Recent works on the asymptotic analysis of Coulomb gases

Djalil CHAFAÏ

CEREMADE
Université Paris-Dauphine / PSL

PDE/Probability Interactions:
Particle Systems, Hyperbolic Conservation Laws
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Outline

Electrostatics

Gases

Random matrices

Dynamics for planar case

Conditioning

Jellium

Coulomb kernel in mathematical physics

- Coulomb kernel in $\mathbb{R}^d$, $d \geq 1$,

$$x \in \mathbb{R}^d \mapsto g(x) = \begin{cases} \log \dfrac{1}{|x|} & \text{if } d = 2, \\ \dfrac{1}{(d-2)|x|^{d-2}} & \text{if not.} \end{cases}$$

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- Fundamental solution of Poisson's equation

$$\Delta g = -c_d \delta_0 \quad \text{where} \quad c_d = |\mathbb{S}^{d-1}| = d\omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}.$$

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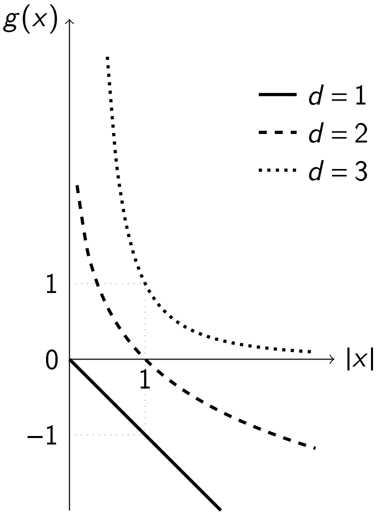
- Riesz kernel $|x|^{-s}$, if $s = d - \alpha$ then fractional Laplacian Δ_α

From now on $d \geq 2$



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- Integration by parts and "carré du champ", $\eta = \mu - \nu$,

$$\mathcal{E}(\eta) = \frac{1}{2}\int \mathrm{U}_\eta \mathrm{d}\eta = -\frac{1}{2c_d}\int \mathrm{U}_\eta\Delta\mathrm{U}_\eta \mathrm{d}x = \frac{1}{2c_d}\int |\nabla\mathrm{U}_\eta|^2\mathrm{d}x.$$

Confinement and equilibrium measure

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- Aka “O. Frostman measure” (PhD student of M. Riesz)

Convexity and Bochner positivity

- Convexity/Positivity for probability measures μ and ν

$$\frac{t\mathcal{E}_V(\mu)+(1-t)\mathcal{E}_V(\nu)-\mathcal{E}_V(t\mu+(1-t)\nu)}{t(1-t)}$$
$$=\mathcal{E}(\mu-\nu)=\frac{1}{2c_d}\int_{\mathbb{R}^d}|\nabla \mathrm{U}_{\mu-\nu}|^2 \mathrm{d}x.$$

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- If $d=2$ and $\mu_r=$ uniform law on $\{x\in\mathbb{R}^2:|x|=r\}$, then

$$U_{\mu_r}(x)=-\log(r)\mathbf{1}_{|x|\leq r}-\log|x|\mathbf{1}_{|x|>r}\quad\text{and}\quad\mathcal{E}(\mu_r)=-\frac{\log(r)}{2}$$

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- Euler–Lagrange: if $c_V = \mathscr{E}(\mu_V) - \int V \mathrm{d}\mu_V$ then q.e.

$$\mathrm{U}_{\mu_V} + V \begin{cases} = c_V & \text{on } \mathrm{supp}(\mu_V) \\ \geq c_V & \text{outside} \end{cases}$$

Examples of equilibrium measures

Dimension d	Potential V	Equilibrium measure μ_V
≥ 1	$\infty\mathbf{1}_{\lvert\cdot\rvert>r}$	Uniform on sphere of radius r
≥ 1	$<\infty$ and $\mathscr{C}^2$	$c_d^{-1}\Delta V$ on interior of support
≥ 1	$\frac{1}{2}\lvert\cdot\rvert^2$	Uniform on unit ball
(Ginibre) 2	$\frac{1}{2}\lvert\cdot\rvert^2$	Uniform on unit disc
(Spherical) 2	$\frac{1}{2}\log(1+\lvert\cdot\rvert^2)$	Heavy-tailed $\frac{1}{\pi(1+\lvert\cdot\rvert^2)^2}$

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(CUE) 2	$\infty \mathbf{1}_{([a,b]\times\{0\})^c}$	Arcsine $s \mapsto \frac{\mathbf{1}_{s\in[a,b]}}{\pi\sqrt{(s-a)(b-s)}}$
(GUE) 2	$\frac{\lvert\cdot\rvert^2}{2}\mathbf{1}_{\mathbb{R}\times\{0\}} + \infty \mathbf{1}_{(\mathbb{R}\times\{0\})^c}$	Semicircle $s \mapsto \frac{\sqrt{4-s^2}}{2\pi}\mathbf{1}_{s\in[-2,2]}$

