Few words around the circular law

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AMS Short Course
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Elementary matrix model

- Random variable X taking values in $\mathcal{M}_n(\mathbb{C})$

$$\begin{pmatrix} X_{11} & \cdots & X_{1n} \\ \vdots & \ddots & \vdots \\ X_{n1} & \cdots & X_{nn} \end{pmatrix}$$

- Independent and equally distributed entries X_{ij}
- Behavior of the spectrum of X ?

Algebraic and geometric spectra of $A \in \mathcal{M}_n(\mathbb{C})$

- Algebraic spectrum: eigenvalues (complex)
  - roots in $\mathbb{C}$ of characteristic polynomial $P_A(z) := \det(A - zI)$
  - $A = UTU^*$ and $\mathrm{diag}(T) = \lambda_1(A), \ldots, \lambda_n(A)$
  - $|\lambda_1(A)| \geqslant \cdots \geqslant |\lambda_n(A)|$
  - Spectral radius: $|\lambda_1(A)|$

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  - $|\lambda_1(A)| \geqslant \cdots \geqslant |\lambda_n(A)|$
  - Spectral radius: $|\lambda_1(A)|$
- Geometric spectrum: singular values (real $\geqslant 0$)
  - half lengths of principal axes of ellipsoid $\{Ax : \|x\|_2 = 1\}$
  - $A = UDV^*$ and $D = \mathrm{diag}(s_1(A), \ldots, s_n(A))$
  - $s_1(A) \geqslant \cdots \geqslant s_n(A)$
  - Operator norm: $s_1(A) = \max_{\|x\|_2=1} \|Ax\|_2$
  - $s_k(A) = \lambda_k(\sqrt{AA^*})$

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  - $s_k(A) = \lambda_k(\sqrt{AA^*})$
- If $AA^* = A^*A$ (normal matrix) then $s_k(A) = |\lambda_k(A)|$

Weyl inequalities and determinental rigidity

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$$|\lambda_1(A)\cdots\lambda_k(A)| \leqslant s_1(A)\cdots s_k(A)$$

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$$\mu_A = \frac{\delta_{\lambda_1(A)} + \cdots + \delta_{\lambda_n(A)}}{n} \quad \text{et} \quad \nu_A = \frac{\delta_{s_1(A)} + \cdots + \delta_{s_n(A)}}{n}$$

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- Determinental rigidity:

$$\begin{aligned} |\lambda_1(A)\cdots\lambda_n(A)| &= s_1(A)\cdots s_n(A) = |\det(A)| \\ \int \log(|\lambda|)\,d\mu_A(\lambda) &= \int \log(s)\,d\mu_{\sqrt{AA^*}}(s) = \frac{1}{n}\log|\det(A)| \end{aligned}$$

Sensitivity to perturbations

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \\ 0 & \cdots & & \ddots & 1 \\ 0 & \cdots & & & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \\ 0 & \cdots & & \ddots & 1 \\ \varepsilon_n & \cdots & & & 0 \end{pmatrix}$$

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$$AA^* = \mathrm{diag}(1,\ldots,1,0) \qquad BB^* = \mathrm{diag}(1,\ldots,1,\varepsilon_n)$$

$$A^n = 0, \lambda_k(A) = 0 \qquad B^n = \varepsilon_n I_n, \lambda_k(B) = \varepsilon_n^{1/n} e^{i2\pi k/n}$$

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$$\begin{cases} \nu_A & \to \delta_1 \\ \mu_A & = \delta_0 \end{cases} \qquad \begin{cases} \nu_B & \to \delta_1 \\ \mu_B & \to \mathrm{Uniform}(C(0,1)) \end{cases}$$

