_{n-1}) \cdots (2, U_2), \]

where the \(U_i\)'s are independent random variables with values in \(\{1, \ldots, i\}\).
Specific example: the Ewens distribution on the symmetric group

$\mu^\theta$ : the Ewens distribution of parameter $\theta > 0$ on the symmetric group $S_n$,

$$
\mu^\theta(\sigma) = \frac{\theta|\sigma|}{\theta(n)}, \sigma \in S_n,
$$

where

- $|\sigma|$ is the number of cycles in the cycle decomposition of $\sigma$,
- $\theta(n)$ is the Pochhammer symbol defined by $\theta(n) = \frac{\Gamma(\theta + n)}{\Gamma(\theta)}$,

$$
\Gamma(\theta) = \int_0^{+\infty} s^{\theta-1} e^{-s} ds.
$$

Result: The Chinese restaurant process. $\mu^\theta$ is the law of the product of transpositions

$$(n, U_n)(n - 1, U_{n-1}) \cdots (2, U_2),$$

where the $U_i$’s are independent random variables with values in $\{1, \ldots, i\}$ and

$$
P(U_i = i) = \frac{\theta}{\theta + i - 1}, \ P(U_i = 1) = \cdots = P(U_i = i - 1) = \frac{1}{\theta + i - 1}.
$$
Specific example: the Ewens distribution on the symmetric group

\( \mu^\theta \) : the Ewens distribution of parameter \( \theta > 0 \) on the symmetric group \( S_n \),

\[
\mu^\theta(\sigma) = \frac{\theta^{|\sigma|}}{\theta(n)}, \sigma \in S_n,
\]

where

- \(|\sigma|\) is the number of cycles in the cycle decomposition of \( \sigma \),
- \( \theta(n) \) is the Pochhammer symbol defined by \( \theta(n) = \frac{\Gamma(\theta + n)}{\Gamma(\theta)} \),

\[
\Gamma(\theta) = \int_0^{+\infty} s^{\theta-1} e^{-s} ds.
\]

Result: The Chinese restaurant process. \( \mu^\theta \) is the law of the product of transpositions

\((n, U_n)(n-1, U_{n-1}) \cdots (2, U_2)\),

where the \( U_i \)'s are independent random variables with values in \( \{1, \ldots, i\} \) and

\[
\mathbb{P}(U_i = i) = \frac{\theta}{\theta + i - 1}, \quad \mathbb{P}(U_i = 1) = \cdots = \mathbb{P}(U_i = i - 1) = \frac{1}{\theta + i - 1}.
\]

Particular case: \( \theta = 1 \),
Specific example: the Ewens distribution on the symmetric group

\( \mu^\theta \) : the Ewens distribution of parameter \( \theta > 0 \) on the symmetric group \( S_n \),

\[
\mu^\theta (\sigma) = \frac{\theta |\sigma|}{\theta(n)}, \quad \sigma \in S_n,
\]

where

- \( |\sigma| \) is the number of cycles in the cycle decomposition of \( \sigma \),
- \( \theta(n) \) is the Pochhammer symbol defined by \( \theta(n) = \frac{\Gamma(\theta + n)}{\Gamma(\theta)} \),

\[
\Gamma(\theta) = \int_0^{+\infty} s^{\theta-1} e^{-s} \, ds.
\]

Result: The Chinese restaurant process. \( \mu^\theta \) is the law of the product of transpositions

\[(n, U_n)(n-1, U_{n-1}) \cdots (2, U_2),\]

where the \( U_i \)'s are independent random variables with values in \( \{1, \ldots, i\} \) and

\[
P(U_i = i) = \frac{\theta}{\theta + i - 1}, \quad P(U_i = 1) = \cdots = P(U_i = i - 1) = \frac{1}{\theta + i - 1}.
\]

Particular case: \( \theta = 1 \), \( \mu^\theta \) is the uniform distribution on \( S_n \), \( \mu^\theta = \mu_o \).
Weak transport inequality for the Ewens distribution

Let us define the weak-transport cost: $u$ with $c_{\nu,1} \geq \nu \sigma$, $T |\nu|_2 \leq \nu |p_{\mu} S^\sigma d H \Pi \hat{\zeta}$, one has $\nu |p_{\mu} S^\sigma d H \Pi \hat{\zeta}$.

Key properties for the proof:

- Marton's inequality
- Talagrand's concentration

Kantorovich duality
- for classical costs
- for weak costs

Examples of weak cost
- Marton's type of cost
- Barycentric cost
- Strassen's result
- Martingale costs

Weak transport inequalities
- Dual characterization
to concentration

Universal transport inequalities

Barycentric transport inequalities
- examples
- characterisation on $\mathbb{R}$

Transport inequality on the symmetric group
- introduction

Ewens distribution
- deviation inequalities

The Schrödinger minimization problem
- definition
- curvature in discrete spaces
- functional inequalities
- Examples in discrete
- Weak transport costs
Weak transport inequality for the Ewens distribution

Let us define the weak-transport cost:

\[ \mathcal{T}_2(\nu_2 | \nu_1) \]
Weak transport inequality for the Ewens distribution

Let us define the weak-transport cost:

\[ T_2(\nu_2|\nu_1) := \inf_{\pi \in \Pi(\nu_1, \nu_2)} \int \sum_{i=1}^{n} \left( \int \mathbb{1}_{\sigma(i) \neq \tau(i)} \, d\rho_{\sigma}(\tau) \right)^2 \, d\nu_1(\sigma). \]
Weak transport inequality for the Ewens distribution

Let us define the weak-transport cost :

$$\mathcal{T}_2(\nu_2|\nu_1) := \inf_{\pi \in \Pi(\nu_1, \nu_2)} \int \sum_{i=1}^n \left( \int 1_{\sigma(i) \neq \tau(i)} d\nu_2(\tau) \right)^2 d\nu_1(\sigma).$$

By Cauchy-Schwarz inequality

$$\frac{1}{n} W_1^2(\nu_1, \nu_2) \leq \mathcal{T}_2(\nu_2|\nu_1) \leq W_1(\nu_1, \nu_2),$$

where $W_1$ is the Wasserstein distance on $\mathcal{P}(S_n)$ associated to $d_H$. 
Weak transport inequality for the Ewens distribution

Let us define the weak-transport cost:

$$\tilde{T}_2(\nu_2|\nu_1) := \inf_{\pi \in \Pi(\nu_1, \nu_2) \atop \pi = \nu_1 \otimes \rho} \int \sum_{i=1}^{n} \left( \int 1_{\sigma(i) \neq \tau(i)} d\rho_{\sigma}(\tau) \right)^2 d\nu_1(\sigma).$$

By Cauchy-Schwarz inequality

$$\frac{1}{n} W_1^2(\nu_1, \nu_2) \leq \tilde{T}_2(\nu_2|\nu_1) \leq W_1(\nu_1, \nu_2),$$

where $W_1$ is the Wasserstein distance on $\mathcal{P}(S_n)$ associated to $d_H$.

**Theorem : [S. 2017]**

For all $s \in (0, 1)$,

$$\frac{1}{20} \tilde{T}_2(\nu_2|\nu_1) \leq \frac{1}{s} H(\nu_1|\mu^\theta) + \frac{1}{1-s} H(\nu_2|\mu^\theta), \forall \nu_1, \nu_2 \in \mathcal{P}(S_n),$$

where $\mu^\theta$ is the Ewens distribution.
Weak transport inequality for the Ewens distribution

Let us define the weak-transport cost:

\[
\tilde{T}_2(\nu_2|\nu_1) := \inf_{\pi \in \Pi(\nu_1, \nu_2)} \int \sum_{i=1}^{n} \left( \int 1_{\sigma(i) \neq \tau(i)} d\rho_\sigma(\tau) \right)^2 d\nu_1(\sigma).
\]

By Cauchy-Schwarz inequality

\[
\frac{1}{n} W_1^2(\nu_1, \nu_2) \leq \tilde{T}_2(\nu_2|\nu_1) \leq W_1(\nu_1, \nu_2),
\]

where \( W_1 \) is the Wasserstein distance on \( \mathcal{P}(S_n) \) associated to \( d_H \).

**Theorem : [S. 2017]**

For all \( s \in (0, 1) \),

\[
\frac{1}{20} \tilde{T}_2(\nu_2|\nu_1) \leq \frac{1}{s} H(\nu_1|\mu^\theta) + \frac{1}{1-s} H(\nu_2|\mu^\theta), \quad \forall \nu_1, \nu_2 \in \mathcal{P}(S_n),
\]

or equivalently, for all function \( \varphi : S_n \rightarrow \mathbb{R} \), one has

\[
\left( \int_{S_n} e^{s\tilde{Q}\varphi} d\mu^\theta \right)^{1/s} \left( \int_{S_n} e^{-(1-s)\varphi} d\mu^\theta \right)^{1/(1-s)} \leq 1,
\]
Weak transport inequality for the Ewens distribution

Let us define the weak-transport cost:

\[ \tilde{T}_2(\nu_2|\nu_1) := \inf_{\pi \in \Pi(\nu_1, \nu_2)} \int \sum_{i=1}^{n} \left( \int 1_{\sigma(i) \neq \tau(i)} \, dp_{\sigma}(\tau) \right)^2 \, d\nu_1(\sigma). \]

By Cauchy-Schwarz inequality \( \frac{1}{n} W_1^2(\nu_1, \nu_2) \leq \tilde{T}_2(\nu_2|\nu_1) \leq W_1(\nu_1, \nu_2) \),
where \( W_1 \) is the Wasserstein distance on \( \mathcal{P}(S_n) \) associated to \( d_H \).

**Theorem** [S. 2017]

For all \( s \in (0, 1) \),

\[ \frac{1}{20} \tilde{T}_2(\nu_2|\nu_1) \leq \frac{1}{s} H(\nu_1|\mu^\theta) + \frac{1}{1-s} H(\nu_2|\mu^\theta), \quad \forall \nu_1, \nu_2 \in \mathcal{P}(S_n), \]

or equivalently, for all function \( \varphi : S_n \to \mathbb{R} \), one has

\[ \left( \int_{S_n} e^{s \tilde{Q}\varphi} \, d\mu^\theta \right)^{1/s} \left( \int_{S_n} e^{-(1-s)\varphi} \, d\mu^\theta \right)^{1/(1-s)} \leq 1, \]

where \( \tilde{Q}\varphi(\sigma) = \inf_{\sigma' \in \mathcal{P}(S_n)} \left\{ \int \varphi(\tau) \, dp(\tau) + c(\sigma, \sigma') \right\}, \quad \sigma \in S_n, \)
Weak transport inequality for the Ewens distribution

Let us define the weak-transport cost:

$$\mathcal{T}_2(\nu_2|\nu_1) := \inf_{\pi \in \Pi(\nu_1, \nu_2)} \int \sum_{i=1}^{n} \left( \int 1_{\sigma(i) \neq \tau(i)} d\nu_1(\tau) \right)^2 d\nu_1(\sigma).$$

By Cauchy-Schwarz inequality

$$\frac{1}{n} W_1^2(\nu_1, \nu_2) \leq \mathcal{T}_2(\nu_2|\nu_1) \leq W_1(\nu_1, \nu_2),$$

where $W_1$ is the Wasserstein distance on $\mathcal{P}(S_n)$ associated to $d_H$.

**Theorem: [S. 2017]**

For all $s \in (0, 1)$,

$$\frac{1}{20} \mathcal{T}_2(\nu_2|\nu_1) \leq \frac{1}{s} H(\nu_1|\mu^\theta) + \frac{1}{1-s} H(\nu_2|\mu^\theta), \ \forall \nu_1, \nu_2 \in \mathcal{P}(S_n),$$

or equivalently, for all function $\varphi : S_n \to \mathbb{R}$, one has

$$\left( \int_{S_n} e^{s\hat{Q}_\varphi} d\mu^\theta \right)^{1/s} \left( \int_{S_n} e^{-(1-s)\varphi} d\mu^\theta \right)^{1/(1-s)} \leq 1,$$

where $\hat{Q}_\varphi(\sigma) = \inf_{p \in \mathcal{P}(S_n)} \left\{ \int \varphi(\tau) dp(\tau) + \hat{c}(\sigma, p) \right\}, \ \sigma \in S_n,$

with $\hat{c}(\sigma, p) = \frac{1}{20} \sum_{i=1}^{n} \left( \int 1_{\sigma(i) \neq \tau(i)} dp(\tau) \right)^2$.
Weak transport inequality for the Ewens distribution

Let us define the weak-transport cost:

$$\tilde{T}_2(\nu_2|\nu_1) := \inf_{\pi \in \Pi(\nu_1, \nu_2)} \int \sum_{i=1}^{n} \left( \int 1_{\sigma(i) \neq \tau(i)} d\rho_\sigma(\tau) \right)^2 d\nu_1(\sigma).$$

By Cauchy-Schwarz inequality:

$$\frac{1}{n} W_1^2(\nu_1, \nu_2) \leq \tilde{T}_2(\nu_2|\nu_1) \leq W_1(\nu_1, \nu_2),$$

where $W_1$ is the Wasserstein distance on $\mathcal{P}(S_n)$ associated to $d_H$.

**Theorem: [S. 2017]**

For all $s \in (0, 1)$,

$$\frac{1}{20} \tilde{T}_2(\nu_2|\nu_1) \leq \frac{1}{s} H(\nu_1|\mu^\theta) + \frac{1}{1-s} H(\nu_2|\mu^\theta), \ \forall \nu_1, \nu_2 \in \mathcal{P}(S_n),$$

or equivalently, for all function $\varphi : S_n \to \mathbb{R}$, one has

$$\left( \int_{S_n} e^{s\tilde{Q}\varphi} d\mu^\theta \right)^{1/s} \left( \int_{S_n} e^{-(1-s)\varphi} d\mu^\theta \right)^{1/(1-s)} \leq 1,$$

where

$$\tilde{Q}\varphi(\sigma) = \inf_{\rho \in \mathcal{P}(S_n)} \left\{ \int \varphi(\tau) d\rho(\tau) + \tilde{c}(\sigma, \rho) \right\}, \ \sigma \in S_n,$$

with

$$\tilde{c}(\sigma, \rho) = \frac{1}{20} \sum_{i=1}^{n} \left( \int 1_{\sigma(i) \neq \tau(i)} d\rho(\tau) \right)^2.$$

**Key properties for the proof:** The Chinese restaurant process, $\mu^\theta(\sigma) = \mu^\theta(\sigma^{-1})$ and $\mu^\theta(\sigma) = \mu^\theta(t^{-1}\sigma t)$, $\forall t \in S_n$. 

---

**Introduction**

Marton’s inequality

Talagrand’s concentration

Kantorovich duality

for classical costs

for weak costs

Examples of weak cost

Marton’s type of cost

Barycentric cost

Strassen’s result

Martingale costs

Weak transport

inequalities

Dual characterization to concentration

Universal transport inequalities

Barycentric transport inequalities examples

characterisation on $\mathbb{R}$

Transport inequality on the symmetric group introduction

Ewens distribution

deviation inequalities

The Schrödinger minimization problem definition

curvature in discrete spaces

functional inequalities

Examples in discrete

Weak transport costs.25
Weak transport inequality for the Ewens distribution

Let us define the weak-transport cost:

$$\hat{T}_2(\nu_2|\nu_1) := \inf_{\pi \in \Pi(\nu_1, \nu_2)} \int \sum_{i=1}^{\pi(\nu_1, \nu_2)} \left( \int 1_{\sigma(i) \neq \tau(i)} \, d\rho(\tau) \right)^2 \, d\nu_1(\sigma).$$

By Cauchy-Schwarz inequality

$$\frac{1}{n} W_1^2(\nu_1, \nu_2) \leq \hat{T}_2(\nu_2|\nu_1) \leq W_1(\nu_1, \nu_2),$$

where $W_1$ is the Wasserstein distance on $\mathcal{P}(S_n)$ associated to $d_H$.

**Theorem : [S. 2017]**

For all $s \in (0, 1)$,

$$\frac{1}{20} \hat{T}_2(\nu_2|\nu_1) \leq \frac{1}{s} H(\nu_1|\mu^\theta) + \frac{1}{1-s} H(\nu_2|\mu^\theta), \quad \forall \nu_1, \nu_2 \in \mathcal{P}(S_n),$$

or equivalently, for all function $\varphi : S_n \to \mathbb{R}$, one has

$$\left( \int_{S_n} e^{s\hat{Q}\varphi} \, d\mu^\theta \right)^{1/s} \left( \int_{S_n} e^{-(1-s)\varphi} \, d\mu^\theta \right)^{1/(1-s)} \leq 1,$$

where

$$\hat{Q}\varphi(\sigma) = \inf_{p \in \mathcal{P}(S_n)} \left\{ \int \varphi(\tau) \, dp(\tau) + \hat{c}(\sigma, p) \right\}, \quad \sigma \in S_n,$$

with

$$\hat{c}(\sigma, p) = \frac{1}{20} \sum_{i=1}^{n} \left( \int 1_{\sigma(i) \neq \tau(i)} \, dp(\tau) \right)^2.$$

**Key properties for the proof :** The Chinese restaurant process,

$$\mu^\theta(\sigma) = \mu^\theta(\sigma^{-1}) \quad \text{and} \quad \mu^\theta(\sigma) = \mu^\theta(t^{-1}\sigma t) \quad \forall t \in S_n.$$
Weak transport inequality for the Ewens distribution

Let us define the weak-transport cost:

\[
\tilde{T}_2(\nu_2|\nu_1) := \inf_{\pi \in \Pi(\nu_1, \nu_2)} \int \sum_{i=1}^{n} \left( \int 1_{\sigma(i) \neq \tau(i)} d\nu_1(\tau) \right)^2 d\nu_1(\sigma).
\]

By Cauchy-Schwarz inequality

\[
\frac{1}{n} W_1^2(\nu_1, \nu_2) \leq \tilde{T}_2(\nu_2|\nu_1) \leq W_1(\nu_1, \nu_2),
\]

where \( W_1 \) is the Wasserstein distance on \( \mathcal{P}(S_n) \) associated to \( d_H \).

Theorem: [S. 2017]

For all \( s \in (0, 1) \),

\[
\frac{1}{20} \tilde{T}_2(\nu_2|\nu_1) \leq \frac{1}{s} H(\nu_1|\mu^\theta) + \frac{1}{1-s} H(\nu_2|\mu^\theta), \quad \forall \nu_1, \nu_2 \in \mathcal{P}(S_n),
\]

or equivalently, for all function \( \varphi : S_n \to \mathbb{R} \), one has

\[
\left( \int_{S_n} e^{s\tilde{Q}_{\varphi}} d\mu^\theta \right)^{1/s} \left( \int_{S_n} e^{-(1-s)\varphi} d\mu^\theta \right)^{1/(1-s)} \leq 1,
\]

where \( \tilde{Q}_{\varphi}(\sigma) = \inf_{p \in \mathcal{P}(S_n)} \left\{ \int \varphi(\tau) d\nu(\tau) + \tilde{c}(\sigma, p) \right\}, \quad \sigma \in S_n, \)

with \( \tilde{c}(\sigma, p) = \frac{1}{20} \sum_{i=1}^{n} \left( \int 1_{\sigma(i) \neq \tau(i)} d\nu(\tau) \right)^2. \)

Key properties for the proof: The Chinese restaurant process,

\[
\mu^\theta(\sigma) = \mu^\theta(\sigma^{-1}) \quad \text{and} \quad \mu^\theta(\sigma) = \mu^\theta(t^{-1}\sigma t) \quad \forall t \in S_n.
\]
Weak transport inequality for the Ewens distribution

Let us define the weak-transport cost:

$$\tilde{T}_2(\nu_2|\nu_1) := \inf_{\pi \in \Pi(\nu_1, \nu_2) \atop \pi = \nu_1 \otimes \rho} \int \sum_{i=1}^{\infty} \left( \int 1_{\sigma(i) \neq \tau(i)} \, dp_\sigma(\tau) \right)^2 \, d\nu_1(\sigma).$$

By Cauchy-Schwarz inequality

$$\frac{1}{n} W_1^2(\nu_1, \nu_2) \leq \tilde{T}_2(\nu_2|\nu_1) \leq W_1(\nu_1, \nu_2),$$

where $W_1$ is the Wasserstein distance on $\mathcal{P}(S_n)$ associated to $d_H$.

