

## Random Matrix Diagonalization—Some Numerical Computations

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in which boundary conditions can be conveniently set up in terms of some form of  $F$ , instead of a more conventional "amplitude." Furthermore, analogues of the Kirchhoff-Helmholtz integral formula may be available in other areas. It may be of some interest to note that, by the use of this formula, one can set up a generalized Huygens' principle for the propagation of any  $n$ th power of the amplitude

in a light field ( $n$  an integer) or of the  $n$ th moment of the amplitude.

#### ACKNOWLEDGMENTS

During the progress of this work, the writer has had the benefit of useful comments by some of his colleagues, including, among others, D. E. Royal, R. Graves, and P. Stehle.

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## Random Matrix Diagonalization—Some Numerical Computations\*

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Numerical results of Monte-Carlo calculations of spacing and eigenvalue distributions for the invariant and independent Gaussian orthogonal ensemble of Hamiltonian matrices are presented. Many of the histograms should be useful for comparison with experimental data. A table of the first few moments of each distribution is given. For the spacing distributions, such moments are equivalent to spacing correlation coefficients, and hence these are also made available indirectly.

### I. INTRODUCTION

THE use of random matrix models to explain the observed statistical fluctuations in the energy level spacings, energy-level widths, and expectation values of complex spectra is now a well-established point of view.<sup>1-12</sup> However, the difficulties of extracting analytical results from such

models to compare to experimental data are considerable, and this task has so far been accomplished only in special cases and with a tremendous display of analytical virtuosity.<sup>3-9</sup> It is the purpose of this article to present a complete set of numerical results for one model to indicate what might in the future be obtained analytically and to provide much needed theoretical results to compare to experimental data. (We do not go into the data analysis in this paper.)

\* Research supported by the United States Atomic Energy Commission.

<sup>1</sup> For a complete list of early references, see C. E. Porter and N. Rosenzweig, *Ann. Acad. Sci. Fennicae* **AVI**, No. 44 (1960).

<sup>2</sup> N. Rosenzweig and C. E. Porter, *Phys. Rev.* **120**, 1698 (1960); *Phys. Rev.* **123**, 853 (1961).

<sup>3</sup> M. L. Mehta, *Nucl. Phys.* **18**, 395 (1960); M. L. Mehta and M. Gaudin, *Nucl. Phys.* **18**, 420 (1960); **22**, 340 (1960); M. Gaudin, *Nucl. Phys.* **25**, 447 (1960).

<sup>4</sup> F. J. Dyson, *J. Math. Phys.* **3**, 140 (1962); **3**, 157 (1962); **3**, 166 (1962); **3**, 1191 (1962); **3**, 1199 (1962).

<sup>5</sup> F. J. Dyson and M. L. Mehta, *J. Math. Phys.* **4**, 701 (1963); M. L. Mehta and F. J. Dyson, *J. Math. Phys.* **4**, 713 (1963).

<sup>6</sup> J. Gunson, *J. Math. Phys.* **3**, 752 (1962).

<sup>7</sup> K. G. Wilson, *J. Math. Phys.* **3**, 1040 (1962).

<sup>8</sup> L. D. Favro, P. B. Kahn, and M. L. Mehta, "Concerning Polynomials Encountered in the Study of the Distribution Function of Spacings," Brookhaven National Laboratory Report BNL 757 (T-280), September 1962 (unpublished).

<sup>9</sup> P. B. Kahn, *Nucl. Phys.* **41**, 159 (1963).

<sup>10</sup> N. Rosenzweig, *Bull. Am. Phys. Soc.* **7**, 91 (1962) and *Brandeis Summer Institute Lecture Notes* (W. A. Benjamin, Inc., New York, 1962).

<sup>11</sup> H. S. Leff, *Bull. Am. Phys. Soc.* **8**, 31 (1963).

<sup>12</sup> C. E. Porter, *Phys. Letters* **2**, 292 (1962); *Nucl. Phys.* **40**, 167 (1963); *Physics Today* **16**, No. 2, 26 (1963), **16** No. 4, 90 (1963).