Random matrix model

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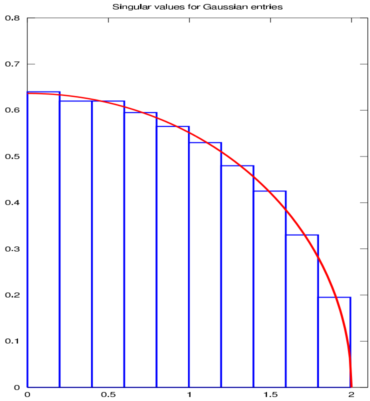
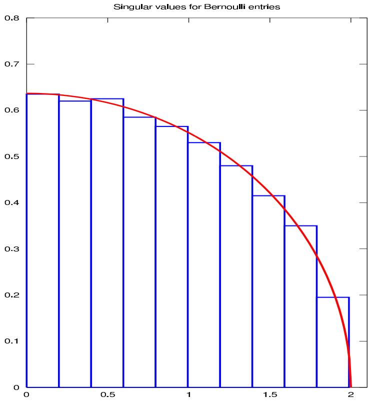
- Independent and equally distributed entries X_{ij}
- $\boxed{\text{Behavior of } \mu_X \text{ and } \nu_X \text{ when } n \to \infty \text{ ?}}$

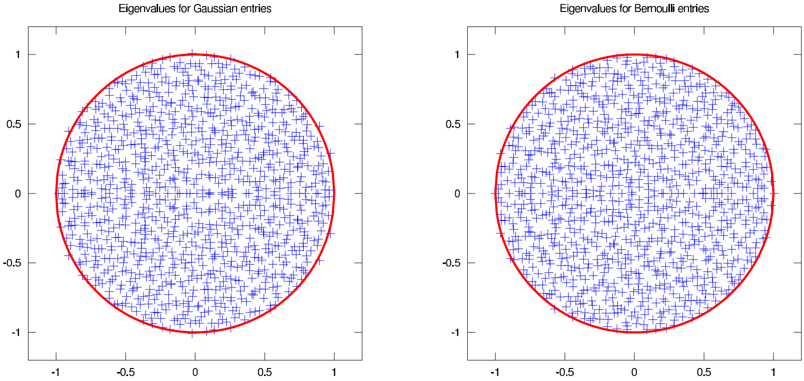
Quarter circular law (Universality)





Singular values of $\frac{1}{\sqrt{n}}X$

Circular law (Universality)



Eigenvalues $\frac{1}{\sqrt{n}}X$

Theorem (Quarter circular law – Marchenko-Pastur)

If $\mathrm{Var}(X_{11}) = 1$ *then*

$$\nu_{\frac{1}{\sqrt{n}}X} \xrightarrow[n\to\infty]{} \frac{\sqrt{4-x^2}\mathbf{1}_{[0,2]}}{\pi}\,dx$$

Theorem (Circular law – Girko, Bai, G.-T, Pan-Zou, Tao-Vu)

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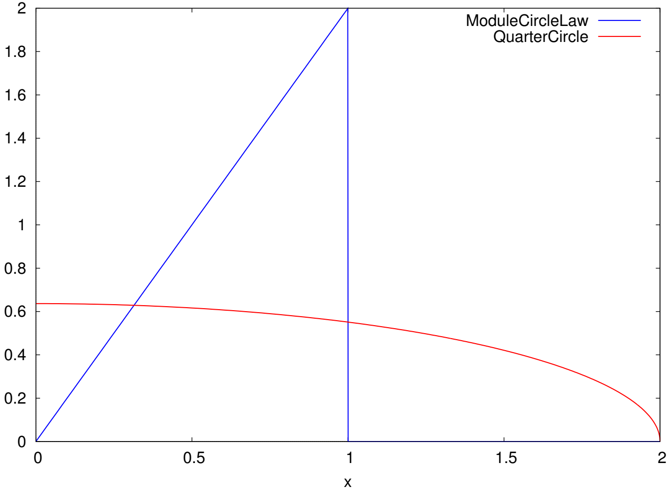
ModuleCircleLaw
QuarterCircle
2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
0.5
1
1.5
2
x

Support convergence and edge behavior

If $\mathrm{Var}(X_{11}) = 1$ then quatercircular and circular laws give a.s.

$$\varliminf_{n\to\infty} s_1\left(\frac{1}{\sqrt{n}}X\right) \geqslant 2 \quad \text{and} \quad \varliminf_{n\to\infty} \left|\lambda_1\left(\frac{1}{\sqrt{n}}X\right)\right| \geqslant 1$$

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Idea: Gelfand spectral radius formula: for any matrix norm $\|\cdot\|$,

$$|\lambda_1(A)| = \lim_{k\to\infty} \left\|A^k\right\|^{1/k}$$

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Second moment stabilization:

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Law of Large Numbers!