**Theorem** [S. 2017]

For all $s \in (0, 1)$,

$$\frac{1}{20} \tilde{T}_2(\nu_2|\nu_1) \leq \frac{1}{s} H(\nu_1|\mu^\theta) + \frac{1}{1-s} H(\nu_2|\mu^\theta), \ \forall \nu_1, \nu_2 \in \mathcal{P}(S_n),$$

or equivalently, for all function $\varphi : S_n \to \mathbb{R}$, one has

$$\left( \int_{S_n} e^{s\hat{Q}\varphi} \, d\mu^\theta \right)^{1/s} \left( \int_{S_n} e^{-(1-s)\varphi} \, d\mu^\theta \right)^{1/(1-s)} \leq 1,$$

where

$$\hat{Q}\varphi(\sigma) = \inf_{p \in \mathcal{P}(S_n)} \left\{ \int \varphi(\tau) \, dp(\tau) + \hat{c}(\sigma, p) \right\}, \ \sigma \in S_n,$$

with

$$\hat{c}(\sigma, p) = \frac{1}{20} \sum_{i=1}^{n} \left( \int 1_{\sigma(i) \neq \tau(i)} \, dp(\tau) \right)^2.$$

**Key properties for the proof**:

The Chinese restaurant process,

$$\mu^\theta(\sigma) = \mu^\theta(\sigma^{-1}) \quad \text{and} \quad \mu^\theta(\sigma) = \mu^\theta(t^{-1}\sigma t) \quad \forall t \in S_n.$$
Weak transport inequality for the Ewens distribution

Let us define the weak-transport cost:

$$\mathcal{T}_2(\nu_2|\nu_1) := \inf_{\pi \in \Pi(\nu_1, \nu_2)} \int \sum_{i=1}^n \left( \int \mathbb{1}_{\sigma(i) \neq \tau(i)} d\pi(\tau) \right)^2 d\nu_1(\sigma).$$

By Cauchy-Schwarz inequality, $$\frac{1}{n} W_1^2(\nu_1, \nu_2) \leq \mathcal{T}_2(\nu_2|\nu_1) \leq W_1(\nu_1, \nu_2),$$ where $$W_1$$ is the Wasserstein distance on $$\mathcal{P}(S_n)$$ associated to $$d_H$$.

**Theorem : [S. 2017]**

For all $$s \in (0, 1)$$,

$$\frac{1}{20} \mathcal{T}_2(\nu_2|\nu_1) \leq \frac{1}{s} H(\nu_1|\mu^\theta) + \frac{1}{1-s} H(\nu_2|\mu^\theta), \ \forall \nu_1, \nu_2 \in \mathcal{P}(S_n),$$

or equivalently, for all function $$\varphi : S_n \to \mathbb{R}$$, one has

$$\left( \int_{S_n} e^{s\hat{Q}_\varphi} d\mu^\theta \right)^{1/s} \left( \int_{S_n} e^{-(1-s)\varphi} d\mu^\theta \right)^{1/(1-s)} \leq 1,$$

where $$\hat{Q}_\varphi(\sigma) = \inf_{p \in \mathcal{P}(S_n)} \left\{ \int_{S_n} \varphi(\tau) d\pi(\tau) + \hat{c}(\sigma, p) \right\}, \ \sigma \in S_n,$$

with $$\hat{c}(\sigma, p) = \frac{1}{20} \sum_{i=1}^n \left( \int \mathbb{1}_{\sigma(i) \neq \tau(i)} d\pi(\tau) \right)^2.$$

**Key properties for the proof:** The Chinese restaurant process,

$$\mu^\theta(\sigma) = \mu^\theta(\sigma^{-1}) \quad \text{and} \quad \mu^\theta(\sigma) = \mu^\theta(t^{-1}\sigma t) \quad \forall t \in S_n.$$
Application to concentration on $S_n$
Application to concentration on $S_n$

\[
\left( \int_{S_n} e^{s \bar{Q}\varphi} \, d\mu^{\theta} \right)^{1/s} \left( \int_{S_n} e^{-(1-s)\varphi} \, d\mu^{\theta} \right)^{1/(1-s)} \leq 1,
\]
Application to concentration on $S_n$

\[
\left( \int_{S_n} e^{s\tilde{Q}\varphi} d\mu^\theta \right)^{1/s} \left( \int_{S_n} e^{-(1-s)\varphi} d\mu^\theta \right)^{1/(1-s)} \leq 1,
\]

Assume $\varphi$ is a configuration function:
Application to concentration on $S_n$

$$
\left( \int_{S_n} e^{s\hat{Q}\varphi} \, d\mu^\theta \right)^{1/s} \left( \int_{S_n} e^{-(1-s)\varphi} \, d\mu^\theta \right)^{1/(1-s)} \leq 1,
$$

Assume $\varphi$ is a configuration function: there exist functions $\alpha_i : S_n \to \mathbb{R}^+$ such that

$$
\varphi(\tau) \geq \varphi(\sigma) - \sum_{i=1}^{n} \alpha_i(\sigma) \mathbb{1}_{\sigma(i) \neq \tau(i)} \quad \forall \sigma, \tau \in S_n.
$$
Application to concentration on $S_n$

$$
\left( \int_{S_n} e^{s \hat{Q} \varphi} d\mu^\theta \right)^{1/s} \left( \int_{S_n} e^{-(1-s)\varphi} d\mu^\theta \right)^{1/(1-s)} \leq 1,
$$

Assume $\varphi$ is a configuration function: there exist functions $\alpha_i : S_n \to \mathbb{R}^+$ such that

$$
\varphi(\tau) \geq \varphi(\sigma) - \sum_{i=1}^n \alpha_i(\sigma) \mathbb{1}_{\sigma(i) \neq \tau(i)} \quad \forall \sigma, \tau \in S_n.
$$

It follows that

$$
\hat{Q} \varphi(\sigma) = \inf_{p \in \mathcal{P}(S_n)} \left\{ \int \varphi(\tau) dp(\tau) + \frac{1}{20} \sum_{i=1}^n \left( \int \mathbb{1}_{\sigma(i) \neq \tau(i)} dp(\tau) \right)^2 \right\}
$$

$$
\geq \varphi(\sigma) - \sup_p \sum_{i=1}^n \left[ \alpha_i(\sigma) \int \mathbb{1}_{\sigma(i) \neq \tau(i)} dp(\tau) - \frac{1}{20} \left( \int \mathbb{1}_{\sigma(i) \neq \tau(i)} dp(\tau) \right)^2 \right]
$$

$$
\geq \varphi(\sigma) - \sum_{i=1}^n \sup_{l \geq 0} \left\{ \alpha_i(\sigma) l - \frac{l^2}{20} \right\}
$$

$$
= \varphi(\sigma) - 5 \sum_{i=1}^n \alpha_i^2(\sigma)
$$

$$
= \varphi(\sigma) - 5 |\alpha(\sigma)|_2^2,
$$
Application to concentration on $S_n$

\[
\left( \int_{S_n} e^{s\tilde{Q}\varphi} d\mu^\theta \right)^{1/s} \left( \int_{S_n} e^{-(1-s)\varphi} d\mu^\theta \right)^{1/(1-s)} \leq 1,
\]

Assume $\varphi$ is a configuration function: there exist functions $\alpha_i : S_n \rightarrow \mathbb{R}^+$ such that

\[
\varphi(\tau) \geq \varphi(\sigma) - \sum_{i=1}^{n} \alpha_i(\sigma) \mathbb{1}_{\sigma(i) \neq \tau(i)} \quad \forall \sigma, \tau \in S_n.
\]

It follows that

\[
\tilde{Q}\varphi(\sigma) \geq \varphi(\sigma) - 5|\alpha(\sigma)|^2,
\]
Application to concentration on $S_n$

\[
\left( \int_{S_n} e^{s \tilde{Q} \varphi} \, d\mu^\theta \right)^{1/s} \left( \int_{S_n} e^{-(1-s)\varphi} \, d\mu^\theta \right)^{1/(1-s)} \leq 1,
\]

Assume $\varphi$ is a configuration function: there exist functions $\alpha_i : S_n \to \mathbb{R}^+$ such that

\[
\varphi(\tau) \geq \varphi(\sigma) - \sum_{i=1}^n \alpha_i(\sigma) \mathbb{1}_{\sigma(i) \neq \tau(i)} \quad \forall \sigma, \tau \in S_n.
\]

It follows that \(\tilde{Q} \varphi(\sigma) \geq \varphi(\sigma) - 5|\alpha(\sigma)|^2\),

Let $\mu = \mu^\theta$ and $\mu(\varphi) = \int \varphi \, d\mu$. 
Application to concentration on $S_n$

$$\left( \int_{S_n} e^{s\tilde{Q}\varphi} d\mu^\theta \right)^{1/s} \left( \int_{S_n} e^{-(1-s)\varphi} d\mu^\theta \right)^{1/(1-s)} \leq 1,$$

Assume $\varphi$ is a configuration function: there exist functions $\alpha_i : S_n \to \mathbb{R}^+$ such that

$$\varphi(\tau) \geq \varphi(\sigma) - \sum_{i=1}^n \alpha_i(\sigma) \mathbb{1}_{\sigma(i) \neq \tau(i)} \quad \forall \sigma, \tau \in S_n.$$

It follows that

$$\tilde{Q}\varphi(\sigma) \geq \varphi(\sigma) - 5|\alpha(\sigma)|^2,$$

Let $\mu = \mu^\theta$ and $\mu(\varphi) = \int \varphi \, d\mu$.

As $s \to 0$, 

$$\left( \int_{S_n} e^{s\tilde{Q}\varphi} d\mu^\theta \right)^{1/s} \left( \int_{S_n} e^{-(1-s)\varphi} d\mu^\theta \right)^{1/(1-s)} \leq 1,$$
Application to concentration on $S_n$

\[
\left( \int_{S_n} e^{s\tilde{Q}\varphi} \, d\mu^\theta \right)^{1/s} \left( \int_{S_n} e^{-(1-s)\varphi} \, d\mu^\vartheta \right)^{1/(1-s)} \leq 1,
\]

Assume $\varphi$ is a configuration function: there exist functions $\alpha_i : S_n \to \mathbb{R}^+$ such that

\[
\varphi(\tau) \geq \varphi(\sigma) - \sum_{i=1}^{n} \alpha_i(\sigma) \mathbb{1}_{\sigma(i) \neq \tau(i)} \quad \forall \sigma, \tau \in S_n.
\]

It follows that

\[
\tilde{Q}\varphi(\sigma) \geq \varphi(\sigma) - 5|\alpha(\sigma)|^2,
\]

Let $\mu = \mu^\theta$ and $\mu(\varphi) = \int \varphi \, d\mu$.

As $s \to 0$, \( e^{\mu(\tilde{Q}\varphi)} \int e^{-\varphi} \, d\mu \leq 1 \).
Application to concentration on $S_n$

$$\left( \int_{S_n} e^{s\tilde{Q} \varphi} \, d\mu^\theta \right)^{1/s} \left( \int_{S_n} e^{-(1-s)\varphi} \, d\mu^\theta \right)^{1/(1-s)} \leq 1,$$

Assume $\varphi$ is a configuration function: there exist functions $\alpha_i : S_n \to \mathbb{R}^+$ such that

$$\varphi(\tau) \geq \varphi(\sigma) - \sum_{i=1}^{n} \alpha_i(\sigma) \mathbb{1}_{\sigma(i) \neq \tau(i)} \quad \forall \sigma, \tau \in S_n.$$

It follows that

$$\tilde{Q} \varphi(\sigma) \geq \varphi(\sigma) - 5|\alpha(\sigma)|_2^2,$$

Let $\mu = \mu^\theta$ and $\mu(\varphi) = \int \varphi \, d\mu$.

As $s \to 0$, 

$$e^{\mu(\tilde{Q} \varphi)} \int e^{-\varphi} \, d\mu \leq 1, \quad \int e^{-\varphi} \, d\mu \leq e^{-\mu(\varphi) + 5\mu(|\alpha|_2^2)}.$$
Application to concentration on $S_n$

\[
\left(\int_{S_n} e^{s\tilde{Q}\varphi} d\mu^\theta\right)^{1/s} \left(\int_{S_n} e^{-(1-s)\varphi} d\mu^\theta\right)^{1/(1-s)} \leq 1,
\]

Assume $\varphi$ is a configuration function: there exist functions $\alpha_i : S_n \to \mathbb{R}^+$ such that

\[
\varphi(\tau) \geq \varphi(\sigma) - \sum_{i=1}^n \alpha_i(\sigma) 1_{\sigma(i) \neq \tau(i)} \quad \forall \sigma, \tau \in S_n.
\]

It follows that

\[
\tilde{Q}\varphi(\sigma) \geq \varphi(\sigma) - 5|\alpha(\sigma)|^2_2,
\]

Let $\mu = \mu^\theta$ and $\mu(\varphi) = \int \varphi \, d\mu$.

As $s \to 0$, 

\[
e^{\mu(\tilde{Q}\varphi)} \int e^{-\varphi} \, d\mu \leq 1, \quad \int e^{-\varphi} \, d\mu \leq e^{-\mu(\varphi)} + 5\mu(|\alpha|^2_2).
\]

As $s \to 1$,
Application to concentration on $S_n$

$$\left( \int_{S_n} e^{s\tilde{Q}\varphi} d\mu^\theta \right)^{1/s} \left( \int_{S_n} e^{-(1-s)\varphi} d\mu^\theta \right)^{1/(1-s)} \leq 1,$$

Assume $\varphi$ is a configuration function: there exist functions $\alpha_i : S_n \to \mathbb{R}^+$ such that

$$\varphi(\tau) \geq \varphi(\sigma) - \sum_{i=1}^{n} \alpha_i(\sigma) \mathbb{1}_{\sigma(i) \neq \tau(i)} \quad \forall \sigma, \tau \in S_n.$$

It follows that

$$\tilde{Q}\varphi(\sigma) \geq \varphi(\sigma) - 5|\alpha(\sigma)|_2^2,$$

Let $\mu = \mu^\theta$ and $\mu(\varphi) = \int \varphi \, d\mu$.

As $s \to 0$, 

$$e^{\mu(\tilde{Q}\varphi)} \int e^{-\varphi} \, d\mu \leq 1, \quad \int e^{-\varphi} \, d\mu \leq e^{-\mu(\varphi)} + 5\mu(|\alpha|_2^2).$$

As $s \to 1$, 

$$\int e^{\tilde{Q}\varphi} \, d\mu \, e^{-\mu(\varphi)} \leq 1,$$
Application to concentration on $S_n$

\[
\left( \int_{S_n} e^{s \tilde{Q} \varphi} d\mu^\theta \right)^{1/s} \left( \int_{S_n} e^{-(1-s) \varphi} d\mu^\theta \right)^{1/(1-s)} \leq 1,
\]

Assume $\varphi$ is a configuration function: there exist functions $\alpha_i : S_n \to \mathbb{R}^+$ such that

\[
\varphi(\tau) \geq \varphi(\sigma) - \sum_{i=1}^n \alpha_i(\sigma) \mathbb{1}_{\sigma(i) \neq \tau(i)} \quad \forall \sigma, \tau \in S_n.
\]

It follows that

\[
\tilde{Q} \varphi(\sigma) \geq \varphi(\sigma) - 5|\alpha(\sigma)|_2^2,
\]

Let $\mu = \mu^\theta$ and $\mu(\varphi) = \int \varphi \, d\mu$.

As $s \to 0$,

\[
e^{\mu(\tilde{Q} \varphi)} \int e^{-\varphi} \, d\mu \leq 1, \quad \int e^{-\varphi} \, d\mu \leq e^{-\mu(\varphi) + 5\mu(|\alpha|_2^2)}.
\]

As $s \to 1$,

\[
\int e^{\tilde{Q} \varphi} \, d\mu \, e^{-\mu(\varphi)} \leq 1, \quad \int e^{\varphi} \, d\mu \leq e^{\mu(\varphi)}.
\]
Examples of configuration functions on $S_n$. 
Examples of configuration functions on $S_n$.

- $\varphi(\sigma) = |\sigma|_k$: number of cycles of length $k$ in the cycle decomposition of $\sigma$. 
Examples of configuration functions on $S_n$.

- $\varphi(\sigma) = |\sigma|_k$: number of cycles of length $k$ in the cycle decomposition of $\sigma$.
- $|\sigma|_1$: the number of fixed points by $\sigma$.
Examples of configuration functions on $S_n$.

- $\varphi(\sigma) = |\sigma|_k$ : number of cycles of length $k$ in the cycle decomposition of $\sigma$.
- $|\sigma|_1$ : the number of fixed points by $\sigma$.

Let $m_k = \int |\sigma|_k \, d\mu(\sigma)$.
Examples of configuration functions on $S_n$.

- $\varphi(\sigma) = |\sigma|_k$ : number of cycles of length $k$ in the cycle decomposition of $\sigma$.
- $|\sigma|_1$ : the number of fixed points by $\sigma$.

Let $m_k = \int |\sigma|_k \, d\mu(\sigma)$. We get for all $t \geq 0$,

$$\mu^\theta(|\sigma|_k \leq m_k - t) \leq \exp \left( -\frac{t^2}{20km_k} \right),$$
Examples of configuration functions on $S_n$.

- $\varphi(\sigma) = |\sigma|_k$ : number of cycles of length $k$ in the cycle decomposition of $\sigma$.
- $|\sigma|_1$ : the number of fixed points by $\sigma$.

Let $m_k = \int |\sigma|_k \, d\mu(\sigma)$. We get for all $t \geq 0$,

$$\mu^\theta( |\sigma|_k \leq m_k - t ) \leq \exp \left( -\frac{t^2}{20km_k} \right),$$

and

$$\mu^\theta( |\sigma|_k \geq m_k + t ) \leq \exp \left( -\frac{t^2}{20k(m_k + t)} \right).$$
Examples of configuration functions on $S_n$.

- $\varphi(\sigma) = |\sigma|_k$: number of cycles of length $k$ in the cycle decomposition of $\sigma$.
- $|\sigma|_1$: the number of fixed points by $\sigma$.

Let $m_k = \int |\sigma|_k \, d\mu(\sigma)$. We get for all $t \geq 0$,

$$\mu^\theta(|\sigma|_k \leq m_k - t) \leq \exp \left( -\frac{t^2}{20km_k} \right),$$

and

$$\mu^\theta(|\sigma|_k \geq m_k + t) \leq \exp \left( -\frac{t^2}{20k(m_k + t)} \right).$$

- $\varphi(\sigma) = \sup_{t \in F} \sum_{k=1}^{n} a_{k,\sigma(k)}^t$
Examples of configuration functions on $S_n$.

- $\varphi(\sigma) = |\sigma|_k$ : number of cycles of length $k$ in the cycle decomposition of $\sigma$.
  $|\sigma|_1$ : the number of fixed points by $\sigma$.

Let $m_k = \int |\sigma|_k \, d\mu(\sigma)$. We get for all $t \geq 0$,

$$\mu^\theta(|\sigma|_k \leq m_k - t) \leq \exp\left( -\frac{t^2}{20km_k} \right),$$

and

$$\mu^\theta(|\sigma|_k \geq m_k + t) \leq \exp\left( -\frac{t^2}{20k(m_k + t)} \right).$$

- $\varphi(\sigma) = \sup_{t \in \mathcal{F}} \sum_{k=1}^{n} a_{k,\sigma(k)}^t$, with $0 \leq a_{k,\sigma(k)}^t \leq M$, then for all $t \geq 0$,

$$\mu^\theta(\varphi \leq \mu^\theta(\varphi) - t) \leq \exp\left( -\frac{t^2}{20\mu^\theta(\psi)} \right),$$

and

$$\mu^\theta(\varphi \geq \mu^\theta(\varphi) + t) \leq \exp\left( -\frac{t^2}{20(\mu^\theta(\psi) + Mt)} \right),$$

where $\psi(\sigma) = \sup_{t \in \mathcal{F}} \sum_{k=1}^{n} (a_{k,\sigma(k)}^t)^2 \leq M\varphi(\sigma)$. 
The Schrödinger minimization problem

introduction
Marton's inequality
Talagrand's concentration

Kantorovich duality
for classical costs
for weak costs

Examples of weak cost
Marton's type of cost
Barycentric cost
Strassen's result
Martingale costs

Weak transport inequalities
Dual characterization
to concentration

Universal transport inequalities

Barycentric transport inequalities
examples
characterisation on $\mathbb{R}$

Transport inequality on the symmetric group
introduction
Ewens distribution
deviation inequalities

The Schrödinger minimization problem

definition
curvature in discrete spaces
functional inequalities
Examples in discrete
Weak transport costs.28

Notations:

$X$: the state space of an homogenous Markov process,
$X$ is discrete.

$L$: the infinitesimal generator.

$L_\gamma$, $\gamma \geq 0$ is the temperature.

$m$: a reversible measure on $X$, $m_{\gamma} > 0$,
for all $x, y \in X$, $m_{\gamma} = L_{\gamma} m_{\gamma}$, $x \neq y$.

$P_t = e^{tL}$: the Markov semi-group,
$P_\gamma t = e^{tL_\gamma}$, $P_\gamma t$,

$\Omega \subset X_{r_0, 1}$: the set of left-limited, right-continuous, piecewise constant paths $\omega$.

$\omega_t, \omega_t' \in \Omega_{r_0, 1}$:
$\omega_t^X$, $X_t: \omega \mapsto \omega_t$.

For any $Q \ll P$, $Q_t = X_t # Q$,
$R_\gamma$ be the Markov path measure

$R_\gamma$ is the Markov path measure

Since $m$ is an invariant measure for the Markov semi-group,
$R_\gamma t = m, \forall t \in \mathbb{R}$.