### II. THEORETICAL MODEL

The model used in this paper is the invariant, independent Gaussian orthogonal ensemble in 10 dimensions.<sup>1</sup> Thus the real symmetric Hamiltonian matrix ( $10 \times 10$ ) is distributed according to the form

$$P(H) = C \exp(-\text{Tr } H^2/4\sigma^2). \quad (1)$$

where  $C$  is a normalization constant and  $\sigma$  is the root-mean-square dispersion of the off-diagonal matrix elements. The parameter  $\sigma$  is a scale factor, and is related to the mean distance  $D$  between levels by the approximate connection (valid for large  $N$ )

$$D = \pi\sigma/N^{\frac{1}{2}}, \quad (2)$$

where in this paper we are concerned with  $N = 10$ .

The details of the numerical computations are

described elsewhere.<sup>13</sup> It is sufficient to note that a straightforward Monte-Carlo calculation was performed in which 10 000 random  $10 \times 10$  matrices were generated according to the distribution (1) and then diagonalized, and the resulting spectra were sorted to yield the results which are plotted in the graphs and tabulated in Table I of this paper. The eigenvector components were, of course, obtained also, but since the eigenvector component distribution is well understood in this (invariant) case (for example, the marginal distribution of a single component is that of a component of a randomly oriented unit vector in 10 dimensions—see reference 1), we do not discuss the eigenvector components further since they feed naturally into the distributions of widths (transition probabilities) and expectation values which are discussed elsewhere.<sup>2,14</sup>

It has been shown by Mehta<sup>5</sup> that an intimate connection exists between the circular orthogonal and circular symplectic ensembles. The connection is such that the spacing distributions in the circular symplectic ensembles are identical with alternate spacing distributions in the circular orthogonal ensemble, i.e., the nearest-neighbor spacing distribution in the circular symplectic ensemble is the next-nearest-neighbor spacing distribution in the circular orthogonal ensemble with the mean nearest-neighbor spacing of the circular orthogonal ensemble doubled, etc.

It is suspected that a similar connection exists, at least in the infinite-dimensional limit, between the

Gaussian symplectic and the Gaussian orthogonal ensembles, but this has not yet been proven. If this connection does exist, then our computations for the Gaussian orthogonal ensemble may apply to the Gaussian symplectic ensemble as well.

### III. NUMERICAL RESULTS

The numerical results are shown in Figs. 1-6. In Figs. 1-3 are plotted the histograms for the nine possible spacing distributions  $P^k$  for  $10 \times 10$  matrices. The superscript  $k$  is the number of levels "in between" and ranges from zero to eight. The first four moments of these distributions are given in Table I.

It is perhaps worth noting that an "edge" effect was arbitrarily normalized out of the numerical spacing distribution computations. To understand what this means, consider a  $4 \times 4$  matrix. In the resulting spectrum there are four levels, three nearest-neighbor spacings, two next-nearest-neighbor spacings, and one next-next-nearest-neighbor spacing for each matrix diagonalized. It is obvious that if the mean nearest-neighbor spacing is  $D$ , then the mean next-next-nearest-neighbor spacing is  $3D$ . However, because of the finite size of a  $4 \times 4$  matrix, it is *not* true that the mean next-nearest-neighbor spacing is  $2D$  since the two next-nearest-neighbor

TABLE I. Tabulation of Moments.

Moment	1	2	3	4
Spacings Neighbor				
0	1.001	1.316	2.105	3.925
1	2.001	4.484	11.09	29.93
2	3.001	9.590	32.48	116.1
3	4.001	16.67	72.21	324.4
4	5.001	25.73	136.1	738.7
5	6.001	36.78	230.1	1468.
6	7.001	49.84	360.6	2651.
7	8.001	64.90	533.7	4448.
8	9.001	82.02	756.7	7065.
Eigenvalues All	-.001347	.4327	-.0004436	.2515
1, 10	.8404	.7257	.6432	.5845
2, 9	.6047	.3795	.2463	.1648
3, 8	.4156	.1845	.08655	.04257
4, 7	.2455	.07075	.02264	.007850
5, 6	.1066	.01701	.003376	.0007729

