An elementary proof of some functional inequalities on paths space for Lévy processes by a cylindrical method

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Abstract

This note gives an very short proof of the Poisson log-Sobolev inequality, by a mixture of what can be found in [Wu00] and [AL00].

Let $(X_t)_{t\geqslant 0}$ be the Markov process on $\mathbb{R}^d$ with infinitesimal generator $\mathbf{L}$ of the form

$$\forall f \in \mathcal{C}_c^\infty(\mathbb{R}^d,\mathbb{R}),\ (\mathbf{L}f)(x) := b\cdot\nabla f + \lambda \int_{\mathbb{R}^d} \left[\mathrm{D}_y f(x) - \frac{y}{1+|y|^2}\nabla f\right]\nu(dy),$$

where $(b,\lambda)\in\mathbb{R}^d\times\mathbb{R}_+^*$, $\mathrm{D}_y g(x) := f(x+y)-f(x)$ and ν is a Lévy measure, i.e. a positive measure such that:

$$\int_{\mathbb{R}^d} (|y|^2\wedge 1)\,\nu(dy) < +\infty.$$

The process $(X_t)_{t\geqslant 0}$ is nothing else but a Lévy process without Gaussian part. When ν is a probability measure, and $b = -\lambda\int_{\mathbb{R}^d} y(1+|y|^2)^{-1}\,\nu(dy)$, $(X_t)_{t\geqslant 0}$ is a compound Poisson process of intensity λ and jump law ν. In particular, the case $\nu=\delta_1$ corresponds to the simple Poisson process. Let us give a *short and elementary* proof of the following inequality: $\forall f\in\mathcal{C}_b(\mathbb{R}^d,\mathbb{R}_+)$,

$$\mathbf{Ent}(f(X_t)) := \mathbf{E}(\Phi(f(X_t))) - \Phi(\mathbf{E}(f(X_t))) \leqslant \lambda t\,\mathbf{E}\left(\int_{\mathbb{R}^d}\Theta_y(f)(X_t)\,\nu(dy)\right),$$

where $\Phi(u) := u\log u$ for $u>0$ and $\Phi(0):=0$ and where

$$\Theta_y(f) := \min\left(\frac{(\mathrm{D}_y f)^2}{f}\,,\,\mathrm{D}_y f\,\mathrm{D}_y\log f\right).$$

In particular, this gives a modified logarithmic Sobolev inequality for infinitly divisible laws (take $t=1$). The method used is a mix of what can be found in [Wu00] and [AL00], see also [Pri00], [Ané01] and [CHL97]. In the sequel, we denote by $(\mathbf{P}_t)_{t\geqslant 0}$ the associated Markov semi-group acting on $\mathcal{C}_b(\mathbb{R}^d,\mathbb{R})$ and defined for any $x\in\mathbb{R}^d$ and $f\in\mathcal{C}_b(\mathbb{R}^d,\mathbb{R})$ by

$$\mathbf{P}_t(f)\,(x) := \mathbf{E}(f(X_t)\,|\,X_0=x) = \mathbb{E}_{\mathcal{L}(X_t\,|\,X_0=x)}(f).$$

Namely, one have $\mathbf{P}_t(\mathbf{P}_s(f)) = \mathbf{P}_{s+t}(f)$ for any $t, s \in \mathbb{R}_+$, and $\partial_t \mathbf{P}_t(f) = \mathbf{L}\mathbf{P}_t(f) = \mathbf{P}_t(\mathbf{L}f)$. The key property is that $\mathbf{L}$ in translation invariant, and hence D_y and $\mathbf{P}_t$ commute. We have for any $t > 0$ and any $f \in \mathcal{C}_c^\infty(\mathbb{R}^d, \mathbb{R}^+)$:

$$\mathbf{E}(\Phi(f(X_t))) - \Phi(\mathbf{E}(f(X_t))) = \mathbf{P}_t(\Phi(f)) - \Phi(\mathbf{P}_t(f)) = \alpha(t) - \alpha(0) = \int_0^t \alpha'(s)\,ds,$$

where $\alpha(s) := \mathbf{P}_s(\Phi(\mathbf{P}_{t-s}(f)))$. But now $\alpha'(s) = \mathbf{P}_s(\mathbf{L}(\Phi(g)) - \Phi'(g)\mathbf{L}g)$ where $g := \mathbf{P}_{t-s}(f)$. At this stage, we notice that by definition of $\mathbf{L}$:

$$\mathbf{L}(\Phi(g)) - \Phi'(g)\mathbf{L}g = \lambda \int_{\mathbb{R}^d} [\mathrm{D}_y\Phi(g) - \Phi'(g)\mathrm{D}_y g]\,\nu(dy),$$