$R_\gamma 0, 1$: $\omega \in \Omega_{r_0, 1}$.

$(\omega^X, X_t)$:
$X_t: \omega \rightarrow \omega_t$.

For any $Q \ll P$, $Q_t = X_t # Q$,
$R_\gamma$ be the Markov path measure

$R_\gamma$ is the Markov path measure

Since $m$ is an invariant measure for the Markov semi-group,
$R_\gamma t = m, \forall t \in \mathbb{R}$.

$R_\gamma 0, 1$: $\omega \in \Omega_{r_0, 1}$.
The Schrödinger minimization problem

Notations:

\[ X \]: the state space of an homogenous Markov process, \( X \) is discrete.

\[ L \]: the infinitesimal generator.

\[ L^\gamma = e^{t L^\gamma} \]: the Markov semi-group, \( L^\gamma \) is the temperature.

\[ m \]: a reversible mesure on \( X \), \( m(x, q) \geq 0 \), for all \( x, y \in X \), \( m(x, q) L^\gamma \)

\[ \rho(X) \]: the projection map, \( \rho(X) : \omega \mapsto \omega(t) \).

For any \( Q \in \mathcal{P}(\Omega) \), \( Q(t) = \rho(X(t)) \).

\[ \kappa(X) \]: As reference measure on \( \Omega \), let \( \kappa(X) \) be the Markov path measure with initial measure \( \kappa(X) \) and generator \( L^\gamma \).

Since \( m \) is an invariant measure for the Markov semi-group, \( \kappa(X) \) is the Markov path measure with initial measure \( \kappa(X) \) and generator \( L^\gamma \).
The Schrödinger minimization problem

Notations:
\[ X \] : the state space of an homogeneous Markov process,

\[ \gamma : \] temperature.

\[ \rho : \] reversible measure on \[ X \], \[ \rho(x,y) > 0 \] for all \[ x, y \in X \],

\[ \mathbf{P}_t \] : the Markov semi-group,

\[ \mathbf{P}_\gamma t \] : the Markov semi-group with generator \[ \gamma \].

\[ \Omega \] : the set of left-limited, right-continuous, piecewise constant paths \[ \omega \]

\[ X_t \] : the projection map, \[ X_t : \omega \rightarrow \omega(t) \].

For any \[ Q \in \mathcal{M}(\Omega) \], \[ Q_t = X_t # Q_{0,1} \].

\[ \mathbf{R}_\gamma \] : As reference measure on \[ \Omega \], let \[ \mathbf{R}_\gamma \] be the Markov path measure with initial measure \[ \mathbf{R}_\gamma 0 = \rho \] and generator \[ \gamma \]. Since \[ \rho \] is an invariante measure for the Markov semi-group, \[ \mathbf{R}_\gamma t = \rho \] for all \[ t \].

\[ \mathbf{R}_\gamma 0,1 : \mathcal{M}(X_0,X_1) \] for all \[ x, y \in X \],

\[ \mathbf{R}_\gamma 0,1(x,y) = \rho(x,y) P_{\gamma} x,y \],

\[ \mathbf{R}_\gamma 0,1 = \mathbf{R}_\gamma 0,1 \].
The Schrödinger minimization problem

Notations:

$\mathcal{X}$: the state space of an homogenous Markov process, $\mathcal{X}$ is discrete.

$L$: the infinitesimal generator.
The Schrödinger minimization problem

Notations:

$\mathcal{X}$: the state space of an homogenous Markov process, $\mathcal{X}$ is discrete.

$L$: the infinitesimal generator.

$L^\gamma = \gamma L$, $\gamma > 0$ is the temperature.
The Schrödinger minimization problem

Notations:

\( \mathcal{X} \) : the state space of an homogenous Markov process, \( \mathcal{X} \) is discrete.

\( L \) : the infinitesimal generator.

\( L^\gamma = \gamma L, \ \gamma > 0 \) is the temperature.

\( m \) : a reversible measure on \( \mathcal{X} \),

\( P_t = e^{tL} \) : the Markov semi-group,

\( P^\gamma_t = e^{tL^\gamma} \) : the Markov semi-group for \( \gamma > 0 \),

\( \Omega \) : the set of left-limited, right-continuous, piecewise constant paths \( \omega \),

\( X_t \) : the projection map,

For any \( Q \in \mathcal{M}(\Omega) \),

\( Q_t = X_t # Q \),

\( R^\gamma_0, 1_s \) : as reference measure on \( \Omega \), let \( R^\gamma_0 \) be the Markov path measure with initial measure \( R^\gamma_0 \),

Since \( m \) is an invariente measure for the Markov semi-group, \( R^\gamma_t = m \),

\( R^\gamma_0, 1_s \) : is \( m \),

\( R^\gamma_0, 1_s \) : is the Schrödinger minimization problem

Examples of weak cost

Marton's type of cost

Barycentric cost

Strassen's result

Martingale costs

Weak transport inequalities

Dual characterization to concentration

Universal transport inequalities

Barycentric transport inequalities

Examples

Characterisation on \( \mathbb{R} \)

Transport inequality on the symmetric group

Introduction

Ewens distribution

Deviation inequalities

The Schrödinger minimization problem

Definition

Curvature in discrete spaces

Functional inequalities

Examples in discrete

Weak transport costs
The Schrödinger minimization problem

Notations:

\( \mathcal{X} \): the state space of an homogenous Markov process, \( \mathcal{X} \) is discrete.

\( L \): the infinitesimal generator.

\( L^\gamma = \gamma L, \, \gamma > 0 \) is the temperature.

\( m \): a reversible mesure on \( \mathcal{X} \), \( m(x) > 0, \forall x \in \mathcal{X} \),
The Schrödinger minimization problem

Notations:

\( \mathcal{X} \): the state space of an homogenous Markov process, \( \mathcal{X} \) is discrete.

\( L \): the infinitesimal generator.

\( L^\gamma = \gamma L, \ \gamma > 0 \) is the temperature.

\( m \): a reversible measure on \( \mathcal{X} \), \( m(x) > 0, \ \forall x \in \mathcal{X} \), for all \( x, y \in \mathcal{X} \),

\[ m(x)L(x, y) = m(y)L(y, x). \]
The Schrödinger minimization problem

Notations:

\(\mathcal{X}\) : the state space of an homogenous Markov process, \(\mathcal{X}\) is discrete.

\(L\) : the infinitesimal generator.

\(L^\gamma = \gamma L, \; \gamma > 0\) is the temperature.

\(m\) : a reversible measure on \(\mathcal{X}\), \(m(x) > 0, \forall x \in \mathcal{X}\), for all \(x, y \in \mathcal{X}\),

\[m(x)L(x, y) = m(y)L(y, x).\]

\(P_t = e^{tL}\) : the Markov semi-group,
The Schrödinger minimization problem

Notations:

\( X \) : the state space of an homogenous Markov process, \( X \) is discrete.

\( L \) : the infinitesimal generator.

\( L^\gamma = \gamma L, \ \gamma > 0 \) is the temperature.

\( m \) : a reversible mesure on \( X \), \( m(x) > 0, \ \forall x \in X \), for all \( x, y \in X \),

\[ m(x)L(x, y) = m(y)L(y, x). \]

\( P_t = e^{tL} \) : the Markov semi-group,

\[ P_t^\gamma = e^{tL^\gamma} = P_{\gamma t}, \]

\( \Omega \) : the set of left-limited, right-continuous, piecewise constant paths

\( \omega \) : \( \omega \mapsto \omega_t \) is the projection map,

\( X : \omega \mapsto \omega_t \) for any \( Q \in \mathcal{P}(\Omega) \),

\( R_\gamma \) : the Markov path measure with initial measure \( R_\gamma_0 \) and generator \( L_\gamma \).

Since \( m \) is an invariant measure for the Markov semi-group, \( R_\gamma_t = m, \ \forall t \geq 0 \).

\( R_\gamma_0, 1_s : (x, y) \mapsto R_\gamma_0, 1_s (x, y) \) for all \( x, y \in X \).

\( R_\gamma_0, 1_s (x, y) = m(x)P_{\gamma t, \gamma t} \).
The Schrödinger minimization problem

Notations:

\( \mathcal{X} \) : the state space of an homogenous Markov process, \( \mathcal{X} \) is discrete.

\( L \) : the infinitesimal generator.

\( L^\gamma = \gamma L, \ \gamma > 0 \) is the temperature.

\( m \) : a reversible measure on \( \mathcal{X} \), \( m(x) > 0, \forall x \in \mathcal{X} \), for all \( x, y \in \mathcal{X} \),

\[ m(x)L(x,y) = m(y)L(y,x). \]

\( P_t = e^{tL} \) : the Markov semi-group,

\( P_t^\gamma = e^{tL^\gamma} = P_\gamma t, \)

\( \Omega \subset \mathcal{X}^{[0,1]} \) : the set of left-limited, right-continuous, piecewise constant paths

\[ \omega = (\omega_t)_{t \in [0,1]} \in \mathcal{X}^{[0,1]} . \]
The Schrödinger minimization problem

**Notations:**

- \( \mathcal{X} \): the state space of an homogenous Markov process, \( \mathcal{X} \) is discrete.
- \( L \): the infinitesimal generator.
- \( L^\gamma = \gamma L, \ \gamma > 0 \) is the temperature.
- \( m \): a reversible measure on \( \mathcal{X} \), \( m(x) > 0, \ \forall x \in \mathcal{X} \), for all \( x, y \in \mathcal{X} \),
  \[ m(x)L(x, y) = m(y)L(y, x). \]
- \( P_t = e^{tL} \): the Markov semi-group, \( P_t^\gamma = e^{tL^\gamma} = P_{\gamma t} \),
- \( \Omega \subset \mathcal{X}[0,1] \): the set of left-limited, right-continuous, piecewise constant paths
  \[ \omega = (\omega_t)_{t\in[0,1]} \in \mathcal{X}[0,1]. \]
- \( X_t \): the projection map, \( X_t : \omega \mapsto \omega_t \)
The Schrödinger minimization problem

Notations:

\(\mathcal{X}\) : the state space of an homogenous Markov process, \(\mathcal{X}\) is discrete.

\(L\) : the infinitesimal generator.

\(L^\gamma = \gamma L\), \(\gamma > 0\) is the temperature.

\(m\) : a reversible measure on \(\mathcal{X}\), \(m(x) > 0\), \(\forall x \in \mathcal{X}\), for all \(x, y \in \mathcal{X}\),

\[ m(x)L(x, y) = m(y)L(y, x). \]

\(P_t = e^{tL}\) : the Markov semi-group,

\(P_\gamma^t = e^{tL^\gamma} = P_\gamma t\),

\(\Omega \subset \mathcal{X}^{[0,1]}\) : the set of left-limited, right-continuous, piecewise constant paths

\[ \omega = (\omega_t)_{t \in [0,1]} \in \mathcal{X}^{[0,1]}\]

\(X_t\) : the projection map,

\[ X_t : \omega \mapsto \omega_t \]

For any \(Q \in \mathcal{M}(\Omega)\),

\[ Q_t = X_t \neq Q \]
The Schrödinger minimization problem

Notations:

\( \mathcal{X} \) : the state space of an homogenous Markov process, \( \mathcal{X} \) is discrete.

\( L \) : the infinitesimal generator.

\( L^\gamma = \gamma L, \ \gamma > 0 \) is the temperature.

\( m \) : a reversible measure on \( \mathcal{X} \), \( m(x) > 0, \ \forall x \in \mathcal{X} \), for all \( x, y \in \mathcal{X} \),

\[ m(x)L(x, y) = m(y)L(y, x). \]

\( P_t = e^{tL} \) : the Markov semi-group, \( P_t^\gamma = e^{tL^\gamma} = P_{\gamma t} \),

\( \Omega \subset \mathcal{X}^{[0,1]} \) : the set of left-limited, right-continuous, piecewise constant paths

\[ \omega = (\omega_t)_{t\in[0,1]} \in \mathcal{X}^{[0,1]} \]

\( X_t \) : the projection map, \( X_t : \omega \mapsto \omega_t \)

For any \( Q \in \mathcal{M}(\Omega) \), \( Q_t = X_t \# Q \)

\( R^\gamma \) : As reference measure on \( \Omega \), let \( R^\gamma \) be the Markov path measure
The Schrödinger minimization problem

Notations:

\( \mathcal{X} \) : the state space of an homogenous Markov process, \( \mathcal{X} \) is discrete.

\( L \) : the infinitesimal generator.
\( L^\gamma = \gamma L, \ gamma > 0 \) is the temperature.

\( m \) : a reversible measure on \( \mathcal{X} \), \( m(x) > 0, \forall x \in \mathcal{X}, \) for all \( x, y \in \mathcal{X} \),

\[ m(x)L(x,y) = m(y)L(y,x). \]

\( P_t = e^{tL} \) : the Markov semi-group, \( P_t^\gamma = e^{tL^\gamma} = P_{\gamma t}, \)

\( \Omega \subset \mathcal{X}[0,1] \) : the set of left-limited, right-continuous, piecewise constant paths

\( \omega = (\omega_t)_{t \in [0,1]} \in \mathcal{X}[0,1]. \)

\( X_t \) : the projection map, \( X_t : \omega \mapsto \omega_t \)

For any \( Q \in \mathcal{M}(\Omega), \quad Q_t = X_t \neq Q \)

\( R^\gamma \) : As reference measure on \( \Omega \), let \( R^\gamma \) be the Markov path measure with initial measure \( R_0^\gamma = m \) and generator \( L^\gamma \).
**The Schrödinger minimization problem**

**Notations:**

- \( \mathcal{X} \): the state space of an homogenous Markov process, \( \mathcal{X} \) is discrete.

- \( L \): the infinitesimal generator.

- \( L^\gamma = \gamma L, \quad \gamma > 0 \) is the temperature.

- \( m: \) a reversible measure on \( \mathcal{X} \), \( m(x) > 0, \forall x \in \mathcal{X}, \) for all \( x, y \in \mathcal{X}, \)

\[
m(x)L(x, y) = m(y)L(y, x).
\]

- \( P_t = e^{tL} \): the Markov semi-group, \( P_t^\gamma = e^{tL^\gamma} = P_\gamma t, \)

- \( \Omega \subset \mathcal{X}^{[0,1]} \): the set of left-limited, right-continuous, piecewise constant paths

\[
\omega = (\omega_t)_{t \in [0,1]} \quad \in \mathcal{X}^{[0,1]}
\]

- \( X_t \): the projection map, \( X_t : \omega \mapsto \omega_t \)

- For any \( Q \in \mathcal{M}(\Omega) \), \( Q_t = X_t \# Q \)

- \( R^\gamma \): As reference measure on \( \Omega \), let \( R^\gamma \) be the Markov path measure with initial measure \( R_0^\gamma = m \) and generator \( L^\gamma \).

Since \( m \) is an invariante measure for the Markov semi-group,

\[
R_t^\gamma = m, \quad \forall t \in [0, 1].
\]
The Schrödinger minimization problem

Notations:

\( \mathcal{X} \): the state space of an homogenous Markov process, \( \mathcal{X} \) is discrete.

\( L \): the infinitesimal generator.

\( L^\gamma = \gamma L, \ \gamma > 0 \) is the temperature.

\( m \): a reversible measure on \( \mathcal{X} \), \( m(x) > 0, \forall x \in \mathcal{X} \), for all \( x, y \in \mathcal{X} \),

\[ m(x)L(x, y) = m(y)L(y, x). \]

\( P_t = e^{tL} \): the Markov semi-group, \( P_t^\gamma = e^{tL^\gamma} = P_{\gamma t} \),

\( \Omega \subset \mathcal{X}^{[0,1]} \): the set of left-limited, right-continuous, piecewise constant paths

\[ \omega = (\omega_t)_{t \in [0,1]} \in \mathcal{X}^{[0,1]} \]

\( X_t \): the projection map, \( X_t : \omega \mapsto \omega_t \)

For any \( Q \in \mathcal{M}(\Omega), \quad Q_t = X_t # Q \)

\( R^\gamma \): As reference measure on \( \Omega \), let \( R^\gamma \) be the Markov path measure with initial measure \( R_0^\gamma = m \) and generator \( L^\gamma \).

Since \( m \) is an invariente measure for the Markov semi-group,

\[ R_t^\gamma = m, \quad \forall t \in [0,1]. \]

\[ R_{0,1}^\gamma := (X_0, X_1) # R^\gamma, \]
The Schrödinger minimization problem

Notations:

\( \mathcal{X} \) : the state space of an homogenous Markov process, \( \mathcal{X} \) is discrete.

\( L \) : the infinitesimal generator.

\( L^\gamma = \gamma L, \ \gamma > 0 \) is the temperature.

\( m \) : a reversible measure on \( \mathcal{X} \), \( m(x) > 0, \forall x \in \mathcal{X} \), for all \( x, y \in \mathcal{X} \),

\[ m(x)L(x, y) = m(y)L(y, x). \]

\( P_t = e^{tL} \) : the Markov semi-group, \( P^\gamma_t = e^{tL^\gamma} = P_{\gamma t} \),

\( \Omega \subset \mathcal{X}^{[0,1]} \) : the set of left-limited, right-continuous, piecewise constant paths

\[ \omega = (\omega_t)_{t \in [0,1]} \in \mathcal{X}^{[0,1]} . \]

\( X_t \) : the projection map, \( X_t : \omega \mapsto \omega_t \)

For any \( Q \in \mathcal{M}(\Omega) \), \( Q_t = X_t \# Q \)

\( R^\gamma \) : As reference measure on \( \Omega \), let \( R^\gamma \) be the Markov path measure with initial measure \( R_0^\gamma = m \) and generator \( L^\gamma \).

Since \( m \) is an invariente measure for the Markov semi-group,

\[ R_t^\gamma = m, \quad \forall t \in [0, 1]. \]

\[ R_{0,1}^\gamma := (X_0, X_1) \# R^\gamma, \text{ for all } x, y \in \mathcal{X}, \]

\[ R_{0,1}^\gamma(x, y) = m(x)P_\gamma(x, y), \quad R_{0,1}^\gamma = m \otimes P_\gamma. \]
The Schrödinger minimization problem

Let $\mu_0, \mu_1 \dashv P \in P_\infty(X)$ with density $h_0$ and $h_1$ with respect to $\mu$.

• The dynamic Schrödinger problem associated to $R_\gamma$ is to minimize $H_{\pi}^\gamma$ over all $\pi \in \Pi(\mu_0, \mu_1)$.

• The static Schrödinger problem associated to $R_\gamma$ is to minimize $H_{\pi}^\gamma$ over all $\pi \in \Pi(\mu_0, \mu_1)$.

Theorem: [see C. Léonard 2013]

1. The dynamic and static Schrödinger problems have same minimum value, $T_{\gamma}^S_{\mu_0, \mu_1}$.

2. The dynamic problem is reached for the so-called Schrödinger bridge $\pi_{\gamma}^{X_{0}, X_{1}}$ with density $f_{X_{0}, X_{1}}^\gamma$ with respect to $R_\gamma$, where $f, g : X \to \mathbb{R}$ satisfy the so-called Schrödinger system

\[
 f_{x}^\gamma E_{R_\gamma}^\gamma g_{y}^\gamma |_{X_{0}} = h_{0, x}^\gamma,
\]

\[
 g_{y}^\gamma E_{R_\gamma}^\gamma f_{x}^\gamma |_{X_{1}} = h_{1, y}^\gamma.
\]
The Schrödinger minimization problem
Let $\mu_0, \mu_1 \in \mathcal{P}(X)$ (with finite support) with density $h_0$ and $h_1$ with respect to $m$. 
The Schrödinger minimization problem

Let $\mu_0, \mu_1 \in \mathcal{P}(X)$ (with finite support) with density $h_0$ and $h_1$ with respect to $m$.

- The dynamic Schrödinger problem associated to $R^\gamma$ is

\[
\inf_{\pi \in \Pi_{\mu_0, \mu_1}} H^\gamma_{\pi} \mid \gamma = \int_X \gamma(x) \, d\pi.
\]

Theorem: (see C. Léonard 2013)

1. The dynamic and static Schrödinger problems have the same minimum value.

2. The dynamic problem is reached for the so-called Schrödinger bridge $Q^\gamma_{\mu_0, \mu_1}$, with density $f_x^0 \gamma g_x^1$ with respect to $\gamma$, where $f_x^0, g_x^1 : X \to \mathbb{R}$ satisfy the so-called Schrödinger system:

\[
\begin{align*}
\int_X f_x^0 \gamma h_0 \, d\pi & = \int_X f_x^0 \gamma \, d\mu_0, \\
\int_X g_y^1 \gamma h_1 \, d\pi & = \int_X g_y^1 \gamma \, d\mu_1.
\end{align*}
\]

The static problem is reached for $Q^\gamma_{\mu_0, \mu_1}$.
The Schrödinger minimization problem
Let $\mu_0, \mu_1 \in \mathcal{P}(X)$ (with finite support) with density $h_0$ and $h_1$ with respect to $m$.