Proof of the quarter circular law

- H Hermitian $n \times n$ and $\eta_H := \frac{1}{n}\sum_{k=1}^{n} \delta_{\lambda_k(H)}$

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- Enough on $\mathbb{R}$ for the quarter circular law ($H = AA^*$)
- $\boxed{\text{Not enough on } \mathbb{C} \text{ for the circular law!}}$

Tightness for free

From the strong law of large numbers (SLLN):

$$\int s^2 \, d\nu_{\frac{1}{\sqrt{n}}X}(s) = \frac{1}{n^2}\sum_{k=1}^{n} s_k(X)^2 = \frac{1}{n^2}\sum_{i,j=1}^{n} |X_{ij}|^2 \xrightarrow[n\to\infty]{a.s.} \mathbb{E}(|X_{11}|^2).$$

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From Weyl's majorization inequalities:

$$\int |\lambda|^2 \, d\mu_{\frac{1}{\sqrt{n}}X}(\lambda) = \frac{1}{n^2}\sum_{k=1}^{n} |\lambda_k(X)|^2 \leqslant \frac{1}{n^2}\sum_{k=1}^{n} s_k(X)^2 \xrightarrow[n\to\infty]{a.s.} \mathbb{E}(|X_{11}|^2).$$

Conclusion: a.s. $(\mu_{\frac{1}{\sqrt{n}}X})_{n\geqslant 1}$ is tight

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$$\pi^{-n^2} e^{-\sum_{i,j=1}^{n} |G_{ij}|^2} = \pi^{-n^2} e^{-\mathrm{Tr}(GG^*)} = \pi^{-n^2} e^{-\sum_{k=1}^{n} s_k(G)^2}$$

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- $\mathrm{Tr}(GG^*) = \mathrm{Tr}(TT^*) = \mathrm{Tr}(DD^*) + \mathrm{Tr}(NN^*)$
- $(\lambda_1(G), \ldots, \lambda_n(G))$ has density

$$\varphi_n(z_1, \ldots, z_n) = \frac{n!}{1!2! \cdots n! \pi^{n^2}} \exp\left(-\sum_{k=1}^n |z_k|^2\right) \prod_{1 \leqslant i < j \leqslant n} |z_i - z_j|^2.$$

Analysis of a Gaussian case (2/3)

- $\gamma(z) := e^{-|z|^2}$, $H_\ell(z) := \frac{1}{\sqrt{\ell!}} z^\ell$,

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- Then the k-points correlation is

$$\varphi_n^{(k)}(z_1,\ldots,z_k) = \frac{(n-k)!}{n!\pi^{k^2}} \gamma(z_1)\cdots\gamma(z_k)\,\mathrm{det}\,[K(z_i,z_j)]_{1\leqslant i,j\leqslant k}$$

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- The 1-point correlation is the density of $\mathbb{E}(\mu_G)$:

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- Following Mehta, this gives the mean circular law:

$$\lim_{n\to\infty} n\varphi_n^{(1)}(\sqrt{n}z) = \pi^{-1}\mathbf{1}_{[0,1]}(|z|).$$

Analysis of a Gaussian case (3/3)

- Kostlan's observation:

$$(|\lambda_1(G)|, \dots, |\lambda_n(G)|) \overset{\text{law}}{=} (Z_{(1)}, \dots, Z_{(n)})$$

where $Z_1, \dots, Z_n$ are independent with $Z_k^2 \sim \mathrm{Gamma}(k, 1)$

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$$|\lambda_1(\tfrac{1}{\sqrt{n}}G)| \xrightarrow[n\to\infty]{\text{a.s.}} 1$$

- Moreover if $\gamma_n := \log(n/2\pi) - 2\log(\log(n))$ then

$$\sqrt{4n\gamma_n}\left(|\lambda_1(\tfrac{1}{\sqrt{n}}G)| - 1 - \sqrt{\frac{\gamma_n}{4n}}\right) \xrightarrow[n\to\infty]{\text{law}} \mathrm{Gumbel}.$$

(the Gumbel law has cdf $x \mapsto e^{-e^{-x}}$ on $\mathbb{R}$)

Large deviations (1/2)

