REMARKS ON THE LOG-SOBOLEV INEQUALITY FOR THE CONTINUOUS CUBE

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ABSTRACT. Nothing else but a short discussion about some logarithmic Sobolev inequality on the unit continuous cube for various choices of gradients.

CONTENTS

Let μ be a probability measure on a domain $\Omega \in \mathbb{R}^d$, $\mathcal{F} : \mathbb{R} \to \mathbb{R}$ be a smooth bounded function and $A \in \mathcal{M}_d(\mathbb{R})$ be a constant symmetric real matrix of size $d \times d$. We are interested in the best (greater) positive constant $\tau := \tau(\mu, d, \mathcal{F}, A)$ such that for any $f \in \mathcal{C}^\infty(\Omega, \mathbb{R})$,

$$\tau \left\{ \mathbf{E}_\mu(\mathcal{F}(f)) - \mathcal{F}(\mathbf{E}_\mu(f)) \right\} \leqslant \mathbf{E}_\mu\left(\nabla f^\top A \, \nabla f\right). \tag{1}$$

The cases $(\mathcal{F}(t), A) := (t^2, \mathrm{I}_d)$ and $(\mathcal{F}(t), A) := (2t^2 \log t^2, \mathrm{I}_d)$ correspond to Poincaré (or spectral gap) and to log-Sobolev inequalities respectively, like in (2) and (3) below.

1. LEBESGUE'S MEASURE ON THE CONTINUOUS UNIT CUBE

In this section, we denote by σ the Lebesgue measure on the continuous cube $[0,1]^d \in \mathbb{R}^d$ of dimension d. Hence, σ is the uniform law on $[0,1]^d$.

1.1. Spectral gap or Poincaré inequality.

We are interested in best (greater) non negative constant λ such that for any $f \in \mathcal{C}^\infty\big([0,1]^d, \mathbb{R}\big)$,

$$\lambda \, \mathbf{Var}_\sigma(f) \leqslant \mathbf{E}_\sigma\left(|\nabla f|^2\right). \tag{2}$$

Constant λ is nothing else but the spectral gap of the Laplacian $\boldsymbol{\Delta}$ on $[0,1]^d$ with Neumann boundary conditions (i.e. with domain such that the derivative vanishes at the boundary). The associated Markov process is the reflected Brownian motion. Inequality (2) is a purely spectral property and it is not hard to show that $\lambda = \pi^2$ (take $f(x) = \cos(\pi x_i)$ for the upper bound, and use for example Fourier transform for the equality). The dimension does not play any role since the desired inequality in dimension d can be obtained from the one dimensional one, with the same constant, by tensorisation (product stability property).

Notice that if we consider the best non negative constant such that inequality (2) remains valid for any f in $\mathcal{C}_c^\infty\big([0,1]^d, \mathbb{R}\big)$, we are dealing with the spectral gap of the Laplacian $\boldsymbol{\Delta}$ on $[0,1]^d$ with Dirichlet boundary conditions (i.e. with domain consisting in functions that vanishe at the boundary). The associated Markov process is the killed Brownian motion at the boundary and the best constant remains equal to π^2 (associated with the eigenfunctions $\sin(\pi x_i)$).

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1.2. **Logarithmic Sobolev inequality.** We are interested in best (greater) non negative constant ρ such that for any $f \in \mathcal{C}^\infty\big([0,1]^d, \mathbb{R}\big)$,

$$2\rho\,\mathbf{Ent}_\sigma\big(f^2\big) \leqslant \mathbf{E}_\sigma\big(|\nabla f|^2\big)\,. \tag{3}$$

Here again, on can consider $f \in \mathcal{C}_c^\infty\big([0,1]^d, \mathbb{R}\big)$, in which case the best constant is different (a priori greater or equal).

Inequality (3) is not associated, as I know, to a spectral property. But one can easily show that $\rho \leqslant \lambda$ (just take $f = 1 + \varepsilon g$ in (3) and let ε tends to 0, which gives (2) for f with constant ρ). Actually, it is not difficult to guess that (3) holds with a non trivial ρ (i.e. > 0). More hard is to find the optimal ρ.

Notice that we cannot use the $\mathbf{\Gamma_2}$ criterion since we are dealing with $[0,1]^d$, which is a manifold with boundary. Here again, the dimension does not plays any role since the desired inequality in dimension d can be obtained from the one dimensional one, with the same constant, by tensorisation.