But $\mathrm{D}_y\Phi(g) - \Phi'(g)\mathrm{D}_y g = \Psi(g, \mathrm{D}_y g)$, where

$$\begin{aligned}\Psi(u,v) :&= \Phi(u+v) - \Phi(u) - \Phi'(u)v \\ &= (u+v)\log(u+v) - u\log u - (1+\log u)v,\end{aligned}$$

for any $(u,v) \in \mathbb{R}^2$ with $u > 0$ and $u + v > 0$. Hence, by the Fubini Theorem,

$$\alpha'(s) = \lambda \int_{\mathbb{R}^d} \mathbf{P}_s(\Psi(g, \mathrm{D}_y g))\ \nu(dy).$$

Now, since $g = \mathbf{P}_{t-s}(f)$ and since the process have independent increments, we have $\mathrm{D}_y g = \mathrm{D}_y\mathbf{P}_{t-s}(f) = \mathbf{P}_{t-s}(\mathrm{D}_y f)$. Then, by the Jensen inequality for the bivariate convex function Ψ and the probability measure $\mathbf{P}_{t-s}(\cdot)(x) = \mathcal{L}(X_{t-s} \,|\, X_0 = x)$:

$$\Psi(g, \mathrm{D}_y g) = \Psi(\mathbf{P}_{t-s}(f)\,, \mathbf{P}_{t-s}(\mathrm{D}_y f)) \leqslant \mathbf{P}_{t-s}(\Psi(f, \mathrm{D}_y f))\,,$$

Hence, we have:

$$\alpha'(s) \leqslant \lambda \int_{\mathbb{R}^d} \mathbf{P}_s(\mathbf{P}_{t-s}(\Psi(f, \mathrm{D}_y f)))\ \nu(dy) = \lambda \int_{\mathbb{R}^d} \mathbf{P}_t(\Psi(f, \mathrm{D}_y f))\ \nu(dy).$$

Therefore, again by the Fubini Theorem:

$$\alpha'(s) \leqslant \lambda\,\mathbf{P}_t\biggl(\int_{\mathbb{R}^d} \Psi(f, \mathrm{D}_y f)\,\nu(dy)\biggr).$$

This yields finally to:

$$\mathbf{E}(\Phi(f(X_t))) - \Phi(\mathbf{E}(f(X_t))) \leqslant \lambda t\,\mathbf{E}\biggl(\int_{\mathbb{R}^d} \Psi(f(X_t), \mathrm{D}_y f(X_t))\,\nu(dy)\biggr).$$

This inequality gives the desired two bounds in terms of $(\mathrm{D}_y f)^2/f$ and $\mathrm{D}_y f\,\mathrm{D}_y \log f$ since

$$\Psi(u,v) \leqslant \frac{v^2}{u} \quad \text{and} \quad \Psi(u,v) \leqslant v(\log(u+v) - \log u).$$

The same argument gives the Poincaré inequality in few lines: $\forall f \in \mathcal{C}_c^\infty(\mathbb{R}^d,\mathbb{R})$,