- The dynamic Schrödinger problem associated to $R^\gamma$ is to minimize $H(Q|R^\gamma)$ over all $Q \in \mathcal{P}(\Omega)$ such that $Q_0 = \mu_0$, $Q_1 = \mu_1$. 
The Schrödinger minimization problem
Let $\mu_0, \mu_1 \in \mathcal{P}(X)$ (with finite support) with density $h_0$ and $h_1$ with respect to $m$.

- The dynamic Schrödinger problem associated to $R^\gamma$ is to minimize $H(Q|R^\gamma)$ over all $Q \in \mathcal{P}(\Omega)$ such that $Q_0 = \mu_0$, $Q_1 = \mu_1$.
- The static Schrödinger problem associated to $R^\gamma$ is
The Schrödinger minimization problem
Let $\mu_0, \mu_1 \in \mathcal{P}(X)$ (with finite support) with density $h_0$ and $h_1$ with respect to $m$.

- The **dynamic Schrödinger** problem associated to $R^\gamma$ is to minimize $H(Q|R^\gamma)$ over all $Q \in \mathcal{P}(\Omega)$ such that $Q_0 = \mu_0$, $Q_1 = \mu_1$.
- The **static Schrödinger** problem associated to $R^\gamma$ is to minimize $H(\pi|R^\gamma_{0,1})$ over all $\pi \in \Pi(\mu_0, \mu_1)$.

---

**Theorem:** [see C. Léonard 2013]

1. The dynamic and static Schrödinger problems have the same minimum value,

\[
\inf_{\pi \in \Pi(\mu_0, \mu_1)} H(\pi|R^\gamma_{0,1}) = \inf_{Q \in \mathcal{P}(\Omega)} H(Q|R^\gamma).
\]

2. The dynamic problem is reached for the so-called Schrödinger bridge $Q^\gamma$, with density $f x^0 q g x^1 q$ with respect to $R^\gamma$, where $f, g : X \rightarrow \mathbb{R}$ satisfy the so-called Schrödinger system:

\[
\begin{align*}
\int_x f x^0 q E_{R^\gamma} g x^1 q | x^0 q &= h_0 x^0 q, \\
\int_y f x^0 q E_{R^\gamma} g x^1 q | x^1 q &= h_1 x^1 q.
\end{align*}
\]

3. The static problem is reached for $Q^\gamma_{0,1}$, with density $f x^0 q g x^1 q$ where $| \cdot | : X^2 \rightarrow \mathbb{R}$ satisfies the so-called Schrödinger system:

\[
\begin{align*}
\int_x f x^0 q E_{R^\gamma} g x^1 q | x^0 q &= h_0 x^0 q, \\
\int_y f x^0 q E_{R^\gamma} g x^1 q | x^1 q &= h_1 x^1 q.
\end{align*}
\]
The Schrödinger minimization problem
Let $\mu_0, \mu_1 \in \mathcal{P}(X)$ (with finite support) with density $h_0$ and $h_1$ with respect to $m$.

- The dynamic Schrödinger problem associated to $R^\gamma$ is to minimize $H(Q|R^\gamma)$ over all $Q \in \mathcal{P}(\Omega)$ such that $Q_0 = \mu_0$, $Q_1 = \mu_1$.
- The static Schrödinger problem associated to $R^\gamma$ is to minimize $H(\pi|R^\gamma_{0,1})$ over all $\pi \in \Pi(\mu_0, \mu_1)$.

Theorem: [see C. Léonard 2013]
**The Schrödinger minimization problem**

Let $\mu_0, \mu_1 \in \mathcal{P}(X)$ (with finite support) with density $h_0$ and $h_1$ with respect to $m$.

- The **dynamic** Schrödinger problem associated to $R^\gamma$ is to minimize $H(Q|R^\gamma)$ over all $Q \in \mathcal{P}(\Omega)$ such that $Q_0 = \mu_0$, $Q_1 = \mu_1$.
- The **static** Schrödinger problem associated to $R^\gamma$ is to minimize $H(\pi|R^\gamma_{0,1})$ over all $\pi \in \Pi(\mu_0, \mu_1)$.

---

**Theorem** : [see C. Léonard 2013]

1. The **dynamic** and **static** Schrödinger problems have **same minimum value**,
The Schrödinger minimization problem

Let $\mu_0, \mu_1 \in \mathcal{P}(X)$ (with finite support) with density $h_0$ and $h_1$ with respect to $m$.

- The dynamic Schrödinger problem associated to $R^\gamma$ is to minimize $H(Q|R^\gamma)$ over all $Q \in \mathcal{P}(\Omega)$ such that $Q_0 = \mu_0$, $Q_1 = \mu_1$.
- The static Schrödinger problem associated to $R^\gamma$ is to minimize $H(\pi|R^\gamma_{0,1})$ over all $\pi \in \Pi(\mu_0, \mu_1)$.

**Theorem : [see C. Léonard 2013]**

1. The dynamic and static Schrödinger problems have same minimum value,

$$T^\gamma_S(\mu_0, \mu_1) = \inf_{\pi \in \Pi(\mu_0, \mu_1)} H(\pi|R^\gamma_{0,1}).$$
The Schrödinger minimization problem

Let $\mu_0, \mu_1 \in \mathcal{P}(X)$ (with finite support) with density $h_0$ and $h_1$ with respect to $m$.

- The dynamic Schrödinger problem associated to $R^\gamma$ is to minimize $H(Q|R^\gamma)$ over all $Q \in \mathcal{P}(\Omega)$ such that $Q_0 = \mu_0$, $Q_1 = \mu_1$.
- The static Schrödinger problem associated to $R^\gamma$ is to minimize $H(\pi|R^\gamma_{0,1})$ over all $\pi \in \Pi(\mu_0, \mu_1)$.

**Theorem** : [see C. Léonard 2013]

1. The dynamic and static Schrödinger problems have same minimum value,

\[
T^\gamma_S(\mu_0, \mu_1) = \inf_{\pi \in \Pi(\mu_0, \mu_1)} H(\pi|R^\gamma_{0,1}).
\]

2. The dynamic problem is reached for the so called Schrödinger bridge $\hat{Q}^\gamma \in \mathbb{P}(\Omega)$,
The Schrödinger minimization problem
Let $\mu_0, \mu_1 \in \mathcal{P}(X)$ (with finite support) with density $h_0$ and $h_1$ with respect to $m$.

- The dynamic Schrödinger problem associated to $R^\gamma$ is
to minimize $H(Q|R^\gamma)$ over all $Q \in \mathcal{P}(\Omega)$ such that $Q_0 = \mu_0, Q_1 = \mu_1$.
- The static Schrödinger problem associated to $R^\gamma$ is
to minimize $H(\pi|R^\gamma_{0,1})$ over all $\pi \in \Pi(\mu_0, \mu_1)$.

Theorem: [see C. Léonard 2013]

1. The dynamic and static Schrödinger problems have same minimum value,

$$T^\gamma_S(\mu_0, \mu_1) = \inf_{\pi \in \Pi(\mu_0, \mu_1)} H(\pi|R^\gamma_{0,1}).$$

2. The dynamic problem is reached for the so called Schrödinger bridge
$
\hat{Q}^\gamma \in \mathbb{P}(\Omega),
$ with density $f(X_0)g(X_1)$ with respect to $R^\gamma$,
The Schrödinger minimization problem

Let \( \mu_0, \mu_1 \in \mathcal{P}(X) \) (with finite support) with density \( h_0 \) and \( h_1 \) with respect to \( m \).

- The dynamic Schrödinger problem associated to \( R^\gamma \) is to minimize \( H(Q|R^\gamma) \) over all \( Q \in \mathcal{P}(\Omega) \) such that \( Q_0 = \mu_0, Q_1 = \mu_1 \).
- The static Schrödinger problem associated to \( R^\gamma \) is to minimize \( H(\pi|R^\gamma_{0,1}) \) over all \( \pi \in \Pi(\mu_0, \mu_1) \).

**Theorem** : [see C. Léonard 2013]

1. The dynamic and static Schrödinger problems have same minimum value,

\[
T_S^\gamma(\mu_0, \mu_1) = \inf_{\pi \in \Pi(\mu_0, \mu_1)} H(\pi|R^\gamma_{0,1}).
\]

2. The dynamic problem is reached for the so called Schrödinger bridge \( \hat{Q}^\gamma \in \mathbb{P}(\Omega) \), with density \( f(X_0)g(X_1) \) with respect to \( R^\gamma \), where \( f, g : \mathcal{X} \to \mathbb{R} \) satisfy the so called Schrödinger system

\[
\begin{align*}
  f(x) \mathbb{E}_{R^\gamma}(g(X_1)|X_0 = x) &= h_0(x), \\
  g(y) \mathbb{E}_{R^\gamma}(f(X_0)|X_1 = y) &= h_1(y).
\end{align*}
\]
The Schrödinger minimization problem

Let $\mu_0, \mu_1 \in \mathcal{P}(X)$ (with finite support) with density $h_0$ and $h_1$ with respect to $m$.

- The dynamic Schrödinger problem associated to $R^\gamma$ is
to minimize $H(Q|R^\gamma)$ over all $Q \in \mathcal{P}(\Omega)$ such that $Q_0 = \mu_0$, $Q_1 = \mu_1$.
- The static Schrödinger problem associated to $R^\gamma$ is
to minimize $H(\pi|R_{0,1}^\gamma)$ over all $\pi \in \Pi(\mu_0, \mu_1)$.

**Theorem** : [see C. Léonard 2013]

1. The dynamic and static Schrödinger problems have same minimum value,
$$T_S^\gamma(\mu_0, \mu_1) = \inf_{\pi \in \Pi(\mu_0, \mu_1)} H(\pi|R_{0,1}^\gamma).$$

2. The dynamic problem is reached for the so called Schrödinger bridge
$\hat{Q}^\gamma \in \mathcal{P}(\Omega)$, with density $f(X_0)g(X_1)$ with respect to $R^\gamma$, where
$f, g : X \to \mathbb{R}$ satisfy the so called Schrödinger system

$$\left\{ \begin{array}{l}
f(x) \mathbb{E}_{R^\gamma}(g(X_1)|X_0 = x) = h_0(x), \\
g(y) \mathbb{E}_{R^\gamma}(f(X_0)|X_1 = y) = h_1(y).
\end{array} \right.$$
The Schrödinger minimization problem
Let \( \mu_0, \mu_1 \in \mathcal{P}(X) \) (with finite support) with density \( h_0 \) and \( h_1 \) with respect to \( m \).

- The dynamic Schrödinger problem associated to \( R^\gamma \) is to minimize \( H(Q|R^\gamma) \) over all \( Q \in \mathcal{P}(\Omega) \) such that \( Q_0 = \mu_0, Q_1 = \mu_1 \).
- The static Schrödinger problem associated to \( R^\gamma \) is to minimize \( H(\pi|R^\gamma_{0,1}) \) over all \( \pi \in \Pi(\mu_0, \mu_1) \).

**Theorem** [see C. Léonard 2013]

1. The dynamic and static Schrödinger problems have same minimum value,
   \[
   T_S^\gamma(\mu_0, \mu_1) = \inf_{\pi \in \Pi(\mu_0, \mu_1)} H(\pi|R^\gamma_{0,1}).
   \]

2. The dynamic problem is reached for the so called Schrödinger bridge \( \hat{Q}^\gamma \in \mathbb{P}(\Omega) \), with density \( f(X_0)g(X_1) \) with respect to \( R^\gamma \), where \( f, g : \mathcal{X} \to \mathbb{R} \) satisfy the so called Schrödinger system
   \[
   \begin{align*}
   f(x) \mathbb{E}_{R^\gamma}(g(X_1)|X_0 = x) &= h_0(x), \\
   g(y) \mathbb{E}_{R^\gamma}(f(X_0)|X_1 = y) &= h_1(y).
   \end{align*}
   \]

   The static problem is reached for
   \[
   \hat{Q}^\gamma_{0,1} = (X_0, X_1) \# \hat{Q}^\gamma.
   \]
The Schrödinger minimization problem
Let $\mu_0, \mu_1 \in \mathcal{P}(X)$ (with finite support) with density $h_0$ and $h_1$ with respect to $m$.

- The dynamic Schrödinger problem associated to $R^\gamma$ is to minimize $H(Q|R^\gamma)$ over all $Q \in \mathcal{P}(\Omega)$ such that $Q_0 = \mu_0, Q_1 = \mu_1$.
- The static Schrödinger problem associated to $R^\gamma$ is to minimize $H(\pi|R^\gamma_{0,1})$ over all $\pi \in \Pi(\mu_0, \mu_1)$.

**Theorem :** [see C. Léonard 2013]

1. The dynamic and static Schrödinger problems have same minimum value,

$$T^\gamma_S(\mu_0, \mu_1) = \inf_{\pi \in \Pi(\mu_0, \mu_1)} H(\pi|R^\gamma_{0,1}).$$

2. The dynamic problem is reached for the so called Schrödinger bridge $\hat{Q}^\gamma \in \mathbb{P}(\Omega)$, with density $f(X_0)g(X_1)$ with respect to $R^\gamma$, where $f, g : X \to \mathbb{R}$ satisfy the so called Schrödinger system

$$\begin{cases}
  f(x) \mathbb{E}_{R^\gamma}(g(X_1)|X_0 = x) = h_0(x), \\
  g(y) \mathbb{E}_{R^\gamma}(f(X_0)|X_1 = y) = h_1(y).
\end{cases}$$

$$\begin{cases}
  f(x) P^\gamma g(x) = h_0(x), \\
  g(y) P^\gamma f(y) = h_1(y).
\end{cases}$$

The static problem is reached for

$$\hat{Q}^\gamma_{0,1} = (X_0, X_1) \# \hat{Q}^\gamma.$$

For all $x, y \in X$,

$$\hat{Q}^\gamma_{0,1}(x, y) = f(x)g(y)R^\gamma_{0,1}(x, y)$$
The Schrödinger minimization problem

Let $\mu_0, \mu_1 \in \mathcal{P}(X)$ (with finite support) with density $h_0$ and $h_1$ with respect to $m$.

- The dynamic Schrödinger problem associated to $R^\gamma$ is to minimize $H(Q|R^\gamma)$ over all $Q \in \mathcal{P}(\Omega)$ such that $Q_0 = \mu_0$, $Q_1 = \mu_1$.
- The static Schrödinger problem associated to $R^\gamma$ is to minimize $H(\pi|R^\gamma_{0,1})$ over all $\pi \in \Pi(\mu_0, \mu_1)$.

**Theorem** [see C. Léonard 2013]

1. The dynamic and static Schrödinger problems have same minimum value,
   
   $$T^\gamma_S(\mu_0, \mu_1) = \inf_{\pi \in \Pi(\mu_0, \mu_1)} H(\pi|R^\gamma_{0,1}).$$

2. The dynamic problem is reached for the so called Schrödinger bridge $\hat{Q}^\gamma \in \mathbb{P}(\Omega)$, with density $f(X_0)g(X_1)$ with respect to $R^\gamma$, where $f, g : X \rightarrow \mathbb{R}$ satisfy the so called Schrödinger system

   \[
   \begin{cases}
   f(x) \mathbb{E}_{R^\gamma}(g(X_1)|X_0 = x) = h_0(x), \\
   g(y) \mathbb{E}_{R^\gamma}(f(X_0)|X_1 = y) = h_1(y).
   \end{cases}
   \]

   The static problem is reached for

   $$\hat{Q}^\gamma_{0,1} = (X_0, X_1) \# \hat{Q}^\gamma.$$

   For all $x, y \in X$,

   $$\hat{Q}^\gamma_{0,1}(x, y) = f(x)g(y)R^\gamma_{0,1}(x, y) = f(x)g(y)m(x)P^\gamma(x, y).$$
The Schrödinger problem as a weak transport cost

From the decomposition $R^\gamma_{0,1} = m \otimes P^\gamma$, the Schrödinger problem can be formulated as a minimization problem.

\[ \inf_{\pi \in \Pi(p, \nu)} \int c_{\pi} d\mu = \inf_{\pi \in \Pi(p, \nu)} \int c_{\pi} d\mu \]

Where $c_{\pi}$ is the weak transport cost associated to the cost $c$, and $\Pi(p, \nu)$ is the set of all measures $\pi$ on $\{0, 1\}$ with marginals $p$ and $\nu$.

The Kantorovich duality theorem holds, which states that the infimum of the left-hand side is equal to the supremum of the right-hand side.

\[ \inf_{\pi \in \Pi(p, \nu)} \int c_{\pi} d\mu = \sup_{\psi \in C(\{0, 1\})} \int \psi c d\mu \]

With $\psi \in C(\{0, 1\})$ being any measurable function on $\{0, 1\}$.

The Schrödinger minimization problem can be seen as a weak transport cost, where the cost is given by the Schrödinger function $c_{\pi}$.
The Schrödinger problem as a weak transport cost

From the decomposition \( R_{0,1}^\gamma = m \otimes P^\gamma \), one has for all \( \pi \in \Pi(\mu_0, \mu_1) \), \( \pi = \mu_0 \otimes p \),
The Schrödinger problem as a weak transport cost

From the decomposition $R_{0,1}^\gamma = m \otimes P^\gamma$, one has for all $\pi \in \Pi(\mu_0, \mu_1)$, $\pi = \mu_0 \otimes p$,

$$H(\pi | R_{0,1}^\gamma) = H(\mu_0 \otimes p | m \otimes P^\gamma)$$
The Schrödinger problem as a weak transport cost

From the decomposition $R_{0,1}^\gamma = m \otimes P^\gamma$, one has for all $\pi \in \Pi(\mu_0, \mu_1)$, $\pi = \mu_0 \otimes p$,

$$H(\pi | R_{0,1}^\gamma) = H(\mu_0 \otimes p | m \otimes P^\gamma) = H(\mu_0 | m) + \int H(p_x | P_x^\gamma) \, d\mu_0(x),$$
The Schrödinger problem as a weak transport cost

From the decomposition $R_{0,1}^{\gamma} = m \otimes P^{\gamma}$, one has for all $\pi \in \Pi(\mu_0, \mu_1)$, $\pi = \mu_0 \otimes \nu$,

$$H(\pi|R_{0,1}^{\gamma}) = H(\mu_0 \otimes \nu|m \otimes P^{\gamma}) = H(\mu_0|m) + \int H(p_x|P_x^{\gamma}) \, d\mu_0(x),$$

and therefore, taking the infimum over all $\pi \in \Pi(\mu, \nu)$

$$T_S^{\gamma}(\mu_0, \mu_1) = H(\mu_0|m) + T_S(\mu_1|\mu_0),$$

with

$$T_S(\mu_1|\mu_0) := \inf_{\pi \in \Pi(\mu_0, \mu_1), \pi = \mu_0 \otimes \nu} \int H(p_x|P_x^{\gamma}) \, d\mu_0(x).$$
The Schrödinger problem as a weak transport cost

From the decomposition \( R_{0,1}^\gamma = m \otimes P^\gamma \), one has for all \( \pi \in \Pi(\mu_0, \mu_1) \), \( \pi = \mu_0 \otimes p \),

\[
H(\pi | R_{0,1}^\gamma) = H(\mu_0 \otimes p | m \otimes P^\gamma) = H(\mu_0 | m) + \int H(p_x | P^\gamma_x) \, d\mu_0(x),
\]

and therefore, taking the infimum over all \( \pi \in \Pi(\mu, \nu) \)

\[
T_S^\gamma(\mu_0, \mu_1) = H(\mu_0 | m) + T_S(\mu_1 | \mu_0),
\]

with

\[
T_S(\mu_1 | \mu_0) := \inf_{\pi \in \Pi(\mu_0, \mu_1), \pi = \mu_0 \otimes p} \int H(p_x | P^\gamma_x) \, d\mu_0(x).
\]

\( T_S(\mu_1 | \mu_0) \) is a weak transport cost associated to the cost

\[
c(x, p) = H(p | P^\gamma_x), \quad x \in \mathcal{X}, p \in \mathcal{P}(\mathcal{X}).
\]
The Schrödinger problem as a weak transport cost

From the decomposition \( R^\gamma_{0,1} = m \otimes P^\gamma \),

one has for all \( \pi \in \Pi(\mu_0, \mu_1) \), \( \pi = \mu_0 \otimes p \),

\[
H(\pi|R^\gamma_{0,1}) = H(\mu_0 \otimes p|m \otimes P^\gamma) = H(\mu_0|m) + \int H(p_x|P^\gamma_x) \, d\mu_0(x),
\]

and therefore, taking the infimum over all \( \pi \in \Pi(\mu, \nu) \)

\[
T^\gamma_S(\mu_0, \mu_1) = H(\mu_0|m) + T_S(\mu_1|\mu_0),
\]

with

\[
T_S(\mu_1|\mu_0) := \inf_{\pi \in \Pi(\mu_0, \mu_1)} \int H(p_x|P^\gamma_x) \, d\mu_0(x).
\]