1.2.1. *The Gaussian contraction method.* It is well known that if γ denotes the standard one dimensional gaussian measure, we have for any f in $\mathcal{C}_b^\infty(\mathbb{R}, \mathbb{R})$

$$\frac{1}{2}\,\mathbf{Ent}_\gamma\big(f^2\big) \leqslant \mathbf{E}_\gamma\big(f'^2\big)\,. \tag{4}$$

If F_γ denotes the density function of γ, then the image measure of γ by F_γ is simply the Lebesgue measure σ on $[0,1]$. Now, since $\big|F_\gamma'(x)\big| \leqslant 1/\sqrt{2\pi}$, we obtain by appliying (4) to $f = g(F_\gamma)$ where $g \in \mathcal{C}^\infty([0,1], \mathbb{R})$,

$$\pi\,\mathbf{Ent}_\sigma\big(g^2\big) \leqslant \mathbf{E}_\sigma\big(g'^2\big)\,.$$

Therefore, we have obtained that $\rho \geqslant \pi$. Finally, we have

$$\pi \leqslant \rho \leqslant \lambda = \pi^2.$$

1.2.2. *The torus identification method.* The Gaussian contraction method gives only a lower bound for ρ. If we imbed $\mathcal{C}^\infty([0,1], \mathbb{R})$ into $\mathcal{C}_c^\infty([0,2], \mathbb{R})$ by taking the symmetry with respect to the vertical line of abscissa 1, we can join the two boundary points 0 and 2 and identify the obtained space with the circle of perimeter 2 (i.e of radius $r = 1/\pi$). It is known that the logarithmic Sobolev inequality holds on the unit circle for the uniform probability with optimal constant 1 (see for example [ÉY87, Rot80, Wei80]). Thus, by translating back this inequality on the unit cube $([0,1], \sigma)$, we get

$$\rho = \lambda = \pi^2.$$

Notice that despite the fact that the Ricci curvature of the circle is zero, one can use the integrated form of the $\mathbf{\Gamma_2}$ criterion to obtain the optimal logarithmic Sobolev inequality on the circle.

Remark 1.1. There are also well known methods in Riemannian geometry which tell that for spectral gap (i.e. Poincaré inequality) and logarithmic Sobolev inequality “convex bodies bahave like manifolds with positive Ricci curvature” . But these methods gives only bounds, and are not relevant to obtain optimal constants. See [Led00].

Remark 1.2 (The generic cube). By a simple change of variable in (2) and (3), it is straightforward that the uniform probability measure on the unit cube $[0,M]^d$ of lenght $M > 0$ satisfies to Poincaré and log-Sobolev inequalities with constants λ/M^2 and ρ/M^2 respectively.

2. From the cube to the simplex

Recall that σ denotes the uniform probability measure on the unit cube $[0,1]^d$. For any M in $[0,d]$, we consider the conditionned probability measure

$$\sigma_M := \sigma(\,\cdot\,|\, x_1 + \cdots + x_d = M).$$

Let $\Sigma_{M,d}$ be the $(d-1)$-dimensional polyhedra obtained by intersecting the unit cube $[0,1]^d$ and the affine plane of equation $x_1 + \cdots + x_d = M$. Since σ is uniform on $[0,1]^d$, σ_M can be viewed as the probability measure with support $\Sigma_{M,d}$ on which it equals the uniform measure. Another way to realise σ_M is to define the probability measure ν_M on $\mathbb{R}^{d-1}$ by

$$d\nu_M(x) := Z_{\nu_M}^{-1}\, \mathrm{I}_{[0,1]}(x_1)\cdots \mathrm{I}_{[0,1]}(x_{d-1})\mathrm{I}_{[0,1]}(M - x_1 - \cdots - x_{d-1})\, dx_1 \cdots dx_{d-1}.$$

Now, for any measurable function $f : [0,1]^d \to \mathbb{R}$, we state

$$\mathbf{E}_{\sigma_M}(f) := \mathbf{E}_{\nu_M}(f(x_M)),$$

where $x_M := (x_1, \ldots, x_{d-1}, M - x_1 - \cdots - x_{d-1})$.