$$\begin{aligned}
\mathbf{Var}(f(X_t)) &= \mathbf{P}_t\big(f^2\big) - \mathbf{P}_t(f)^2 \\
&= \int_0^t \partial_s \mathbf{P}_s\big(\mathbf{P}_{t-s}(f)^2\big)\, ds \\
&= \int_0^t \mathbf{P}_s\big(\mathbf{L}(\mathbf{P}_{t-s}(f)^2) - 2\mathbf{P}_{t-s}(f)\,\mathbf{L}\mathbf{P}_{t-s}(f)\big)\, ds \\
&= \int_0^t \lambda \int_{\mathbb{R}^d} \mathbf{P}_s\big((\mathrm{D}_y \mathbf{P}_{t-s}(f))^2\big)\, \nu(dy)\, ds \\
&= \int_0^t \lambda \int_{\mathbb{R}^d} \mathbf{P}_s\big(\mathbf{P}_{t-s}(\mathrm{D}_y f)^2\big)\, \nu(dy)\, ds \\
&\leqslant \int_0^t \lambda \int_{\mathbb{R}^d} \mathbf{P}_s\big(\mathbf{P}_{t-s}\big((\mathrm{D}_y f)^2\big)\big)\, \nu(dy)\, ds \\
&= \lambda t\, \mathbf{P}_t\left(\int_{\mathbb{R}^d} (\mathrm{D}_y f)^2\, \nu(dy)\right) \\
&= \lambda t\, \mathbf{E}\left(\int_{\mathbb{R}^d} (\mathrm{D}_y f(X_t))^2\, \nu(dy)\right).
\end{aligned}$$

It can be also recovered from the modified logarithmic Sobolev inequality applied to $1 + \varepsilon f$ by letting ε tends to 0^+. Notice that the natural "carré du champ" is given by $\mathbf{\Gamma} f := \frac{1}{2}(\mathbf{L}(f^2) - 2f\mathbf{L}f) = \frac{\lambda}{2}\int_{\mathbb{R}^d}(\mathrm{D}_y f)^2\,\nu(dy)$. Finally, the modified logarithmic Sobolev and Poincaré inequalities derived above may be simply extended to cylindrical functions of the process by using the tensoriation property and the independence of the increments. Namely, for any smooth function F of $(X_{t_1},\dots,X_{t_n})$ where $0 = t_0 < t_1 < \cdots < t_n$:

$$\mathbf{E}(\Phi(F)) - \Phi(\mathbf{E}(F)) \leqslant \lambda \sum_{i=1}^n (t_i - t_{i-1})\,\mathbf{E}\left(\int_{\mathbb{R}^d} \Psi(F, \mathrm{D}_y^{i\cdots n} F)\,\nu(dy)\right),$$

where

$$\mathrm{D}^{i\cdots n} F(x) := (F \circ \tau_i(y))(x) - F(x),$$

and where

$$(F \circ \tau_i(y))(x) := F(x_1, \dots, x_{i-1}, x_i + y, \dots, x_n + y).$$

This gives an inequality on the paths space by letting n tends to $+\infty$ as soon as a suitable Malliavin derivative makes sense (this holds for example for the simple Poisson process). Namely, if F is a smooth function of $(X_t)_{0\leqslant t\leqslant T}$:

$$\mathbf{E}(\Phi(F)) - \Phi(\mathbf{E}(F)) \leqslant \lambda\,\mathbf{E}\left(\int_0^T\int_{\mathbb{R}^d} \Psi(F, \mathrm{D}_y^t F)\,\nu(dy)\,dt\right),$$

where

$$\mathrm{D}_y^t F((x_s)_{0\leqslant s\leqslant T}) := F((x_s + y\,\mathrm{I}_{[t,T]}(s))_{0\leqslant s\leqslant T}) - F((x_s)_{0\leqslant s\leqslant T}).$$

Moreover, a tensorisation with inequalities of the same type satisfied by the Brownian motion yields to similar inequalities at time t and on the paths space for Lévy processes.

1 Reversed inequalities

$$\begin{aligned}\mathbf{P}_t\big(f^2\big) - \mathbf{P}_t(f)^2 &= \int_0^t \lambda \int_{\mathbb{R}^d} \mathbf{P}_s\big(\mathbf{P}_{t-s}(\mathrm{D}_y f)^2\big)\ \nu(dy)\, ds \\ &\geqslant \int_0^t \lambda \int_{\mathbb{R}^d} \mathbf{P}_s(\mathbf{P}_{t-s}(\mathrm{D}_y f))^2\ \nu(dy)\, ds \\ &= \lambda t \int_{\mathbb{R}^d} \mathbf{P}_t(\mathrm{D}_y f)^2\ \nu(dy).\end{aligned}$$

$$\mathbf{P}_t(\Phi(f)) - \Phi(\mathbf{P}_t(f)) \geqslant \lambda t \int_{\mathbb{R}^d} \Psi(\mathbf{P}_t(f)\,, \mathbf{P}_t(\mathrm{D}_y f))\, \nu(dy).$$