\( T_S(\mu_1|\mu_0) \) is a weak transport cost associated to the cost

\[
c(x,p) = H(p|P^\gamma_x), \quad x \in \mathcal{X}, p \in \mathcal{P}(\mathcal{X}).
\]

Since \( p \mapsto H(p|P^\gamma_x) \) is convex,
The Schrödinger problem as a weak transport cost

From the decomposition \( R_{0,1}^\gamma = m \otimes P^\gamma \),
one has for all \( \pi \in \Pi(\mu_0, \mu_1) \), \( \pi = \mu_0 \otimes p \),
\[
H(\pi | R_{0,1}^\gamma) = H(\mu_0 \otimes p | m \otimes P^\gamma) = H(\mu_0 | m) + \int H(p_x | P_x^\gamma) \, d\mu_0(x),
\]
and therefore, taking the infimum over all \( \pi \in \Pi(\mu, \nu) \)
\[
T_S^\gamma(\mu_0, \mu_1) = H(\mu_0 | m) + T_S(\mu_1 | \mu_0),
\]
with
\[
T_S(\mu_1 | \mu_0) := \inf_{\pi \in \Pi(\mu_0, \mu_1)} \int H(p_x | P_x^\gamma) \, d\mu_0(x).
\]
\( T_S(\mu_1 | \mu_0) \) is a weak transport cost associated to the cost
\[
c(x, p) = H(p | P_x^\gamma), \quad x \in \mathcal{X}, \, p \in \mathcal{P}(\mathcal{X}).
\]
Since \( p \mapsto H(p | P_x^\gamma) \) is convex, the Kantorovich duality theorem holds,
\[
T_S(\mu_1 | \mu_0) = \sup_\psi \left\{ \int R_c \psi \, d\mu_0 - \int \psi \, d\mu_1 \right\},
\]
The Schrödinger problem as a weak transport cost

From the decomposition \( R_{0,1}^\gamma = m \otimes P^\gamma \), one has for all \( \pi \in \Pi(\mu_0, \mu_1) \), \( \pi = \mu_0 \otimes p \),

\[
H(\pi | R_{0,1}^\gamma) = H(\mu_0 \otimes p | m \otimes P^\gamma) = H(\mu_0 | m) + \int H(p_x | P_x^\gamma) \, d\mu_0(x),
\]

and therefore, taking the infimum over all \( \pi \in \Pi(\mu, \nu) \)

\[
T_S^\gamma(\mu_0, \mu_1) = H(\mu_0 | m) + T_S(\mu_1 | \mu_0),
\]

with

\[
T_S(\mu_1 | \mu_0) := \inf_{\pi \in \Pi(\mu_0, \mu_1)} \int H(p_x | P_x^\gamma) \, d\mu_0(x).
\]

\( T_S(\mu_1 | \mu_0) \) is a weak transport cost associated to the cost

\[
c(x, p) = H(p | P_x^\gamma), \quad x \in \mathcal{X}, p \in \mathcal{P}(\mathcal{X}).
\]

Since \( p \mapsto H(p | P_x^\gamma) \) is convex, the Kantorovich duality theorem holds,

\[
T_S(\mu_1 | \mu_0) = \sup_{\psi} \left\{ \int R_c^\psi \, d\mu_0 - \int \psi \, d\mu_1 \right\},
\]

with

\[
R_c^\psi(x) = \inf_{p \in \mathcal{P}(\mathcal{X})} \left\{ \int \psi \, dp + H(p | P_x^\gamma) \right\}.
\]
The Schrödinger problem as a weak transport cost

From the decomposition $R_{0,1}^{\gamma} = m \otimes P^{\gamma}$, one has for all $\pi \in \Pi(\mu_0, \mu_1)$, $\pi = \mu_0 \otimes \rho$,

$$H(\pi | R_{0,1}^{\gamma}) = H(\mu_0 \otimes \rho | m \otimes P^{\gamma}) = H(\mu_0 | m) + \int H(p_x | P_x^{\gamma}) \, d\mu_0(x),$$

and therefore, taking the infimum over all $\pi \in \Pi(\mu, \nu)$

$$T_S^{\gamma}(\mu_0, \mu_1) = H(\mu_0 | m) + T_S(\mu_1 | \mu_0),$$

with

$$T_S(\mu_1 | \mu_0) := \inf_{\pi \in \Pi(\mu_0, \mu_1)} \int H(p_x | P_x^{\gamma}) \, d\mu_0(x).$$

$T_S(\mu_1 | \mu_0)$ is a weak transport cost associated to the cost

$$c(x, p) = H(p | P_x^{\gamma}), \quad x \in \mathcal{X}, \, p \in \mathcal{P}(\mathcal{X}).$$

Since $p \mapsto H(p | P_x^{\gamma})$ is convex, the Kantorovich duality theorem holds,

$$T_S(\mu_1 | \mu_0) = \sup_\psi \left\{ \int R_c \psi \, d\mu_0 - \int \psi \, d\mu_1 \right\},$$

with $R_c \psi(x) = \inf_{p \in \mathcal{P}(\mathcal{X})} \left\{ \int \psi dp + H(p | P_x^{\gamma}) \right\} = - \log P^{\gamma}(e^{-\psi})(x)$. 

Since $p \mapsto H(p | P_x^{\gamma})$ is convex, the Kantorovich duality theorem holds,
The Schrödinger problem as a weak transport cost

From the decomposition  \( R_{0,1}^{\gamma} = m \otimes P^{\gamma} \),
one has for all \( \pi \in \Pi(\mu_0, \mu_1) \), \( \pi = \mu_0 \otimes p \),

\[
H(\pi | R_{0,1}^{\gamma}) = H(\mu_0 \otimes p | m \otimes P^{\gamma}) = H(\mu_0 | m) + \int H(p_x | P_x^{\gamma}) \, d\mu_0(x),
\]
and therefore, taking the infimum over all \( \pi \in \Pi(\mu, \nu) \)

\[
T_S^{\gamma}(\mu_0, \mu_1) = H(\mu_0 | m) + T_S(\mu_1 | \mu_0),
\]
with

\[
T_S(\mu_1 | \mu_0) := \inf_{\pi \in \Pi(\mu_0, \mu_1), \pi = \mu_0 \otimes p} \int H(p_x | P_x^{\gamma}) \, d\mu_0(x).
\]

\( T_S(\mu_1 | \mu_0) \) is a weak transport cost associated to the cost

\[
c(x, p) = H(p | P_x^{\gamma}), \quad x \in \mathcal{X}, p \in \mathcal{P}(\mathcal{X}).
\]

Since \( p \mapsto H(p | P_x^{\gamma}) \) is convex, the Kantorovich duality theorem holds,

\[
T_S(\mu_1 | \mu_0) = \sup_{\psi} \left\{ \int R_c \psi \, d\mu_0 - \int \psi \, d\mu_1 \right\},
\]
with 

\[
R_c \psi(x) = \inf_{p \in \mathcal{P}(\mathcal{X})} \left\{ \int \psi \, dp + H(p | P_x^{\gamma}) \right\} = -\log P^{\gamma}(e^{-\psi})(x).
\]
**Curvature in discrete setting**

**Question:** Is there a “good” notion of curvature in discrete setting from which we can recover

- transport-entropy inequalities,
- Poincaré inequalities,
- modified log-Sobolev inequalities, hypercontractivity,
- Prékopa-Leindler types of inequalities,
- concentration properties...

Several notions of curvature have been proposed on discrete spaces to extend the lower bound on Ricci-curvature in Riemannian geometry.

- The Bakry-Emery curvature condition (1985) - $\Gamma^2$-calculus,
- The coarse Ricci curvature, Ollivier (2009), Lin-Lu-Yau (2010),
- Lott-Sturm-Villani definition of curvature.

- Rough curvature bounds, Bonciocat-Sturm (2009),
- The entropic Ricci curvature, Erbar-Maas (2013), Mielke (2013),

We will focus on the approach by C. Leonard in discrete, following the recent approach by G. Conforti (2018) in continuous spaces when $L$ is a diffusion generator $Lf = \frac{1}{2} \Delta f - \nabla U \cdot \nabla q$. 
Curvature in discrete setting

**Question**: Is there a “good” notion of curvature in discrete setting from which we can recover

- transport-entropy inequalities,
- Poincaré inequalities,
- modified log-Sobolev inequalities, hypercontractivity,
- Prékopa-Leindler types of inequalities,
- concentration properties...

Several notions of curvature have been proposed on discrete spaces to extend the lower bound on Ricci-curvature in Riemannian geometry.
Curvature in discrete setting

**Question**: Is there a “good” notion of curvature in discrete setting from which we can recover

- transport-entropy inequalities,
- Poincaré inequalities,
- modified log-Sobolev inequalities, hypercontractivity,
- Prékopa-Leindler types of inequalities,
- concentration properties...

Several notions of curvature have been proposed on discrete spaces to extend the lower bound on Ricci-curvature in Riemannian geometry.

- **The Bakry-Emery curvature condition** (1985) - $\Gamma_2$-calculus,
Curvature in discrete setting

**Question**: Is there a “good” notion of curvature in discrete setting from which we can recover

- transport-entropy inequalities,
- Poincaré inequalities,
- modified log-Sobolev inequalities, hypercontractivity,
- Prékopa-Leindler types of inequalities,
- concentration properties...

Several notions of curvature have been proposed on discrete spaces to extend the lower bound on Ricci-curvature in Riemannian geometry.

- The Bakry-Emery curvature condition (1985) - $\Gamma_2$-calculus, the **exponential curvature-dimension condition**
Curvature in discrete setting

**Question**: Is there a “good” notion of curvature in discrete setting from which we can recover

- transport-entropy inequalities,
- Poincaré inequalities,
- modified log-Sobolev inequalities, hypercontractivity,
- Prékopa-Leindler types of inequalities,
- concentration properties...

Several notions of curvature have been proposed on discrete spaces to extend the lower bound on Ricci-curvature in Riemannian geometry.

- The Bakry-Emery curvature condition (1985) - $\Gamma_2$-calculus, the exponential curvature-dimension condition
Curvature in discrete setting

**Question**: Is there a “good” notion of curvature in discrete setting from which we can recover

- transport-entropy inequalities,
- Poincaré inequalities,
- modified log-Sobolev inequalities, hypercontractivity,
- Prékopa-Leindler types of inequalities,
- concentration properties...

Several notions of curvature have been proposed on discrete spaces to extend the lower bound on Ricci-curvature in Riemannian geometry.

- The Bakry-Emery curvature condition (1985) - $\Gamma_2$-calculus, the exponential curvature-dimension condition
- Lott-Sturm-Villani definition of curvature.
Curvature in discrete setting

**Question:** Is there a “good” notion of curvature in discrete setting from which we can recover

- transport-entropy inequalities,
- Poincaré inequalities,
- modified log-Sobolev inequalities, hypercontractivity,
- Prékopa-Leindler types of inequalities,
- concentration properties...

Several notions of curvature have been proposed on discrete spaces to extend the lower bound on Ricci-curvature in Riemannian geometry.

- The Bakry-Emery curvature condition (1985) - $\Gamma_2$-calculus, the exponential curvature-dimension condition  
- Lott-Sturm-Villani definition of curvature.
  - Rough curvature bounds, Bonciocat-Sturm (2009),
  - The entropic Ricci curvature, Erbar-Maas (2013), Mielke (2013),
  - Geodesic convexity property of entropy along interpolation paths: 
Curvature in discrete setting

**Question**: Is there a “good” notion of curvature in discrete setting from which we can recover

- transport-entropy inequalities,
- Poincaré inequalities,
- modified log-Sobolev inequalities, hypercontractivity,
- Prékopa-Leindler types of inequalities,
- concentration properties...

Several notions of curvature have been proposed on discrete spaces to extend the lower bound on Ricci-curvature in Riemannian geometry.

- The Bakry-Emery curvature condition (1985) - $\Gamma_2$-calculus, the exponential curvature-dimension condition
- Lott-Sturm-Villani definition of curvature.
  - Rough curvature bounds, Bonciocat-Sturm (2009),
  - The entropic Ricci curvature, Erbar-Maas (2013), Mielke (2013),

see also Maas-Erbar-Tetali (2015), Erbar-Fathi (2016), Fathi-Shu (2018),...
Curvature in discrete setting

**Question:** Is there a “good” notion of curvature in discrete setting from which we can recover

- transport-entropy inequalities,
- Poincaré inequalities,
- modified log-Sobolev inequalities, hypercontractivity,
- Prékopa-Leindler types of inequalities,
- concentration properties...

Several notions of curvature have been proposed on discrete spaces to extend the lower bound on Ricci-curvature in Riemannian geometry.

- The Bakry-Emery curvature condition (1985) - $\Gamma_2$-calculus,
  the exponential curvature-dimension condition
- Lott-Sturm-Villani definition of curvature.
  - Rough curvature bounds, Bonciocat-Sturm (2009),
  - The entropic Ricci curvature, Erbar-Maas (2013), Mielke (2013),
  - Geodesic convexity property of entropy along interpolation paths:

see also Maas-Erbar-Tetali (2015), Erbar-Fathi (2016), Fathi-Shu (2018),...

We will focus on the approach by C. Leonard in discrete,
Curvature in discrete setting

**Question**: Is there a “good” notion of curvature in discrete setting from which we can recover

- transport-entropy inequalities,
- Poincaré inequalities,
- modified log-Sobolev inequalities, hypercontractivity,
- Prékopa-Leindler types of inequalities,
- concentration properties...

Several notions of curvature have been proposed on discrete spaces to extend the lower bound on Ricci-curvature in Riemannian geometry.

- The Bakry-Emery curvature condition (1985) - $\Gamma_2$-calculus, the exponential curvature-dimension condition
- Lott-Sturm-Villani definition of curvature.
  - Rough curvature bounds, Bonciocat-Sturm (2009),
  - The entropic Ricci curvature, Erbar-Maas (2013), Mielke (2013),
  - Geodesic convexity property of entropy along interpolation paths:

see also Maas-Erbar-Tetali (2015), Erbar-Fathi (2016), Fathi-Shu (2018),...

We will focus on the approach by C. Leonard in discrete, following the recent approach by G. Conforti (2018) in continuous spaces when $L$ is a diffusion generator $Lf = \frac{1}{2} (\Delta f - \nabla U \cdot \nabla)$. 
A definition of curvature along Schrödinger paths

introduction
Marton's inequality
Talagrand's concentration

Kantorovich duality
for classical costs
for weak costs

Examples of weak cost
Marton's type of cost
Barycentric cost
Strassen's result
Martingale costs

Weak transport inequalities
Dual characterization
to concentration

Universal transport inequalities

Barycentric transport inequalities
examples
characterisation on \( \mathbb{R} \)

Transport inequality on the symmetric group
introduction
Ewens distribution
deveiation inequalities

The Schrödinger minimization problem
definition

curvature in discrete spaces
functional inequalities
Examples in discrete

Weak transport costs
A definition of curvature along Schrödinger paths

Definition: Schrödinger path

Examples of weak cost

Kantorovich duality

Universal transport inequalities

Weak transport inequalities

Transport inequality on the symmetric group

The Schrödinger minimization problem

curvature in discrete spaces
A definition of curvature along Schrödinger paths

**Definition : Schrödinger path**

Given $\mu_0$ and $\mu_1 \in \mathcal{P}(\mathcal{X})$ with finite support. The Schrödinger path associated to $L^\gamma$ with reversible measure $m$, is

$$\hat{Q}_t^\gamma := X_t \# \hat{Q}^\gamma, \quad t \in [0, 1],$$

where $\hat{Q}^\gamma$ is the Schrödinger bridge associated to $L^\gamma$ and $m$ for $\mu_0$ and $\mu_1$. 
A definition of curvature along Schrödinger paths

**Definition : Schrödinger path**

Given $\mu_0$ and $\mu_1 \in \mathcal{P}(X)$ with finite support. The Schrödinger path associated to $L^{\gamma}$ with reversible measure $m$, is

$$\hat{Q}_t^{\gamma} := X_t \# \hat{Q}^{\gamma}, \quad t \in [0, 1],$$

where $\hat{Q}^{\gamma}$ is the Schrödinger bridge associated to $L^{\gamma}$ and $m$ for $\mu_0$ and $\mu_1$.

$(\hat{Q}_t^{\gamma})_{t \in [0, 1]}$ is a path that interpolates between $\hat{Q}_0^{\gamma} = \mu_0$ and $\hat{Q}_1^{\gamma} = \mu_1$. 
A definition of curvature along Schrödinger paths

**Definition: Schrödinger path**

Given \( \mu_0 \) and \( \mu_1 \in \mathcal{P}(\mathcal{X}) \) with finite support. The Schrödinger path associated to \( L^\gamma \) with reversible measure \( m \), is

\[
\hat{Q}^\gamma_t := X_t \# \hat{Q}^\gamma, \quad t \in [0, 1],
\]

where \( \hat{Q}^\gamma \) is the Schrödinger bridge associated to \( L^\gamma \) and \( m \) for \( \mu_0 \) and \( \mu_1 \).

\((\hat{Q}^\gamma_t)_{t \in [0,1]}\) is a path that interpolates between \( \hat{Q}^\gamma_0 = \mu_0 \) and \( \hat{Q}^\gamma_1 = \mu_1 \).

By reversibility, \( \hat{Q}^\gamma_t(z) = m(z) P_t f(z) P_{1-t} g(z), \quad z \in \mathcal{X}. \)
A definition of curvature along Schrödinger paths

**Definition : Schrödinger path**

Given $\mu_0$ and $\mu_1 \in \mathcal{P}(\mathcal{X})$ with finite support. The Schrödinger path associated to $L^\gamma$ with reversible measure $m$, is

$$\hat{Q}_t^\gamma := X_t \# \hat{Q}^\gamma, \quad t \in [0, 1],$$

where $\hat{Q}^\gamma$ is the Schrödinger bridge associated to $L^\gamma$ and $m$ for $\mu_0$ and $\mu_1$.

$(\hat{Q}_t^\gamma)_{t \in [0, 1]}$ is a path that interpolates between $\hat{Q}_0^\gamma = \mu_0$ and $\hat{Q}_1^\gamma = \mu_1$.

By reversibility,

$$\hat{Q}_t^\gamma(z) = m(z) P_t f(z) P_{1-t} g(z), \quad z \in \mathcal{X}.$$
A definition of curvature along Schrödinger paths

**Definition : Schrödinger path**

Given \( \mu_0 \) and \( \mu_1 \in \mathcal{P}(\mathcal{X}) \) with finite support. The Schrödinger path associated to \( L^\gamma \) with reversible measure \( m \), is

\[
\hat{Q}_t^\gamma := X_t \# \hat{Q}^\gamma, \quad t \in [0, 1],
\]

where \( \hat{Q}^\gamma \) is the Schrödinger bridge associated to \( L^\gamma \) and \( m \) for \( \mu_0 \) and \( \mu_1 \).

\((\hat{Q}_t^\gamma)_{t \in [0, 1]}\) is a path that interpolates between \( \hat{Q}_0^\gamma = \mu_0 \) and \( \hat{Q}_1^\gamma = \mu_1 \).

By reversibility, \( \hat{Q}_t^\gamma (z) = m(z) P_t f(z) P_{1-t} g(z), \quad z \in \mathcal{X} \).

\[
\hat{Q}_t^\gamma (z) = \sum_{x, y} \nu_t^{x, y} (z) \hat{Q}_{0,1}^\gamma (x, y), \quad \text{with} \quad \nu_t^{x, y} (z) := \frac{m(z) P_t^\gamma (z, x) P_{1-t}^\gamma (z, y)}{m(x) P_{1-t}^\gamma (x, y)}.
\]

**A curvature definition [Conforti 2018-Léonard 2013]**
A definition of curvature along Schrödinger paths

**Definition : Schrödinger path**

Given $\mu_0$ and $\mu_1 \in \mathcal{P}(\mathcal{X})$ with finite support. The Schrödinger path associated to $L^\gamma$ with reversible measure $m$, is

$$\hat{Q}_t^\gamma := X_t \# \hat{Q}^\gamma, \quad t \in [0, 1],$$

where $\hat{Q}^\gamma$ is the Schrödinger bridge associated to $L^\gamma$ and $m$ for $\mu_0$ and $\mu_1$.

$(\hat{Q}_t^\gamma)_{t \in [0, 1]}$ is a path that interpolates between $\hat{Q}_0^\gamma = \mu_0$ and $\hat{Q}_1^\gamma = \mu_1$.