For $M \leqslant 1$, $\Sigma_{M,d}$ is a simplex with sides of length $\sqrt{2}M$, we get by an affine change of variable

$$\tau(\sigma_M, d, \mathcal{F}, A) = \frac{\tau(\sigma_1, d, \mathcal{F}, A)}{2M^2} \geqslant \frac{\tau(\sigma_1, d, \mathcal{F}, A)}{2}.$$

This is still true for $d-1 \leqslant M \leqslant d$ by replacing M with $d-M$ in the above formula. Notice that $\tau(\sigma_1, d, \mathcal{F}, A)$ does not depend on M and corresponds to a simplex with sides of lenght $\sqrt{2}$. For $1 < M < d-1$, $\Sigma_{d,M}$ is an hexagon in dimension $d = 3$. Similarly, for any $i \in \{1, \ldots, [d/2]\}$ and any $M \in [i, i+1]$, on can check by a simple change of variable that

$$\tau(\sigma_M, d, \mathcal{F}, A) \geqslant \frac{\max\left(\tau(\sigma_i, d, \mathcal{F}, A), \tau(\sigma_{i+1}, d, \mathcal{F}, A)\right)}{2}.$$

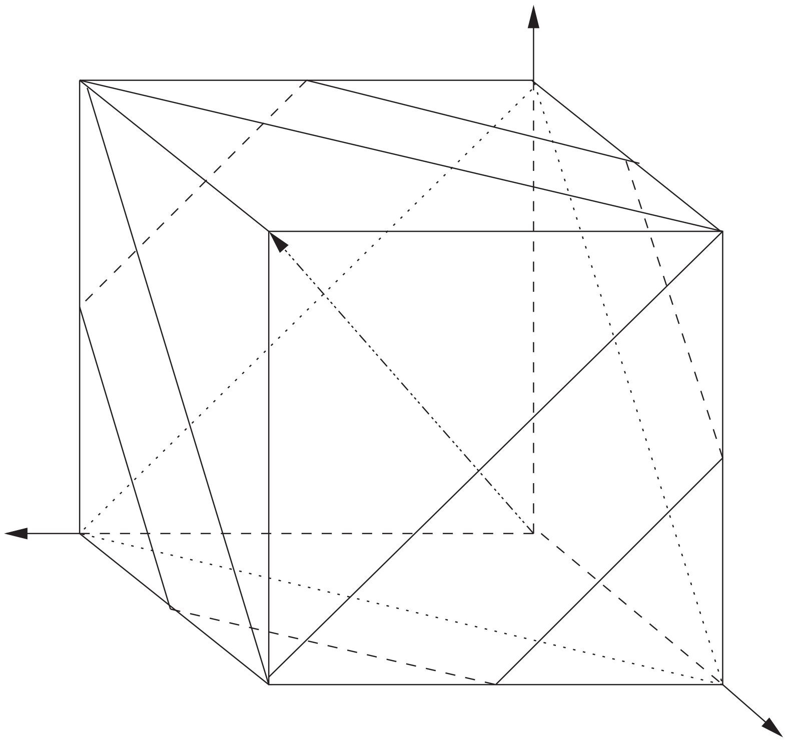

Figure 1. Intersections of the unit cube in dimension 3 with median planes $x + y + z = M$ when M equals 1, 3/2 and 2.

Questions & problems.

- What is the behavior of $\tau(\sigma_i, d, \mathcal{F}, A)$ for $i = 1, \dots, [d/2]$ when d goes to infinity?
- How to compare $\tau(\sigma_1, d, \mathcal{F}, A)$ and $\tau(\sigma, d, \mathcal{F}, A)$?
- Behavior of $\tau(\sigma_1, d, \mathcal{F}, A)$ for various choices of A (especially for $\mathbf{Id}$ and $\mathbf{Id} - \mathbf{J}$)? In statistical mechanics, $\Lambda := \{1, \dots, L\}^d \in \mathbb{Z}^d$, $\mathbb{R}^\Lambda \simeq \mathbb{R}^{L^d =: n}$, the Kawasaki gradient is given by
$$\frac{1}{2} \sum_{\{i \sim j\} \in \Lambda} |\partial_i \bullet - \partial_j \bullet|^2$$
and the associated matrix A is then $(\delta_{(i=j)} - \delta_{(i \sim j)})_{i,j \in \Lambda} =: \mathbf{Id}_{L^d} - \mathbf{J}_\Lambda$.

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