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Outline

Coulomb gas or one component plasma

- We suppose that for all $\beta > 0$ and for n large enough

$$\int_{\mathbb{R}^d} \mathrm{e}^{-n\beta(V(x)-\log(1+|x|)\mathbf{1}_{d=2})}\mathrm{d}x < \infty.$$

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$$\mathrm{Z}_n = \int_{(\mathbb{R}^d)^n} \mathrm{e}^{-\beta \mathrm{E}_n(x_1,\ldots,x_n)}\mathrm{d}x_1 \cdots \mathrm{d}x_n < \infty.$$

where

$$\mathrm{E}_n(x_1,\ldots,x_n) = n\sum_{i=1}^n V(x_i) + \frac{1}{2}\sum_{i\neq j} g(x_i - x_j).$$

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- Particles subject to confinement and singular pair repulsion

Empirical measure

- Empirical energy

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- Empirical measure and off-diagonal energy

$$\beta \mathrm{E}_n(x_1,\ldots,x_n) = \beta n^2 \mathscr{E}_V^{\neq}(\mu_{x_1,\ldots,x_n})$$

where $\mu_{x_1,\ldots,x_n} = \dfrac{1}{n}\displaystyle\sum_{i=1}^{n} \delta_{x_i}$ and

$$\mathscr{E}_V^{\neq}(\mu) = \int V \mathrm{d}\mu + \frac{1}{2}\iint_{\neq} g(u-v)\mathrm{d}\mu(u)\mathrm{d}\mu(v)$$

Laplace method point of view

- Laplace point of view

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- Law of Large Numbers : if $X_n \sim \mathrm{P}_n$ then almost surely

$$\mu_{X_{n,1},\ldots,X_{n,n}} \underset{n\to\infty}{\longrightarrow} \mu_V = \arg\min \mathcal{E}_V$$

Asymptotic analysis of fluctuations

- Quadratic form using $g = -c_d \Delta^{-1}$

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- Central Limit Theorem and Gaussian Free Field (universality)

$$\sum_{i=1}^{n} f(X_{n,i}) - \mathbb{E}\sum_{i=1}^{n} f(X_{n,i}) \xrightarrow[n\to\infty]{\text{law}} \mathcal{N}\left(0, \frac{1}{\beta c_d}\int_{\mathbb{R}^d} |\nabla f|^2 \mathrm{d}x\right).$$

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- Quantitative: concentration of measure inequalities

  ..., Guionnet–Zeitouni, Rougerie–Serfaty, Hardy–C.–Maïda, Berman, ...

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- Sanov: $g = 0$ and $\beta_n = \frac{\kappa}{n}$, then $\mathrm{P}_n = \left(\frac{\mathrm{e}^{-\kappa V}}{Z}\right)^{\otimes n}$.

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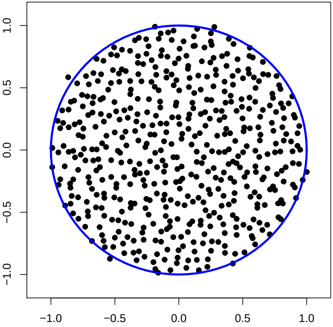
- Coulomb gas P_n with $d = 2,\ \beta = 2,\ V = \frac{1}{2}|\cdot|^2$

High dimensional phenomenon : random matrix spectrum




```
plot(eig(randn(n,n)+i*randn(n,n))/sqrt(2*n)))
```

Ginibre ensemble in nature

- Jean Ginibre (1938 –)
  Statistical Ensembles of Complex, Quaternion, and Real Matrices
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$$P_{n,k} \quad \text{has density} \quad (x_1, \ldots, x_k) \mapsto \det\left(K_{V,n}(x_i, x_j)\right)_{1 \leq i,j \leq n}$$

Outline

Langevin dynamics

- Overdamped Langevin dynamics on $(\mathbb{R}^d)^n$: $X_t \xrightarrow[t\to\infty]{\text{law}} \mathrm{P}_n$

$$\mathrm{d}X_t = \sqrt{2\frac{\alpha}{\beta}}\mathrm{d}B_t - \alpha\nabla\mathrm{E}_n(X_t)\mathrm{d}t, \quad \mathrm{L} = \alpha(\beta^{-1}\Delta - \nabla\mathrm{E}_n\cdot\nabla)$$

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- Lyapunov: Bolley–C.–Fontbona, Lu–Mattingly

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- Do you know an alternative proof?