- Setting $V(z) = |z|^2$, the density of $\lambda_1(\frac{1}{\sqrt{n}}G), \ldots, \lambda_n(\frac{1}{\sqrt{n}}G)$ is

$$c_n e^{-n\sum_{i=1}^n V(z_i)} \prod_{i<j} |z_i - z_j|^2$$

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- Rewriting in terms of $\mu_n := \frac{1}{n}\sum_{k=1}^n \delta_{z_k}$:

$$c_n \exp\left(-n^2\left(\frac{1}{n}\sum_{k=1}^n V(z_k) - \frac{2}{n^2}\sum_{i<j}\log|z_i - z_j|\right)\right)$$

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- Approximation as $n \gg 1$:

$$\approx c_n \exp\left(-n^2\mathcal{I}(\mu_n)\right)$$

where $\mathcal{I}$ is the logarithmic energy with external field:

$$\mathcal{I}(\mu) := \int V(z)\,d\mu + \iint \log\frac{1}{|z-w|}\,d\mu(z)d\mu(w).$$

Large deviations (2/2)

- Hiai-Petz and BenArous-Zeitouni: for every set S

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- Note : logarithmic energy = - Voiculescu free entropy

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- Logarithmic potential of a probability measure μ on $\mathbb{C}$

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-U_{\mu_A}(z) &= \int_{\mathbb{C}} \log|z-\lambda|\, d\mu_A(\lambda) \\
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Hermitization and Brown spectral measure

If $\lim_{n\to\infty} \nu_{A_n - zI} = \nu_z$ weakly then do we have

$$\begin{aligned}\lim_{n\to\infty} \mu_{A_n} &= \lim_{n\to\infty} \frac{1}{2\pi}\Delta \int_0^\infty \log(s)\, d\nu_{A_n - zI}(s) \\ &\stackrel{?}{=} \frac{1}{2\pi}\Delta \int_0^\infty \log(s)\, d\nu_z(s)\end{aligned}$$

Problem: singularity of the logarithm near 0 and ∞

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If A_n is a random variable on $\mathcal{M}_n(\mathbb{C})$ and if for all $z \in \mathbb{C}$

- $\nu_{A_n - zI} \underset{n\to\infty}{\longrightarrow} \nu_z$ *(deterministic)*
- log *is uniformly integrable for* $\nu_{A_n - zI}$

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Large singular values (easy!)

For any $0 < p \leqslant 2$ and any $z \in \mathbb{C}$ we have

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This follows from the strong law of large numbers:

$$\begin{aligned}\int s^2 \, d\nu_{\frac{1}{\sqrt{n}}X - zI}(s) &= \frac{1}{n}\sum_{k=1}^{n} s_k\Big(\frac{1}{\sqrt{n}}X - zI\Big)^2 \\ &= \frac{1}{n}\sum_{i,j=1}^{n} \Big|\frac{1}{\sqrt{n}}X_{ij} - zI_{ij}\Big|^2 \\ &\overset{a.s.}{=} O(1).\end{aligned}$$

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- We need to show that for some $p > 0$,

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- This leads to take $0 < p < \min(\gamma/b, 1)$.

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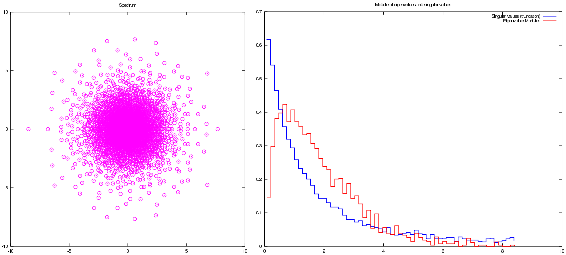

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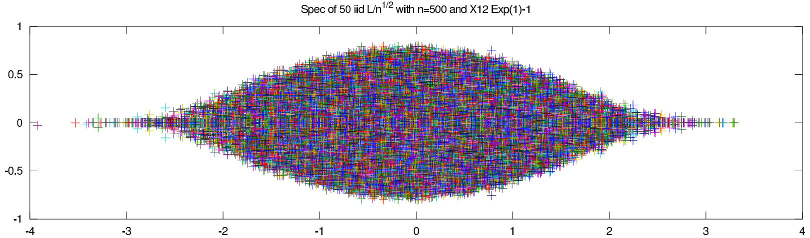

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- Progresses (2012): Rudelson-Vershynin, Basak-Dembo

Thank you!