By reversibility, $\hat{Q}_t^\gamma(z) = m(z) P_t f(z) P_{1-t} g(z)$, $z \in \mathcal{X}$.

$$\hat{Q}_t^\gamma(z) = \sum_{x, y} \nu_t^{x, y}(z) \hat{Q}_{0,1}^\gamma(x, y), \quad \text{with} \quad \nu_t^{x, y}(z) := \frac{m(z) P_t^\gamma(z, x) P_{1-t}^\gamma(z, y)}{m(x) P_1^\gamma(x, y)}.$$

**A curvature definition [Conforti 2018-Léonard 2013]**

The space $(\mathcal{X}, L, m)$ has Ricci-curvature bounded from below by $\kappa$, $\kappa \in \mathbb{R}$,
A definition of curvature along Schrödinger paths

**Definition: Schrödinger path**

Given $\mu_0$ and $\mu_1 \in \mathcal{P}(\mathcal{X})$ with finite support. The Schrödinger path associated to $L^\gamma$ with reversible measure $m$, is

$$\hat{Q}_t^\gamma := X_t \# \hat{Q}^\gamma, \quad t \in [0, 1],$$

where $\hat{Q}^\gamma$ is the Schrödinger bridge associated to $L^\gamma$ and $m$ for $\mu_0$ and $\mu_1$.

$(\hat{Q}_t^\gamma)_{t \in [0, 1]}$ is a path that interpolates between $\hat{Q}_0^\gamma = \mu_0$ and $\hat{Q}_1^\gamma = \mu_1$.

By reversibility, $\hat{Q}_t^\gamma(z) = m(z) P_t f(z) P_{1-t} g(z)$, $z \in \mathcal{X}$.

$$\hat{Q}_t^\gamma(z)) = \sum_{x,y} \nu_{t}^{x,y}(z) \hat{Q}_{0,1}^\gamma(x,y), \quad \text{with} \quad \nu_{t}^{x,y}(z) := \frac{m(z) P_{t}^\gamma(z, x) P_{1-t}^\gamma(z, y)}{m(x) P_{1}^\gamma(x, y)}.$$

**A curvature definition [Conforti 2018-Léonard 2013]**

The space $(\mathcal{X}, L, m)$ has Ricci-curvature bounded from below by $\kappa$, $\kappa \in \mathbb{R}$, if for all $\mu_0, \mu_1 \in \mathcal{P}(\mathcal{X})$ with bounded support, $\mu_0 << m$, $\mu_1 << m$, for all $\gamma > 0$,

$$H(\hat{Q}_t^\gamma | m) \leq a_{\kappa \gamma} (1 - t) H(\mu_0 | m) + a_{\kappa \gamma} (t) H(\mu_1 | m) - c_{\kappa \gamma} (t) T_{L^\gamma} (\mu_0, \mu_1),$$

where $a_{\kappa \gamma}$ and $c_{\kappa \gamma}$ are constants depending on $\kappa$ and $\gamma$.
A definition of curvature along Schrödinger paths

**Definition : Schrödinger path**

Given $\mu_0$ and $\mu_1 \in \mathcal{P}(\mathcal{X})$ with finite support. The Schrödinger path associated to $L^\gamma$ with reversible measure $m$, is

$$\hat{Q}_t^\gamma := X_t \# \hat{Q}^\gamma, \quad t \in [0, 1],$$

where $\hat{Q}^\gamma$ is the Schrödinger bridge associated to $L^\gamma$ and $m$ for $\mu_0$ and $\mu_1$.

$$(\hat{Q}_t^\gamma)_{t \in [0,1]}$$ is a path that interpolates between $\hat{Q}_0^\gamma = \mu_0$ and $\hat{Q}_1^\gamma = \mu_1$.

By reversibility, $\hat{Q}_t^\gamma(z) = m(z) P_t f(z) P_{1-t} g(z), \quad z \in \mathcal{X}$.

$$\hat{Q}_t^\gamma(z) = \sum_{x,y} \nu_{x,y}^z(z) \hat{Q}_{0,1}^\gamma(x, y), \quad \text{with} \quad \nu_{x,y}^z(z) := \frac{m(z) P_t^\gamma(z, x) P_{1-t}^\gamma(z, y)}{m(x) P_1^\gamma(x, y)}.$$

**A curvature definition [Conforti 2018-Léonard 2013]**

The space $(\mathcal{X}, L, m)$ has Ricci-curvature bounded from below by $\kappa$, $\kappa \in \mathbb{R}$, if for all $\mu_0, \mu_1 \in \mathcal{P}(\mathcal{X})$ with bounded support, $\mu_0 << m$, $\mu_1 << m$, for all $\gamma > 0$,

$$H(\hat{Q}_t^\gamma | m) \leq a_{\kappa \gamma} (1 - t) H(\mu_0 | m) + a_{\kappa \gamma}(t) H(\mu_1 | m) - c_{\kappa \gamma}(t) T_L^\gamma (\mu_0, \mu_1),$$

where for $\gamma \neq 0$,

$$a_{\gamma}(t) = \frac{1 - e^{-\gamma t}}{1 - e^{-\gamma}}, \quad c_{\gamma}(t) = \frac{\cosh(\gamma/2) - \cosh(\gamma(2t - 1)/2)}{\sinh(\gamma/2)}, \quad t \in (0, 1).$$
A definition of curvature along Schrödinger paths

**Definition : Schrödinger path**

Given \( \mu_0 \) and \( \mu_1 \in \mathcal{P}(\mathcal{X}) \) with finite support. The Schrödinger path associated to \( L^\gamma \) with reversible measure \( m \), is

\[
\hat{Q}_t^\gamma := X_t \# \hat{Q}^\gamma, \quad t \in [0, 1],
\]

where \( \hat{Q}^\gamma \) is the Schrödinger bridge associated to \( L^\gamma \) and \( m \) for \( \mu_0 \) and \( \mu_1 \).

\( (\hat{Q}_t^\gamma)_{t \in [0,1]} \) is a path that interpolates between \( \hat{Q}_0^\gamma = \mu_0 \) and \( \hat{Q}_1^\gamma = \mu_1 \).

By reversibility,

\[
\hat{Q}_t^\gamma(z) = m(z) P_t f(z) P_{1-t} g(z), \quad z \in \mathcal{X}.
\]

\[
\hat{Q}_t^\gamma(z) = \sum_{x,y} \nu_{x}^{t,y}(z) \hat{Q}_{0,1}^\gamma(x, y), \quad \text{with} \quad \nu_{x}^{t,y}(z) := \frac{m(z) P_t^\gamma(z, x) P_{1-t}^\gamma(z, y)}{m(x) P_{1}^\gamma(x, y)}.
\]

**A curvature definition [Conforti 2018-Léonard 2013]**

The space \((\mathcal{X}, L, m)\) has Ricci-curvature bounded from below by \( \kappa \), \( \kappa \in \mathbb{R} \), if for all \( \mu_0, \mu_1 \in \mathcal{P}(\mathcal{X}) \) with bounded support, \( \mu_0 \ll m, \mu_1 \ll m \), for all \( \gamma > 0 \),

\[
H(\hat{Q}_t^\gamma | m) \leq a_{\kappa \gamma}(1 - t) H(\mu_0 | m) + a_{\kappa \gamma}(t) H(\mu_1 | m) - c_{\kappa \gamma}(t) T_{L^\gamma}(\mu_0, \mu_1),
\]

where for \( \gamma \neq 0 \),

\[
a_{\gamma}(t) = \frac{1 - e^{-\gamma t}}{1 - e^{-\gamma}}, \quad c_{\gamma}(t) = \frac{\cosh(\gamma/2) - \cosh((2t - 1)/\gamma)}{\sinh(\gamma/2)}, \quad t \in (0, 1).
\]

\[
a_{\kappa \gamma}(1 - t) \xrightarrow{\gamma \to 0} 1 - t, \quad a_{\kappa \gamma}(t) \xrightarrow{\gamma \to 0} t, \quad \frac{c_{\kappa \gamma}(t)}{\gamma} \xrightarrow{\gamma \to 0} \kappa t(1 - t).
\]
From curvature to functional inequalities
From curvature to functional inequalities

\[ H(\hat{Q}_t^\gamma | m) \leq a_{\kappa \gamma}(1 - t) H(\mu_0 | m) + a_{\kappa \gamma}(t) H(\mu_1 | m) - c_{\kappa \gamma}(t) T_{L^\gamma}(\mu_0, \mu_1), \]
From curvature to functional inequalities

\[ H(\hat{Q}_t^\gamma \mid m) \leq a_{\kappa \gamma}(1 - t) \ H(\mu_0 \mid m) + a_{\kappa \gamma}(t) \ H(\mu_1 \mid m) - c_{\kappa \gamma}(t) \ T_{L^\gamma}(\mu_0, \mu_1), \]

- In continuous setting (Léonard 2013, Conforti 2018):
  \[ Lf = \frac{1}{2} (\Delta f - \nabla U \cdot \nabla f), \quad dm = e^{-U} dvol, \]
From curvature to functional inequalities

\[ H(\hat{Q}_t^\gamma|m) \leq a_{\kappa\gamma}(1-t)\ H(\mu_0|m) + a_{\kappa\gamma}(t)\ H(\mu_1|m) - c_{\kappa\gamma}(t)\ T_{L^\gamma}(\mu_0, \mu_1), \]

- In continuous setting (Léonard 2013, Conforti 2018):
  \[ Lf = \frac{1}{2} \left( \Delta f - \nabla U \cdot \nabla f \right), \quad dm = e^{-U}dvol, \]

The definition of curvature is equivalent to the Bakry-Emery curvature condition \( CD(\kappa, \infty) \),
From curvature to functional inequalities

\[ H(\hat{Q}_t^\gamma|m) \leq a_{\kappa\gamma}(1 - t) H(\mu_0|m) + a_{\kappa\gamma}(t) H(\mu_1|m) - c_{\kappa\gamma}(t) T_{L\gamma}(\mu_0, \mu_1), \]

- In continuous setting (Léonard 2013, Conforti 2018):

\[ Lf = \frac{1}{2} (\Delta f - \nabla U \cdot \nabla f), \quad dm = e^{-U}dvol, \]

The definition of curvature is equivalent to the Bakry-Emery curvature condition \( CD(\kappa, \infty) \), since

\[ a_{\kappa\gamma}(1 - t) \xrightarrow{\gamma \to 0} 1 - t, \quad \frac{c_{\kappa\gamma}(t)}{\gamma} t T_{L\gamma}(\mu_0, \mu_1) \xrightarrow{\gamma \to 0} \kappa t(1 - t) \frac{W_2^2(\mu_0, \mu_1)}{2}, \]

and \( \hat{Q}_t^\gamma \xrightarrow{\gamma \to 0} \hat{Q}_t^0 \), constant speed geodesic for \( W_2 \).
From curvature to functional inequalities

\[ H(Q^\gamma_t | m) \leq a_{\kappa \gamma}(1 - t) H(\mu_0 | m) + a_{\kappa \gamma}(t) H(\mu_1 | m) - c_{\kappa \gamma}(t) T_{L \gamma}(\mu_0, \mu_1), \]

- In continuous setting (Léonard 2013, Conforti 2018):
  \[ Lf = \frac{1}{2}(\Delta f - \nabla U \cdot \nabla f), \quad dm = e^{-U} dvol, \]

The definition of curvature is equivalent to the Bakry-Emery curvature condition \( CD(\kappa, \infty) \), since

\[ a_{\kappa \gamma}(1 - t) \xrightarrow[\gamma \to 0]{} 1 - t, \quad c_{\kappa \gamma}(t) \xrightarrow[\gamma \to 0]{} \kappa t(1 - t) \frac{W^2_2(\mu_0, \mu_1)}{2}, \]

and \( \hat{Q}^\gamma_t \xrightarrow[\gamma \to 0]{} \hat{Q}^0_t \), constant speed geodesic for \( W_2 \).

It recovers the Lott-Sturm-Villani definition of Ricci curvature \( \geq \kappa \).
From curvature to functional inequalities

\[ H(\hat{Q}_t^\gamma | m) \leq a_{\kappa, \gamma}(1 - t) \, H(\mu_0 | m) + a_{\kappa, \gamma}(t) \, H(\mu_1 | m) - c_{\kappa, \gamma}(t) \, T_{L^\gamma}(\mu_0, \mu_1), \]

- In continuous setting (Léonard 2013, Conforti 2018):

\[
Lf = \frac{1}{2} (\Delta f - \nabla U \cdot \nabla f), \quad dm = e^{-U} dvol,
\]

The definition of curvature is equivalent to the Bakry-Emery curvature condition \( CD(\kappa, \infty) \), since

\[
a_{\kappa, \gamma}(1 - t) \xrightarrow{\gamma \to 0} 1 - t, \quad \frac{c_{\kappa, \gamma}(t)}{\gamma} \xrightarrow{\gamma \to 0} \kappa t(1 - t) \frac{W_2^2(\mu_0, \mu_1)}{2},
\]

and \( \hat{Q}_t^\gamma \xrightarrow{\gamma \to 0} \hat{Q}_t^0 \), constant speed geodesic for \( W_2 \).

It recovers the Lott-Sturm-Villani definition of Ricci curvature \( \geq \kappa \).

- In discrete setting,
From curvature to functional inequalities

\[ H(\frac{\gamma}{t} | m) \leq a_{\kappa, \gamma}(1 - t) H(\mu_0 | m) + a_{\kappa, \gamma}(t) H(\mu_1 | m) - c_{\kappa, \gamma}(t) T_{L^\gamma}(\mu_0, \mu_1), \]

- **In continuous setting** (Léonard 2013, Conforti 2018):
  \[ Lf = \frac{1}{2} (\Delta f - \nabla U \cdot \nabla f), \quad dm = e^{-U} dvol, \]

  The definition of curvature is equivalent to the Bakry-Emery curvature condition \( CD(\kappa, \infty) \), since

  \[ a_{\kappa, \gamma}(1 - t) \xrightarrow{\gamma \to 0} 1 - t, \quad \frac{c_{\kappa, \gamma}(t)}{\gamma} T_{L^\gamma}(\mu_0, \mu_1) \xrightarrow{\gamma \to 0} \kappa t(1 - t) \frac{W_2^2(\mu_0, \mu_1)}{2}, \]

  and

  \[ \frac{\gamma}{t} \xrightarrow{\gamma \to 0} \frac{\gamma}{t}^0, \quad \text{constant speed geodesic for } W_2. \]

  It recovers the Lott-Sturm-Villani definition of Ricci curvature \( \geq \kappa \).

- **In discrete setting**, \[ c_{\kappa, \gamma}(t) T_{L^\gamma}(\mu_0, \mu_1) \xrightarrow{\gamma \to 0} 0 \]

\[ \boxed{\text{Universal transport inequalities}} \]

\[ \boxed{\text{Barycentric transport inequalities examples}} \]

\[ \boxed{\text{Transport inequality on the symmetric group}} \]

\[ \boxed{\text{The Schrödinger minimization problem}} \]

\[ \boxed{\text{functional inequalities}} \]

Examples in discrete
From curvature to functional inequalities

\[ H(\hat{Q}_t^\gamma|m) \leq a_{\kappa\gamma}(1 - t) \, H(\mu_0|m) + a_{\kappa\gamma}(t) \, H(\mu_1|m) - c_{\kappa\gamma}(t) \, T_{L^\gamma}(\mu_0, \mu_1), \]

- In continuous setting (Léonard 2013, Conforti 2018):
  \[ Lf = \frac{1}{2} (\Delta f - \nabla U \cdot \nabla f), \quad dm = e^{\frac{1}{2} U} dvol, \]

The definition of curvature is equivalent to the Bakry-Emery curvature condition \( CD(\kappa, \infty) \), since

\[ a_{\kappa\gamma}(1 - t) \xrightarrow{\gamma \to 0} 1 - t, \quad c_{\kappa\gamma}(t) \xrightarrow{\gamma \to 0} \kappa t (1 - t) \frac{W_2^2(\mu_0, \mu_1)}{2}, \]

and

\[ \hat{Q}_t^\gamma \xrightarrow{\gamma \to 0} \hat{Q}_t^0, \quad \text{constant speed geodesic for } W_2. \]

It recovers the Lott-Sturm-Villani definition of Ricci curvature \( \geq \kappa \).

- In discrete setting,
  \[ c_{\kappa\gamma}(t) \, T_{L^\gamma}(\mu_0, \mu_1) \xrightarrow{\gamma \to 0} 0 \]

If \( \kappa > 0 \), the curvature condition implies
From curvature to functional inequalities

\[ H(\hat{Q}_t^\gamma|m) \leq a_{\kappa \gamma}(1 - t) H(\mu_0|m) + a_{\kappa \gamma}(t) H(\mu_1|m) - c_{\kappa \gamma}(t) T_{L^\gamma}(\mu_0, \mu_1), \]

- In continuous setting (Léonard 2013, Conforti 2018):
  \[ Lf = \frac{1}{2} (\Delta f - \nabla U \cdot \nabla f), \quad dm = e^{-U} d\text{vol}, \]

The definition of curvature is equivalent to the Bakry-Emery curvature condition \( CD(\kappa, \infty) \), since

\[ a_{\kappa \gamma}(1 - t) \xrightarrow{\gamma \to 0} 1 - t, \quad \frac{c_{\kappa \gamma}(t)}{\gamma} t T_{L^\gamma}(\mu_0, \mu_1) \xrightarrow{\gamma \to 0} \kappa t (1 - t) \frac{W^2_2(\mu_0, \mu_1)}{2}, \]

and \( \hat{Q}_t^\gamma \xrightarrow{\gamma \to 0} \hat{Q}_t^0 \), constant speed geodesic for \( W^2_2 \).

It recovers the Lott-Sturm-Villani definition of Ricci curvature \( \geq \kappa \).

- In discrete setting, \( c_{\kappa \gamma}(t) T_{L^\gamma}(\mu_0, \mu_1) \xrightarrow{\gamma \to 0} 0 \) !!!

If \( \kappa > 0 \), the curvature condition implies

- a weak transport inequality:
  \[ T_{L^\gamma}(\mu_0, \mu_1) \leq \frac{a_{\kappa \gamma}(1 - t)}{c_{\kappa \gamma}(t)} H(\mu_0|m) + \frac{a_{\kappa \gamma}(t)}{c_{\kappa \gamma}(t)} H(\mu_1|m). \]
From curvature to functional inequalities

\[ H(\hat{Q}_t^\gamma|m) \leq a_{\kappa\gamma}(1 - t)\ H(\mu_0|m) + a_{\kappa\gamma}(t)\ H(\mu_1|m) - c_{\kappa\gamma}(t)\ T_{L\gamma}(\mu_0, \mu_1), \]

- In continuous setting (Léonard 2013, Conforti 2018):
  \[ Lf = \frac{1}{2} (\Delta f - \nabla U \cdot \nabla f), \quad dm = e^{-U} dvol, \]

The definition of curvature is equivalent to the Bakry-Emery curvature condition \( CD(\kappa, \infty) \), since

\[ a_{\kappa\gamma}(1 - t) \xrightarrow{\gamma \to 0} 1 - t, \quad \frac{c_{\kappa\gamma}(t)}{\gamma} tT_{L\gamma}(\mu_0, \mu_1) \xrightarrow{\gamma \to 0} \kappa t(1 - t) \frac{W_2^2(\mu_0, \mu_1)}{2}, \]

and \( \hat{Q}_t^\gamma \xrightarrow{\gamma \to 0} \hat{Q}_t^0 \), constant speed geodesic for \( W_2 \).

It recovers the Lott-Sturm-Villani definition of Ricci curvature \( \geq \kappa \).

- In discrete setting, \( c_{\kappa\gamma}(t)\ T_{L\gamma}(\mu_0, \mu_1) \xrightarrow{\gamma \to 0} 0 \) !!!

If \( \kappa > 0 \), the curvature condition implies

- a weak transport inequality:
  \[ T_{L\gamma}(\mu_0, \mu_1) \leq \frac{a_{\kappa\gamma}(1 - t)}{c_{\kappa\gamma}(t)} H(\mu_0|m) + \frac{a_{\kappa\gamma}(t)}{c_{\kappa\gamma}(t)} H(\mu_1|m). \]

- the modified log-Sobolev inequality (mLSI):
  \[ H(\nu_0|m) \leq \frac{1}{2\kappa} \sum_{x, w \in \mathcal{X}} (\log h_0(w) - \log h_0(x))(h_0(w) - h_0(x))L(x, w)m(x). \]
From curvature to functional inequalities

\[ H(Q_t^\gamma | m) \leq a_{\kappa \gamma}(1 - t) H(\mu_0 | m) + a_{\kappa \gamma}(t) H(\mu_1 | m) - c_{\kappa \gamma}(t) T_{L^\gamma}(\mu_0, \mu_1), \]

- In continuous setting (Léonard 2013, Conforti 2018):
  \[ Lf = \frac{1}{2} (\Delta f - \nabla U \cdot \nabla f), \quad dm = e^{-U} dvol, \]

The definition of curvature is equivalent to the Bakry-Emery curvature condition \( CD(\kappa, \infty) \), since

\[ a_{\kappa \gamma}(1 - t) \xrightarrow{\gamma \to 0} 1 - t, \quad c_{\kappa \gamma}(t) t T_{L^\gamma}(\mu_0, \mu_1) \xrightarrow{\gamma \to 0} \kappa t(1 - t) \frac{W_2^2(\mu_0, \mu_1)}{2}, \]

and \( \hat{Q}_t^\gamma \xrightarrow{\gamma \to 0} \hat{Q}_t^0 \), constant speed geodesic for \( W_2 \).