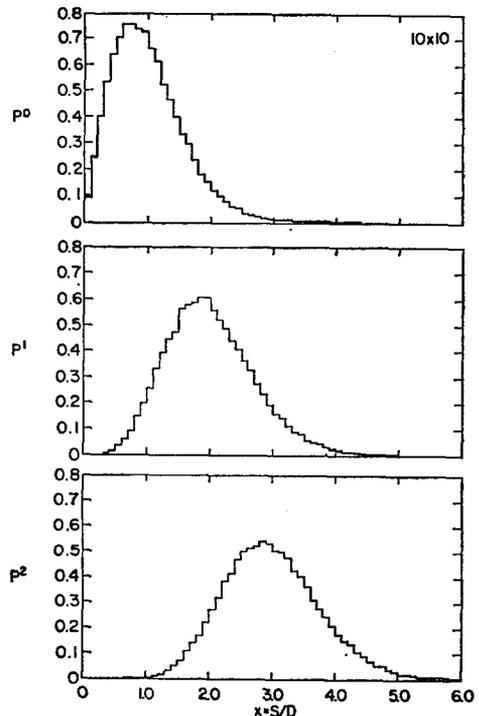


Fig. 1. Histogram plots of the first three ( $k = 0-2$ ) spacing distributions  $K^k$  as a function of  $x = S/D$  for  $10 \times 10$  matrices.

<sup>13</sup> K. Fuchel, Rita J. Greibach, and C. E. Porter, "Random Matrix Diagonalization—A Computer Program," Brookhaven National Laboratory Report BNL 760 (T-282), September 1962 (unpublished).

<sup>14</sup> T. J. Krieger and C. E. Porter, J. Math. Phys. (to be published).

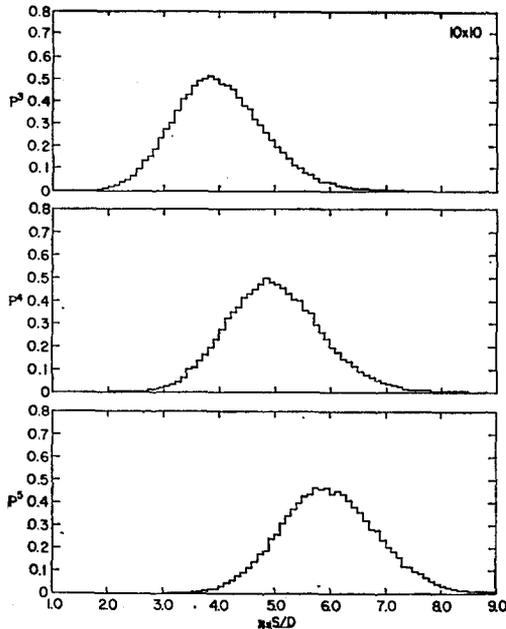


FIG. 2. The same as Fig. 1, except that  $k = 3-5$ . Note that the abscissa scale does not begin at zero.

spacings do not weight equally the three nearest-neighbor spacings. In the computations, a mean spacing was computed for each  $P^k$  and was found to deviate slightly from  $(k + 1)D$ . This deviation was compensated for by scaling  $D$  for each  $P^k$  so that the plots in Figs. 1-3 are arranged such that the mean value of  $x$  is exactly equal to  $k + 1$ . This can be seen from Table I; of course, the higher moments are based on the same scale.

Figure 4 shows the sum of all the  $P^k$ , i.e.,

$$P^{\text{sum}} = \sum_{k=0}^{N-2} P^k. \tag{3}$$

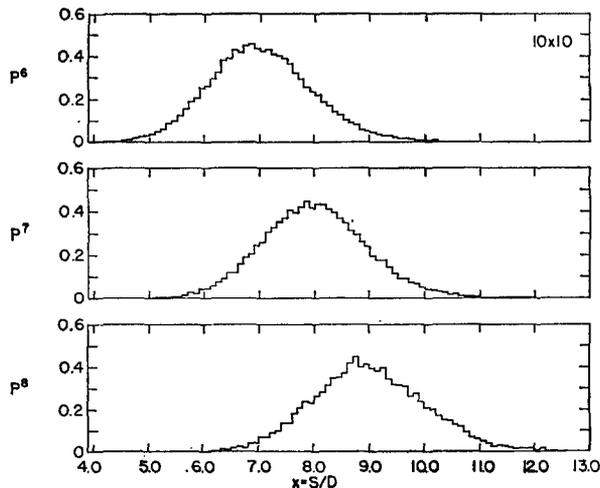


FIG. 3. The same as Fig. 1, except that  $k = 6-8$ . As in Fig. 2, the abscissa scale does not begin at zero.