2 An F.K.G. inequality

The F.K.G. inequality. Let $f, g : \mathbb{R}^d \to \mathbb{R}$ such that $\mathrm{D}_y f \geqslant 0$ and $\mathrm{D}_y g \geqslant 0$ for all y in $\mathrm{supp}(\nu)$, then

$$\mathbf{Cov}(f(X_t), g(X_t)) \geqslant 0.$$

When $-\mathrm{supp}(\nu) \subset \mathrm{supp}(\nu)$, the hypothesis is satisfied only by constant functions and the result is void, whereas for $\mathrm{supp}(\nu) \subset (\mathbb{R}_+)^d$, the hypothesis is satisfied by non decreasing functions.

$$\mathbf{Cov}(f(X_t), g(X_t)) = \mathbf{P}_t(fg) - \mathbf{P}_t(f)\,\mathbf{P}_t(g) = \int_0^t \alpha'(s)\, ds,$$

where $\alpha(s) := \mathbf{P}_s(\mathbf{P}_{t-s}(f)\,\mathbf{P}_{t-s}(g))$. Let $F := \mathbf{P}_{t-s}(f)$ and $G := \mathbf{P}_{t-s}(g)$, then

$$\alpha'(s) = \mathbf{P}_s(\mathbf{L}(FG) - (\mathbf{L}F)G - F(\mathbf{L}G))\,,$$

and by definition of $\mathbf{L}$

$$\mathbf{L}(FG) - (\mathbf{L}F)G - F(\mathbf{L}G) = \int_{\mathbb{R}^d} (\mathrm{D}_y F)(\mathrm{D}_y G)\, \nu(dy).$$

But now $F = \mathrm{D}_y \mathbf{P}_{t-s}(f) = \mathbf{P}_{t-s}(\mathrm{D}_y f)$ and $G = \mathrm{D}_y \mathbf{P}_{t-s}(g) = \mathbf{P}_{t-s}(\mathrm{D}_y g)$ and hence

$$\mathbf{P}_t(fg) - \mathbf{P}_t(f)\,\mathbf{P}_t(g) = \int_0^t\!\!\int_{\mathbb{R}^d} \mathbf{P}_s(\mathbf{P}_{t-s}(\mathrm{D}_y f)\,\mathbf{P}_{t-s}(\mathrm{D}_y g))\ \nu(dy)\, ds.$$

This expression is non negative since $\mathbf{P}_{t-s}(\mathrm{D}_y f) \geqslant 0$ and $\mathbf{P}_{t-s}(\mathrm{D}_y g) \geqslant 0$ for y in $\mathrm{supp}(\nu)$.

3 Martingale

Let us explain how the following formula can be viewed as a result of a martingale representation.

$$\mathbf{P}_t(\Phi(f)) - \Phi(\mathbf{P}_t(f)) = \int_0^t \mathbf{P}_s(-\Phi'(g)\mathbf{L}g + \mathbf{L}\Phi(g))\ ds.$$

For any smooth $h : [0,t] \times \mathbb{R}^d \to \mathbb{R}$, the Itô formula gives that the process $(M_u)_{0 \leqslant u \leqslant t}$ defined by

$$M_u := h(u, X_u) - h(0, X_0) - \int_0^u [\partial_s h(s, X_s) + \mathbf{L}h(s, X_s)]\, ds.$$

is a martingale. At this step, we get by taking $h(s,x) = \Phi(\mathbf{P}_{t-s}(f))$ and $g = \mathbf{P}_{t-s}(f)$:

$$M_t = \Phi(f)(X_t) - \Phi(\mathbf{P}_t(f)) - \int_0^t [-\Phi'(g)\mathbf{L}g + \mathbf{L}\Phi(g)](X_s)\, ds.$$

The desired result follows since

$$0 = \mathbb{E}(M_0) = \mathbb{E}(M_t) = \mathbf{P}_t(\Phi(f)) - \Phi(\mathbf{P}_t(f)) - \int_0^t \mathbf{P}_s(-\Phi'(g)\mathbf{L}g + \mathbf{L}\Phi(g))\, ds.$$

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