It recovers the Lott-Sturm-Villani definition of Ricci curvature \( \geq \kappa \).

- In discrete setting, \( c_{\kappa \gamma}(t) T_{L^\gamma}(\mu_0, \mu_1) \xrightarrow{\gamma \to 0} 0 !!! \)

If \( \kappa > 0 \), the curvature condition implies

- a weak transport inequality:
  \[ T_{L^\gamma}(\mu_0, \mu_1) \leq \frac{a_{\kappa \gamma}(1 - t)}{c_{\kappa \gamma}(t)} H(\mu_0 | m) + \frac{a_{\kappa \gamma}(t)}{c_{\kappa \gamma}(t)} H(\mu_1 | m). \]

- the modified log-Sobolev inequality (mLSI): \( \mu_0 = h_0 m, \)
  \[ H(\nu_0 | m) \leq \frac{1}{2\kappa} \sum_{x, w \in X} (\log h_0(w) - \log h_0(x))(h_0(w) - h_0(x))L(x, w) m(x). \]

(Proof: by differentiating at point \( t = 0 \) and then \( \gamma \to \infty \))
From curvature to functional inequalities

\[ H(Q_t^\gamma | m) \leq a_{\kappa \gamma} (1 - t) H(\mu_0 | m) + a_{\kappa \gamma} (t) H(\mu_1 | m) - c_{\kappa \gamma} (t) T_{L^\gamma} (\mu_0, \mu_1), \]

- In continuous setting (Léonard 2013, Conforti 2018):
  \[ Lf = \frac{1}{2} (\Delta f - \nabla U \cdot \nabla f), \quad dm = e^{-U} dvol, \]

The definition of curvature is equivalent to the Bakry-Emery curvature condition \( CD(\kappa, \infty) \), since

\[ a_{\kappa \gamma} (1 - t) \xrightarrow{\gamma \to 0} 1 - t, \quad c_{\kappa \gamma} (t) T_{L^\gamma} (\mu_0, \mu_1) \xrightarrow{\gamma \to 0} \kappa t (1 - t) \frac{W_2^2 (\mu_0, \mu_1)}{2}, \]

and \( \hat{Q}_t^\gamma \xrightarrow{\gamma \to 0} \hat{Q}_t^0 \), constant speed geodesic for \( W_2 \).

It recovers the Lott-Sturm-Villani definition of Ricci curvature \( \geq \kappa \).

- In discrete setting, \( c_{\kappa \gamma} (t) T_{L^\gamma} (\mu_0, \mu_1) \xrightarrow{\gamma \to 0} 0 \) !!!!

If \( \kappa > 0 \), the curvature condition implies

- a weak transport inequality:
  \[ T_{L^\gamma} (\mu_0, \mu_1) \leq \frac{a_{\kappa \gamma} (1 - t)}{c_{\kappa \gamma} (t)} H(\mu_0 | m) + \frac{a_{\kappa \gamma} (t)}{c_{\kappa \gamma} (t)} H(\mu_1 | m). \]

- the modified log-Sobolev inequality (mLSI):
  \( \mu_0 = h_0 m, \)
  \[ H(\nu_0 | m) \leq \frac{1}{2\kappa} \sum_{x, w \in \mathcal{X}} (\log h_0 (w) - \log h_0 (x)) (h_0 (w) - h_0 (x)) L(x, w) m(x). \]

(Proof : by differentiating at point \( t = 0 \) and then \( \gamma \to \infty \))

**Consequence**: The best constant \( \alpha \) in mLSI satisfies \( \alpha \geq \kappa \).
From curvature to Prékopa-Leindler

Theorem [S. 2018] : Discrete Prékopa-Leindler inequality (PLI)
From curvature to Prékopa-Leindler


Assume \((\mathcal{X}, m, L)\) has Ricci curvature lower bounded by \(\kappa\), \(\kappa \in \mathbb{R}\).
From curvature to Prékopa-Leindler


Assume \((\mathcal{X}, m, L)\) has Ricci curvature lower bounded by \(\kappa, \kappa \in \mathbb{R}\). Let \(t \in (0, 1)\) and \(\gamma > 0\).
Theorem [S. 2018]: Discrete Prékopa-Leindler inequality (PLI)

Assume \((\mathcal{X}, m, L)\) has Ricci curvature lower bounded by \(\kappa\), \(\kappa \in \mathbb{R}\). Let \(t \in (0, 1)\) and \(\gamma > 0\). Then for any positive functions \(F, G, H\) on \(\mathcal{X}\) satisfying

\[
\exp \int \log H \, d\nu_{t}^{x,y} \geq F(x)^{a_{\kappa \gamma}(1-t)} G(y)^{a_{\kappa \gamma}(t)}, \quad x, y, z \in \mathcal{X},
\] (1)
From curvature to Prékopa-Leindler


Assume \((X, m, L)\) has Ricci curvature lower bounded by \(\kappa\), \(\kappa \in \mathbb{R}\). Let \(t \in (0, 1)\) and \(\gamma > 0\). Then for any positive functions \(F, G, H\) on \(X\) satisfying

\[
\exp \int \log H \, d\nu_i^{x,y} \geq F(x)^{a_{\kappa \gamma}(1-t)} G(y)^{a_{\kappa \gamma}(t)} , \quad x, y, z \in X , \tag{1}
\]

one has for any positive function \(K : X \to \mathbb{R}^+\),

\[
\int H \, dm \geq \left( \int \frac{F^{a_{\kappa \gamma}(1-t)} G^{a_{\kappa \gamma}(1-t)-c_{\kappa \gamma}(t)}}{K^{c_{\kappa \gamma}(t)}} \, dm \right)^{a_{\kappa \gamma}(1-t)-c_{\kappa \gamma}(t)} \left( \int G(P_{\gamma} K)^{c_{\kappa \gamma}(t)} \, dm \right)^{a_{\kappa \gamma}(t)}
\]
From curvature to Prékopa-Leindler


Assume $(\mathcal{X}, m, L)$ has Ricci curvature lower bounded by $\kappa$, $\kappa \in \mathbb{R}$. Let $t \in (0, 1)$ and $\gamma > 0$. Then for any positive functions $F, G, H$ on $\mathcal{X}$ satisfying

$$\exp \int \log H \, d\nu^{x,y}_t \geq F(x)^{a_{\kappa\gamma}(1-t)} G(y)^{a_{\kappa\gamma}(t)}, \quad x, y, z \in \mathcal{X}, \quad (1)$$

one has for any positive function $K : \mathcal{X} \to \mathbb{R}^+$,

$$\int H \, dm \geq \left( \int \frac{F^{a_{\kappa\gamma}(1-t)} G^{a_{\kappa\gamma}(1-t)-c_{\kappa\gamma}(t)} \, dm}{K^{\frac{c_{\kappa\gamma}(t)}{a_{\kappa\gamma}(1-t)-c_{\kappa\gamma}(t)}}} \right)^{a_{\kappa\gamma}(1-t)-c_{\kappa\gamma}(t)} \left( \int G(P_\gamma K)^{\frac{c_{\kappa\gamma}(t)}{a_{\kappa\gamma}(t)}} \, dm \right)^{a_{\kappa\gamma}(t)}$$

• Choosing $K = 1$, we get for any $F, G, H$ satisfying (1),

$$\int H \, dm \geq \left( \int F^{\frac{a_{\kappa\gamma}(1-t)}{1-a_{\kappa\gamma}(t)}} \, dm \right)^{1-a_{\kappa\gamma}(t)} \left( \int G \, dm \right)^{a_{\kappa\gamma}(t)}.$$
From curvature to Prékopa-Leindler


Assume \((\mathcal{X}, m, L)\) has Ricci curvature lower bounded by \(\kappa, \kappa \in \mathbb{R}\). Let \(t \in (0, 1)\) and \(\gamma > 0\). Then for any positive functions \(F, G, H\) on \(\mathcal{X}\) satisfying

\[
\exp \int \log H \, d\nu^X_{t, y} \geq F(x)^{a_{\kappa \gamma}(1-t)} G(y)^{a_{\kappa \gamma}(t)}, \quad x, y, z \in \mathcal{X},
\]

one has for any positive function \(K : \mathcal{X} \to \mathbb{R}^+\),

\[
\int H \, dm \geq \left( \int \frac{F^{a_{\kappa \gamma}(1-t)} - c_{\kappa \gamma}(t)}{c_{\kappa \gamma}(t)} \, dm \right) \left( \int G(P_{\gamma}K)^{c_{\kappa \gamma}(t)} \, dm \right)^{a_{\kappa \gamma}(t)}.
\]

- Choosing \(K = 1\), we get for any \(F, G, H\) satisfying (1),

\[
\int H \, dm \geq \left( \int F^{1-a_{\kappa \gamma}(t)} \, dm \right)^{1-a_{\kappa \gamma}(t)} \left( \int G \, dm \right)^{a_{\kappa \gamma}(t)}.
\]

This property also implies mLSI when \(\kappa > 0\).
From curvature to Prékopa-Leindler


Assume \((\mathcal{X}, m, L)\) has Ricci curvature lower bounded by \(\kappa, \kappa \in \mathbb{R}\). Let \(t \in (0, 1)\) and \(\gamma > 0\). Then for any positive functions \(F, G, H\) on \(\mathcal{X}\) satisfying

\[
\exp \left( \int \log H \, d\nu_i^{X,Y} \right) \geq F(x)^{a_{\kappa \gamma} (1-t)} G(y)^{a_{\kappa \gamma} (t)}, \quad x, y, z \in \mathcal{X},
\]

(1)

one has for any positive function \(K : \mathcal{X} \to \mathbb{R}^+\),

\[
\int H \, dm \geq \left( \int \frac{F^{a_{\kappa \gamma} (1-t)}}{K^{a_{\kappa \gamma} (1-t) - c_{\kappa \gamma} (t)}} \, dm \right)^{a_{\kappa \gamma} (1-t) - c_{\kappa \gamma} (t)} \left( \int G(P_{\gamma} K)^{c_{\kappa \gamma} (t)} \, dm \right)^{a_{\kappa \gamma} (t)}
\]

- Choosing \(K = 1\), we get for any \(F, G, H\) satisfying (1),

\[
\int H \, dm \geq \left( \int F^{1 - a_{\kappa \gamma} (t) a_{\kappa \gamma} (t)} \, dm \right)^{1 - a_{\kappa \gamma} (t)} \left( \int G \, dm \right)^{a_{\kappa \gamma} (t)}
\]

This property also implies mLSI when \(\kappa > 0\).

- Choosing \(F = G = H = 1\) we get the following reverse-hypercontractivity result:
From curvature to Prékopa-Leindler


Assume \((\mathcal{X}, m, L)\) has Ricci curvature lower bounded by \(\kappa\), \(\kappa \in \mathbb{R}\). Let \(t \in (0, 1)\) and \(\gamma > 0\). Then for any positive functions \(F, G, H\) on \(\mathcal{X}\) satisfying

\[
\exp \int \log H \, d\nu_i^{X,y} \geq F(x)^{a_{\kappa\gamma}(1-t)} G(y)^{a_{\kappa\gamma}(t)}, \quad x, y, z \in \mathcal{X},
\]

one has for any positive function \(K : \mathcal{X} \to \mathbb{R}^+\),

\[
\int H \, dm \geq \left( \int \frac{F^{a_{\kappa\gamma}(1-t) / c_{\kappa\gamma}(t)} \, dm}{K^{a_{\kappa\gamma}(1-t) / c_{\kappa\gamma}(t)}} \right)^{a_{\kappa\gamma}(1-t) - c_{\kappa\gamma}(t)} \left( \int \frac{G(P_{\gamma}K)^{c_{\kappa\gamma}(t) / a_{\kappa\gamma}(t)} \, dm}{1 - a_{\kappa\gamma}(t)} \right)^{a_{\kappa\gamma}(t)}.
\]

- Choosing \(K = 1\), we get for any \(F, G, H\) satisfying (1),

\[
\int H \, dm \geq \left( \int F^{1/a_{\kappa\gamma}(t)} \, dm \right)^{1 - a_{\kappa\gamma}(t)} \left( \int G \, dm \right)^{a_{\kappa\gamma}(t)}.
\]

This property also implies mLSI when \(\kappa > 0\).
- Choosing \(F = G = H = 1\) we get the following reverse-hypercontractivity result: for any \(t \in (0, 1), \gamma > 0\),

\[
\| P_{\gamma}K \|_{a_{\kappa\gamma}(t) / c_{\kappa\gamma}(t)} \leq \| K \|^\frac{c_{\kappa\gamma}(t) / a_{\kappa\gamma}(t)}{1 - a_{\kappa\gamma}(t)}, \quad (\kappa > 0).
\]
Examples of discrete space with curvature bounded from below [S. 2018]

- **Marton's inequality**
- **Talagrand's concentration**
- **Kantorovich duality**
  - for classical costs
  - for weak costs
- **Examples of weak cost**
  - Marton's type of cost
  - Barycentric cost
  - Strassen's result
  - Martingale costs
- **Weak transport inequalities**
- **Dual characterization to concentration**
- **Universal transport inequalities**
- **Barycentric transport inequalities**
- **Examples**
  - characterisation on $\mathbb{R}$
- **Transport inequality on the symmetric group**
  - introduction
  - Ewens distribution
  - deviation inequalities
- **The Schrödinger minimization problem**
  - definition
  - curvature in discrete spaces
  - functional inequalities
- **Examples in discrete**
  - Weak transport costs.
Examples of discrete space with curvature bounded from below [S. 2018]

- $\mathcal{X} = \mathbb{Z}$, for all $x \in \mathbb{Z}$, $L(x, x + 1) = L(x, x - 1) = 1$, $L(x, x) = -2$.

$m$ : the counting measure.
Examples of discrete space with curvature bounded from below [S. 2018]

- $\mathcal{X} = \mathbb{Z}$, for all $x \in \mathbb{Z}$, $L(x, x + 1) = L(x, x - 1) = 1$, $L(x, x) = -2$. $m$ : the counting measure. Result : $\kappa \geq 0$. 

$m$ : the counting measure.
Examples of discrete space with curvature bounded from below [S. 2018]

- $\mathcal{X} = \mathbb{Z}$, for all $x \in \mathbb{Z}$, $L(x, x + 1) = L(x, x - 1) = 1$, $L(x, x) = -2$.
- $m$: the counting measure. Result: $\kappa \geq 0$.

Observing that

$$
\nu_t^{x,y}(Z) \xrightarrow{\gamma \to 0} \left( \frac{d(z, x)}{d(x, y)} \right) t^{d(z, x)} (1-t)^{d(z, y)} 1_{[x,y]}(Z),
$$
Examples of discrete space with curvature bounded from below [S. 2018]

- $\mathcal{X} = \mathbb{Z}$, for all $x \in \mathbb{Z}$, $L(x, x + 1) = L(x, x - 1) = 1$, $L(x, x) = -2$.
- $m$ : the counting measure. Result : $\kappa \geq 0$.

Observing that

$$\nu_t^{x,y}(z) \xrightarrow{\gamma \to 0} \left( \frac{d(z, x)}{d(x, y)} \right) t^d(z, x) (1 - t)^d(z, y) \mathbb{1}_{[x, y]}(z),$$

one recovers Hillion’s result on $\mathbb{Z}$,
Examples of discrete space with curvature bounded from below [S. 2018]

- $\mathcal{X} = \mathbb{Z}$, for all $x \in \mathbb{Z}$, $L(x, x + 1) = L(x, x - 1) = 1$, $L(x, x) = -2$.
  
  $m$: the counting measure. **Result:** $\kappa \geq 0$.

  Observing that

  $$\nu_{t}^{x,y}(z) \underset{\gamma \to 0}{\longrightarrow} \binom{d(z, x)}{d(x, y)} t^{d(z, x)} (1 - t)^{d(z, y)} 1_{[x, y]} (z),$$

  one recovers Hillion’s result on $\mathbb{Z}$, and we get a new PLI on $\mathbb{Z}$. 
Examples of discrete space with curvature bounded from below [S. 2018]

- $\mathcal{X} = \mathbb{Z}$, for all $x \in \mathbb{Z}$, $L(x, x + 1) = L(x, x - 1) = 1$, $L(x, x) = -2$.  
  $m$: the counting measure.  
  **Result:** $\kappa \geq 0$.

Observing that

$$\nu_t^{x,y}(Z) \xrightarrow{\gamma \to 0} \left( \frac{d(z,x)}{d(x,y)} \right) t^{d(z,x)}(1 - t)^{d(z,y)} 1_{[x,y]}(Z),$$

one recovers Hillion’s result on $\mathbb{Z}$, and we get a new PLI on $\mathbb{Z}$.

- $\mathcal{X} = \{0, 1\}^n$, the discrete cube,
Examples of discrete space with curvature bounded from below [S. 2018]

- $\mathcal{X} = \mathbb{Z}$, for all $x \in \mathbb{Z}$, $L(x, x + 1) = L(x, x - 1) = 1$, $L(x, x) = -2$. $m$ : the counting measure. \textbf{Result : } $\kappa \geq 0$.

  Observing that

  $$
  \nu^x_y(Z) \xrightarrow{\gamma \to 0} \left( \frac{d(z, x)}{d(x, y)} \right) t^{d(z, x)} (1 - t)^{d(z, y)} \mathbb{1}_{[x, y]}(Z),
  $$

  one recovers Hillion’s result on $\mathbb{Z}$, and we get a new PLI on $\mathbb{Z}$.

- $\mathcal{X} = \{0, 1\}^n$, the discrete cube, for all $x \in \{0, 1\}^n$

  $$
  L(x, \sigma_i(x)) = 1 \quad \text{for all } i \in \{1, \ldots, n\}, \quad L(x, x) = -n,
  $$

  $\nu^x_y(Z)$
Examples of discrete space with curvature bounded from below [S. 2018]

- $\mathcal{X} = \mathbb{Z}$, for all $x \in \mathbb{Z}$, $L(x, x + 1) = L(x, x - 1) = 1$, $L(x, x) = -2$.
  
  $m$: the counting measure. Result: $\kappa \geq 0$.

  Observing that
  \[
  \nu_t^{x,y}(z) \xrightarrow{\gamma \to 0} \begin{pmatrix} d(z, x) \\ d(x, y) \end{pmatrix} t^{d(z,x)}(1-t)^{d(z,y)} \mathbb{1}_{[x,y]}(z),
  \]

  one recovers Hillion’s result on $\mathbb{Z}$, and we get a new PLI on $\mathbb{Z}$.

- $\mathcal{X} = \{0, 1\}^n$, the discrete cube, for all $x \in \{0, 1\}^n$
  
  $L(x, \sigma_i(x)) = 1$ for all $i \in \{1, \ldots, n\}$, $L(x, x) = -n$.

  $m$: the uniform probability measure on $\{0, 1\}^n$. 
Examples of discrete space with curvature bounded from below [S. 2018]

- $\mathcal{X} = \mathbb{Z}$, for all $x \in \mathbb{Z}$, $L(x, x + 1) = L(x, x - 1) = 1$, $L(x, x) = -2$.
  
  $m$: the counting measure. \textbf{Result: $\kappa \geq 0$.}

  Observing that

  $$\nu_t^{x,y}(z) \xrightarrow{\gamma \to 0} \begin{pmatrix} d(z, x) \\ d(x, y) \end{pmatrix} t^{d(z,x)} (1 - t)^{d(z,y)} 1_{[x,y]}(z),$$

  one recovers Hillion’s result on $\mathbb{Z}$, and we get a new PLI on $\mathbb{Z}$.

- $\mathcal{X} = \{0, 1\}^n$, the discrete cube, for all $x \in \{0, 1\}^n$

  $$L(x, \sigma_i(x)) = 1 \quad \text{for all } i \in \{1, \ldots, n\}, \quad L(x, x) = -n,$$

  $m$: the uniform probability measure on $\{0, 1\}^n$. \textbf{Result: $\kappa \geq 4$.}
Examples of discrete space with curvature bounded from below [S. 2018]

- $\mathcal{X} = \mathbb{Z}$, for all $x \in \mathbb{Z}$, $L(x, x + 1) = L(x, x - 1) = 1$, $L(x, x) = -2$. $m$ : the counting measure. **Result :** $\kappa \geq 0$.

Observing that

$$\nu^x_y(Z) \xrightarrow[\gamma \to 0]{} \left( \frac{d(z, x)}{d(x, y)} \right) t^{d(z, x)} (1 - t)^{d(z, y)} \mathbb{1}_{[x, y]}(Z),$$

one recovers Hillion’s result on $\mathbb{Z}$, and we get a new PLI on $\mathbb{Z}$.

- $\mathcal{X} = \{0, 1\}^n$, the discrete cube, for all $x \in \{0, 1\}^n$

  $L(x, \sigma_i(x)) = 1$ for all $i \in \{1, \ldots, n\}$, $L(x, x) = -n$.