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Conditioned Coulomb gas

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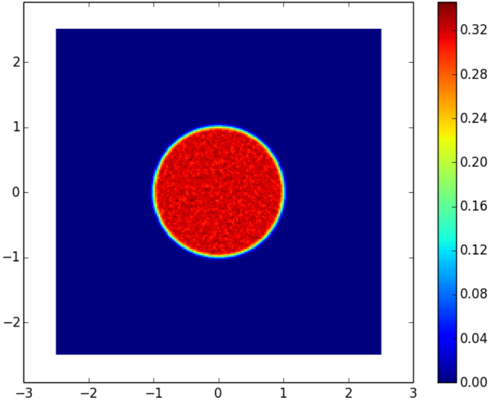
Conditioned Coulomb gases – HMC/Julia



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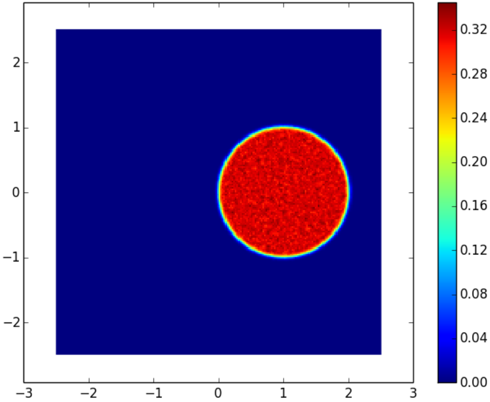

$$V(x) = |x|^2,\ \varphi(x) = a \cdot x + b$$

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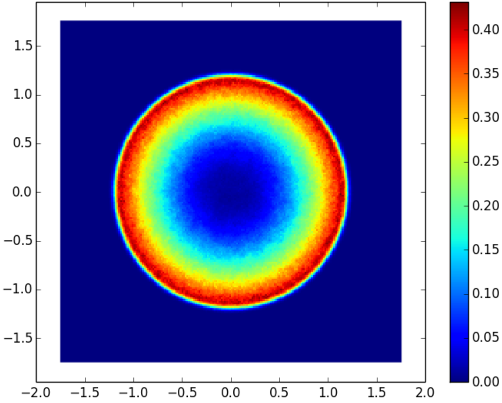

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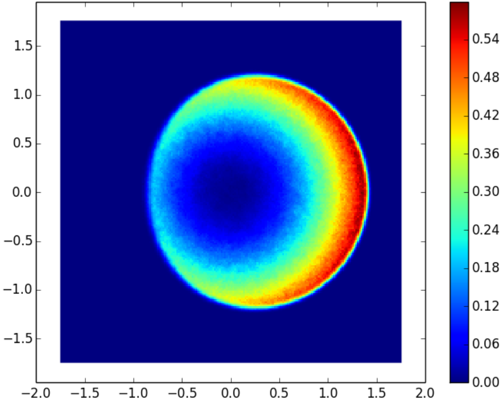

$$V(x) = |x|^4,\ \varphi(x) = ax_1 + b$$

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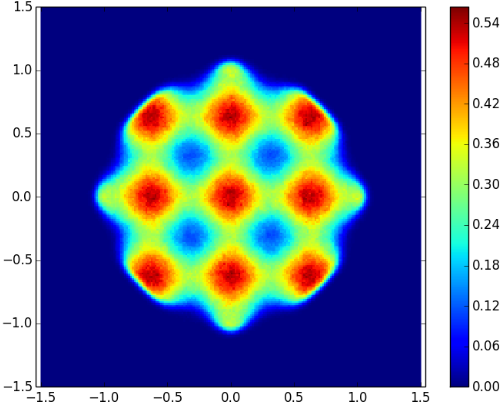

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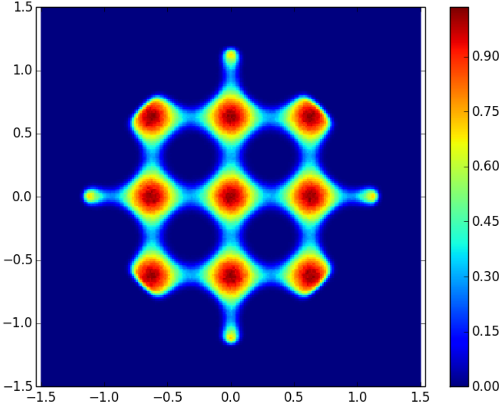

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- Coulomb gas is a Jellium with $\rho = \frac{n}{\alpha c_d}\Delta V$ on $S = \mathbb{R}^d$

Two dimensional jellium with uniform background

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- More: Butez–García-Zelada

Thank you very much for your attention!