$m$ : the uniform probability measure on $\{0, 1\}^n$. **Result :** $\kappa \geq 4$.

It provides the optimal constant in mLSI.
Examples of discrete space with curvature bounded from below [S. 2018]

- \( \mathcal{X} = \mathbb{Z} \), for all \( x \in \mathbb{Z} \), \( L(x, x + 1) = L(x, x - 1) = 1, L(x, x) = -2 \).
  - \( m \): the counting measure. Result: \( \kappa \geq 0 \).

Observing that
\[
\nu^x_y(Z) \xrightarrow{\gamma \to 0} \left( \frac{d(z, x)}{d(x, y)} \right)^t d(z, x) (1 - t)d(z, y) \mathbb{1}_{[x, y]}(Z),
\]

one recovers Hillion’s result on \( \mathbb{Z} \), and we get a new PLI on \( \mathbb{Z} \).

- \( \mathcal{X} = \{0, 1\}^n \), the discrete cube, for all \( x \in \{0, 1\}^n \)

  - \( L(x, \sigma_i(x)) = 1 \) for all \( i \in \{1, \ldots, n\} \), \( L(x, x) = -n \).
  - \( m \): the uniform probability measure on \( \{0, 1\}^n \). Result: \( \kappa \geq 4 \).

It provides the optimal constant in mLSI.

Observing that
\[
\nu^x_z(Z) \xrightarrow{\gamma \to 0} t d(z, x) (1 - t)d(z, y) \mathbb{1}_{[x, y]}(Z),
\]
Examples of discrete space with curvature bounded from below [S. 2018]

- $X = \mathbb{Z}$, for all $x \in \mathbb{Z}$, $L(x, x + 1) = L(x, x - 1) = 1$, $L(x, x) = -2$.
  $m$ : the counting measure. Result : $\kappa \geq 0$.
  Observing that
  \[
  \nu_t^{x,y}(Z) \xrightarrow[\gamma \to 0]{d(z, x)} t^{d(z, x)} (1 - t)^{d(z, y)} 1_{[x, y]}(Z),
  \]
  one recovers Hillion’s result on $\mathbb{Z}$, and we get a new PLI on $\mathbb{Z}$.

- $X = \{0, 1\}^n$, the discrete cube, for all $x \in \{0, 1\}^n$
  $L(\sigma_i(x)) = 1$ for all $i \in \{1, \ldots, n\}$, $L(x, x) = -n$.
  $m$ : the uniform probability measure on $\{0, 1\}^n$. Result : $\kappa \geq 4$.
  It provides the optimal constant in mLSI.
  Observing that
  \[
  \nu_t^{x,y}(Z) \xrightarrow[\gamma \to 0]{d(z, x)} t^{d(z, x)} (1 - t)^{d(z, y)} 1_{[x, y]}(Z),
  \]
  one partially recover a curvature result on $\{0, 1\}^n$ by Gozlan-Roberto-S-Tetali (2014),

\[
\nu_t^{x,y}(Z) \xrightarrow[\gamma \to 0]{d(z, x)} t^{d(z, x)} (1 - t)^{d(z, y)} 1_{[x, y]}(Z),
\]
Examples of discrete space with curvature bounded from below [S. 2018]

- $\mathcal{X} = \mathbb{Z}$, for all $x \in \mathbb{Z}$, $L(x, x + 1) = L(x, x - 1) = 1, L(x, x) = -2$.
  $m$ : the counting measure. \textbf{Result : $\kappa \geq 0$.}

  Observing that
  \[
  \nu_t^x,y(Z) \xrightarrow[\gamma \to 0]{} \left( \frac{d(z, x)}{d(x, y)} \right) t d(z, x) (1 - t) d(z, y) 1_{[x, y]}(Z),
  \]
  one recovers Hillion’s result on $\mathbb{Z}$, and we get a new PLI on $\mathbb{Z}$.

- $\mathcal{X} = \{0, 1\}^n$, the discrete cube, for all $x \in \{0, 1\}^n$
  \[L(x, \sigma_i(x)) = 1 \quad \text{for all } i \in \{1, \ldots, n\}, \quad L(x, x) = -n,\]
  $m$ : the uniform probability measure on $\{0, 1\}^n$. \textbf{Result : $\kappa \geq 4$.}

  It provides the optimal constant in mLSI.

  Observing that
  \[
  \nu_t^x,y(Z) \xrightarrow[\gamma \to 0]{} t d(z, x) (1 - t) d(z, y) 1_{[x, y]}(Z),
  \]
  one partially recover a curvature result on $\{0, 1\}^n$ by Gozlan-Roberto-S-Tetali (2014), \textbf{without curvature term.}
Examples of discrete space with curvature bounded from below [S. 2018]

- $\mathcal{X} = \mathbb{Z}$, for all $x \in \mathbb{Z}$, $L(x, x + 1) = L(x, x - 1) = 1$, $L(x, x) = -2$.
  
  $m$ : the counting measure. **Result : $\kappa \geq 0$.**

  Observing that

  $$
  \nu_{t}^{x,y}(Z) \xrightarrow{\gamma \to 0} \frac{d(z,x)}{d(x,y)} t^{d(z,x)} (1 - t)^{d(z,y)} 1_{[x,y]}(Z),
  $$

  one recovers Hillion’s result on $\mathbb{Z}$, and we get a new PLI on $\mathbb{Z}$.

- $\mathcal{X} = \{0, 1\}^n$, the discrete cube, for all $x \in \{0, 1\}^n$
  
  $L(x, \sigma_i(x)) = 1$ for all $i \in \{1, \ldots, n\}$, $L(x, x) = -n$.

  $m$ : the uniform probability measure on $\{0, 1\}^n$. **Result : $\kappa \geq 4$.**

  It provides the optimal constant in mLSI.

  Observing that

  $$
  \nu_{t}^{x,y}(Z) \xrightarrow{\gamma \to 0} t^{d(z,x)} (1 - t)^{d(z,y)} 1_{[x,y]}(Z),
  $$

  one partially recover a curvature result on $\{0, 1\}^n$ by Gozlan-Roberto-S-Tetali (2014), without curvature term.

- $\mathcal{X} = S_n$, the symmetric group,
Examples of discrete space with curvature bounded from below [S. 2018]

- $\mathcal{X} = \mathbb{Z}$, for all $x \in \mathbb{Z}$, $L(x, x + 1) = L(x, x - 1) = 1$, $L(x, x) = -2$.
  $m$ : the counting measure. Result: $\kappa \geq 0$.
  Observing that
  \[ \nu_t^{x,y}(Z) \xrightarrow{\gamma \to 0} \left(\frac{d(z, x)}{d(x, y)}\right)^t d(z, y) \mathbb{1}_{[x, y]}(Z), \]
  one recovers Hillion’s result on $\mathbb{Z}$, and we get a new PLI on $\mathbb{Z}$.

- $\mathcal{X} = \{0, 1\}^n$, the discrete cube, for all $x \in \{0, 1\}^n$
  $L(x, \sigma_i(x)) = 1$ for all $i \in \{1, \ldots, n\}$, $L(x, x) = -n$.
  $m$ : the uniform probability measure on $\{0, 1\}^n$. Result: $\kappa \geq 4$.
  It provides the optimal constant in mLSI.
  Observing that
  \[ \nu_t^{x,y}(Z) \xrightarrow{\gamma \to 0} t d(z, x) (1 - t) d(z, y) \mathbb{1}_{[x, y]}(Z), \]
  one partially recover a curvature result on $\{0, 1\}^n$ by Gozlan-Roberto-S-Tetali (2014), without curvature term.

- $\mathcal{X} = S_n$, the symmetric group, with for all $x \in S_n$
  $L(x, \tau_{i,j}x) = 1$ for all transposition $\tau_{i,j}$, $L(x, x) = -\frac{n(n - 1)}{2}$,
Examples of discrete space with curvature bounded from below [S. 2018]

- $X = \mathbb{Z}$, for all $x \in \mathbb{Z}$, $L(x, x + 1) = L(x, x - 1) = 1$, $L(x, x) = -2$.
  
  $m$ : the counting measure. **Result :** $\kappa \geq 0$.

Observing that

$$\nu_t^{x,y}(z) \xrightarrow[\gamma \to 0]{} \binom{d(z, x)}{d(x, y)} t^{d(z, x)} (1 - t)^{d(z, y)} 1_{[x,y]}(z),$$

one recovers Hillion’s result on $\mathbb{Z}$, and we get a new PLI on $\mathbb{Z}$.

- $X = \{0, 1\}^n$, the discrete cube, for all $x \in \{0, 1\}^n$

  $L(\sigma_i(x)) = 1$ for all $i \in \{1, \ldots, n\}$, $L(x, x) = -n$.

  $m$ : the uniform probability measure on $\{0, 1\}^n$. **Result :** $\kappa \geq 4$.

It provides the optimal constant in mLSSI.

Observing that

$$\nu_t^{x,y}(z) \xrightarrow[\gamma \to 0]{} t^{d(z, x)} (1 - t)^{d(z, y)} 1_{[x,y]}(z),$$

one partially recover a curvature result on $\{0, 1\}^n$ by Gozlan-Roberto-S-Tetali (2014), without curvature term.

- $X = S_n$, the symmetric group, with for all $x \in S_n$

  $L(\tau_i, jx) = 1$ for all transposition $\tau_i,j$, $L(x, x) = -\frac{n(n-1)}{2}$,

  $m = \mu_0$ : the uniform distribution on $S_n$. 

result : $\kappa \geq 4$.
Examples of discrete space with curvature bounded from below [S. 2018]

- $\mathcal{X} = \mathbb{Z}$, for all $x \in \mathbb{Z}$, $L(x, x + 1) = L(x, x - 1) = 1$, $L(x, x) = -2$.
  
  $m$ : the counting measure.  
  
  **Result :** $\kappa \geq 0$.

  Observing that

  $$\nu_t^{x,y}(Z) \xrightarrow{\gamma \to 0} \left( \frac{d(z, x)}{d(x, y)} \right) t^{d(z, x)} (1 - t)^{d(z, y)} 1_{[x, y]}(Z),$$

  one recovers Hillion’s result on $\mathbb{Z}$, and we get a new PLI on $\mathbb{Z}$.

- $\mathcal{X} = \{0, 1\}^n$, the discrete cube, for all $x \in \{0, 1\}^n$

  $$L(x, \sigma_i(x)) = 1 \quad \text{for all } i \in \{1, \ldots, n\}, \quad L(x, x) = -n,$$

  $m$ : the uniform probability measure on $\{0, 1\}^n$.  
  
  **Result :** $\kappa \geq 4$.

  It provides the optimal constant in mLSI.

  Observing that

  $$\nu_t^{x,y}(Z) \xrightarrow{\gamma \to 0} t^{d(z, x)} (1 - t)^{d(z, y)} 1_{[x, y]}(Z),$$

  one partially recover a curvature result on $\{0, 1\}^n$ by Gozlan-Roberto-S-Tetali (2014), without curvature term.

- $\mathcal{X} = S_n$, the symmetric group, with for all $x \in S_n$

  $$L(x, \tau_{i,j} x) = 1 \quad \text{for all transposition } \tau_{i,j}, \quad L(x, x) = -\frac{n(n - 1)}{2},$$

  $m = \mu_0$ : the uniform distribution on $S_n$.  
  
  **Result :** $\kappa \geq 4$.  

- $\mathcal{X} = \mathbb{R}$, for all $x \in \mathbb{R}$, $L(x, x + 1) = 1$, $L(x, x) = -2$.

  $m$ : the counting measure.  
  
  **Result :** $\kappa \geq 0$.

  Observing that

  $$\nu_t^{x,y}(Z) \xrightarrow{\gamma \to 0} \left( \frac{d(z, x)}{d(x, y)} \right) t^{d(z, x)} (1 - t)^{d(z, y)} 1_{[x, y]}(Z),$$

  one recovers Hillion’s result on $\mathbb{R}$, and we get a new PLI on $\mathbb{R}$.

- $\mathcal{X} = \{0, 1\}^n$, the discrete cube, for all $x \in \{0, 1\}^n$

  $$L(x, \sigma_i(x)) = 1 \quad \text{for all } i \in \{1, \ldots, n\}, \quad L(x, x) = -n,$$

  $m$ : the uniform probability measure on $\{0, 1\}^n$.  
  
  **Result :** $\kappa \geq 4$.

  It provides the optimal constant in mLSI.

  Observing that

  $$\nu_t^{x,y}(Z) \xrightarrow{\gamma \to 0} t^{d(z, x)} (1 - t)^{d(z, y)} 1_{[x, y]}(Z),$$

  one partially recover a curvature result on $\{0, 1\}^n$ by Gozlan-Roberto-S-Tetali (2014), without curvature term.

- $\mathcal{X} = S_n$, the symmetric group, with for all $x \in S_n$

  $$L(x, \tau_{i,j} x) = 1 \quad \text{for all transposition } \tau_{i,j}, \quad L(x, x) = -\frac{n(n - 1)}{2},$$

  $m = \mu_0$ : the uniform distribution on $S_n$.  
  
  **Result :** $\kappa \geq 4$.  

- $\mathcal{X} = \mathbb{R}$, for all $x \in \mathbb{R}$, $L(x, x + 1) = 1$, $L(x, x) = -2$.

  $m$ : the counting measure.  
  
  **Result :** $\kappa \geq 0$.

  Observing that

  $$\nu_t^{x,y}(Z) \xrightarrow{\gamma \to 0} \left( \frac{d(z, x)}{d(x, y)} \right) t^{d(z, x)} (1 - t)^{d(z, y)} 1_{[x, y]}(Z),$$

  one recovers Hillion’s result on $\mathbb{R}$, and we get a new PLI on $\mathbb{R}$.  

  **Example :** $\mathcal{X} = \mathbb{Z}$, for all $x \in \mathbb{Z}$, $L(x, x + 1) = L(x, x - 1) = 1$, $L(x, x) = -2$.

  $m$ : the counting measure.  
  
  **Result :** $\kappa \geq 0$.
Examples of discrete space with curvature bounded from below [S. 2018]

- $\mathcal{X} = \mathbb{Z}$, for all $x \in \mathbb{Z}$, $L(x, x + 1) = L(x, x - 1) = 1$, $L(x, x) = -2$.
  $m$ : the counting measure. **Result :** $\kappa \geq 0$.
  Observing that
  $$\nu_t^{x,y}(z) \xrightarrow[\gamma \to 0]{} \left( \frac{d(z, x)}{d(x, y)} \right) t^{d(z, x)} (1 - t)^{d(z, y)} \mathbb{1}_{[x,y]}(z),$$
  one recovers Hillion’s result on $\mathbb{Z}$, and we get a new PLI on $\mathbb{Z}$.

- $\mathcal{X} = \{0, 1\}^n$, the discrete cube, for all $x \in \{0, 1\}^n$
  $L(x, \sigma_i(x)) = 1$ for all $i \in \{1, \ldots, n\}$, $L(x, x) = -n$.
  $m$ : the uniform probability measure on $\{0, 1\}^n$. **Result :** $\kappa \geq 4$.
  It provides the optimal constant in mLPSI.
  Observing that
  $$\nu_t^{x,y}(z) \xrightarrow[\gamma \to 0]{} t^{d(z, x)} (1 - t)^{d(z, y)} \mathbb{1}_{[x,y]}(z),$$
  one partially recover a curvature result on $\{0, 1\}^n$ by Gozlan-Roberto-S-Tetali (2014), without curvature term.

- $\mathcal{X} = S_n$, the symmetric group, with for all $x \in S_n$
  $L(x, \tau_{i,j}x) = 1$ for all transposition $\tau_{i,j}$, $L(x, x) = -\frac{n(n-1)}{2}$.
  $m = \mu_0$ : the uniform distribution on $S_n$. **Result :** $\kappa \geq 4$.
  Same order of curvature as in Maas-Erbar-Tetali (2015).
Examples of discrete space with curvature bounded from below [S. 2018]

- $\mathcal{X} = \mathbb{Z}$, for all $x \in \mathbb{Z}$, $L(x, x + 1) = L(x, x - 1) = 1$, $L(x, x) = -2$.
  $m$: the counting measure. \textbf{Result:} $\kappa \geq 0$.

  Observing that
  \[
  \nu^x,y_t(Z) \xrightarrow[\gamma \to 0]{} \left(\frac{d(z, x)}{d(x, y)}\right) t^d(z, x) (1 - t)^d(z, y) \mathbb{1}_{[x, y]}(Z),
  \]
  one recovers Hillion’s result on $\mathbb{Z}$, and we get a new PLI on $\mathbb{Z}$.

- $\mathcal{X} = \{0, 1\}^n$, the discrete cube, for all $x \in \{0, 1\}^n$
  \[
  L(x, \sigma_i(x)) = 1 \quad \text{for all } i \in \{1, \ldots, n\}, \quad L(x, x) = -n,
  \]
  $m$: the uniform probability measure on $\{0, 1\}^n$. \textbf{Result:} $\kappa \geq 4$.

  It provides the optimal constant in mLSI.

  Observing that
  \[
  \nu^x,y_t(Z) \xrightarrow[\gamma \to 0]{} t^d(z, x) (1 - t)^d(z, y) \mathbb{1}_{[x, y]}(Z),
  \]
  one partially recover a curvature result on $\{0, 1\}^n$ by Gozlan-Roberto-S-Tetali (2014), without curvature term.

- $\mathcal{X} = S_n$, the symmetric group, with for all $x \in S_n$
  \[
  L(x, \tau_{i,j}x) = 1 \quad \text{for all transposition } \tau_{i,j}, \quad L(x, x) = -\frac{n(n - 1)}{2},
  \]
  $m = \mu_0$: the uniform distribution on $S_n$. \textbf{Result:} $\kappa \geq 4$.

  Same order of curvature as in Maas-Erbar-Tetali (2015).

  $\alpha \geq \kappa$, however $\alpha \geq 4$ is not the good order in mLSI,
Examples of discrete space with curvature bounded from below [S. 2018]

- $\mathcal{X} = \mathbb{Z}$, for all $x \in \mathbb{Z}$, $L(x, x + 1) = L(x, x - 1) = 1$, $L(x, x) = -2$. $m$ : the counting measure. Result : $\kappa \geq 0$.

Observing that

$$
\nu_t^{x,y}(Z) \xrightarrow[\gamma \to 0]{} \binom{d(z, x)}{d(x, y)} t^{d(z, x)} (1 - t)^{d(z, y)} 1_{[x, y]}(Z),
$$

one recovers Hillion’s result on $\mathbb{Z}$, and we get a new P LI on $\mathbb{Z}$.

- $\mathcal{X} = \{0, 1\}^n$, the discrete cube, for all $x \in \{0, 1\}^n$

$L(x, \sigma_i(x)) = 1$ for all $i \in \{1, \ldots, n\}$, $L(x, x) = -n$. $m$ : the uniform probability measure on $\{0, 1\}^n$. Result : $\kappa \geq 4$.

It provides the optimal constant in mLSI.

Observing that

$$
\nu_t^{x,y}(Z) \xrightarrow[\gamma \to 0]{} t^{d(z, x)} (1 - t)^{d(z, y)} 1_{[x, y]}(Z),
$$

one partially recover a curvature result on $\{0, 1\}^n$ by Gozlan-Roberto-S-Tetali (2014), without curvature term.

- $\mathcal{X} = S_n$, the symmetric group, with for all $x \in S_n$

$L(x, \tau_{i,j} x) = 1$ for all transposition $\tau_{i,j}$, $L(x, x) = -\frac{n(n - 1)}{2}$.

$m = \mu_o$ : the uniform distribution on $S_n$. Result : $\kappa \geq 4$.

Same order of curvature as in Maas-Erbar-Tetali (2015).

$\alpha \geq \kappa$, however $\alpha \geq 4$ is not the good order in mLSI, According to Gao-Quastel (2003) Bobkov-Tetali (2006), $\alpha \geq Cn$. 

- $\mathcal{X} = \mathbb{R}$, for all $x \in \mathbb{R}$, $L(x, x + 1) = L(x, x - 1) = 1$, $L(x, x) = -2$. $m$ : the Lebesgue measure. Result : $\kappa \geq 0$.

Observing that

$$
\nu_t^{x,y}(Z) \xrightarrow[\gamma \to 0]{} \binom{d(z, x)}{d(x, y)} t^{d(z, x)} (1 - t)^{d(z, y)} 1_{[x, y]}(Z),
$$

one recovers Hillion’s result on $\mathbb{R}$, and we get a new P LI on $\mathbb{R}$.

Examples of weak cost

Marton’s type of cost

Barycentric cost

Martingale costs

Weak transport inequalities

Dual characterization to concentration

Universal transport inequalities

Barycentric transport inequalities examples

characterisation on $\mathbb{R}$

Transport inequality on the symmetric group

introduction

Ewens distribution

deviation inequalities

The Schrödinger minimization problem

definition

curvature in discrete spaces

Examples in discrete

Weak transport costs.
Thank you.