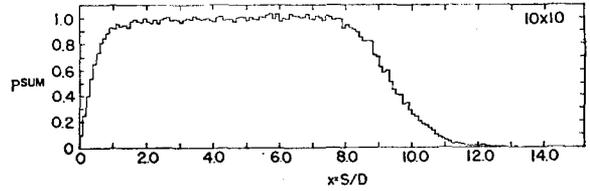


FIG. 4. Histogram plot of  $P^{\text{sum}}$  as a function of  $x = S/D$ .

For a Poisson distribution which is appropriate for levels of many different symmetries,

$$P^k(x) = (x^k/k!) \exp(-x), \tag{4}$$

so that in this case

$$P^{\text{sum}}_{\text{Poisson}} \xrightarrow{N \rightarrow \infty} 1. \tag{5}$$

The repulsion effect for small  $x$  is clearly evident in the figure.

Fig. 5 shows the semicircle law of Wigner<sup>15</sup> for the single eigenvalue distribution. The semicircle law holds asymptotically for large  $N$  and has the form

$$P(E/2\sigma N^{\frac{1}{2}}) = (2/\pi)[1 - E^2/4\sigma^2 N]^{1/2}. \tag{6}$$

The asymptotic law is clearly already very good for ten dimensions. Analytical expressions for the single eigenvalue distribution for all dimensions in the Gaussian ensemble have been obtained by Mehta and Gaudin.<sup>3</sup> We show the Monte-Carlo results here mainly to indicate the correctness of the computer program.

Because of the plus-minus symmetry of the input matrix element distributions, the first and tenth (in numerical order) eigenvalues are statistically equivalent except for sign. Thus, to increase the counting rate, the sign of the first eigenvalue was changed and the result grouped with the tenth eigenvalue. (We could have likewise plotted only half of the semicircle in Fig. 5 since it is symmetrical about zero.) Similarly the other eigenvalues can be paired off. The resulting histograms are shown in

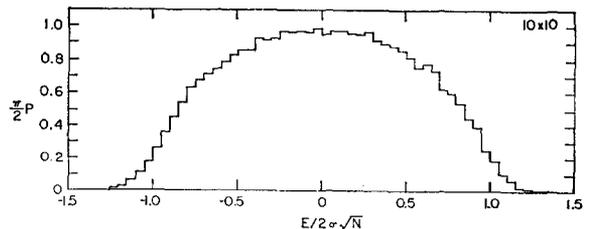


FIG. 5. Histogram plot of the semicircle law. Note that the ordinate is  $(\frac{1}{2}\pi)P$  and the abscissa is  $E/2\sigma N^{\frac{1}{2}}$ .

<sup>15</sup> E. P. Wigner, "Statistical Properties of Real Symmetric Matrices with Many Dimensions" (invited address) Proc. Canadian Math. Congr. (1957), p. 174.

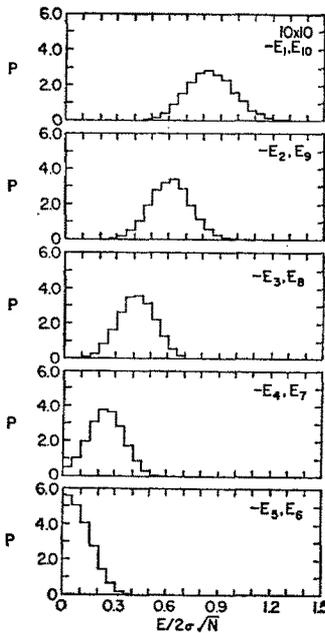


FIG. 6. Histogram plots of the individual eigenvalue distributions as explained in the text. Note that the abscissa is  $E/2\sigma N^{1/2}$ .

Fig. 6. (Moments of the distributions are given in Table I.) Thus we can see how the semicircle breaks down into its component parts. Another way of saying this is that the separate eigenvalue distributions merge smoothly to form the semicircle, i.e., there is no energy gap in the spectrum. The absence of an energy gap is related to the symmetry of the matrix element distribution.

It is planned to report in the future computations which show that breaking the symmetry of the matrix element distribution leads to an energy gap in the spectrum, with a characteristic departure from the semicircle law.

#### ACKNOWLEDGMENTS

The computer program for this work was written by Kurt Fuchel and Rita J. Greibach, for which they deserve special thanks. Miss Frances Pope was helpful in plotting some of the preliminary output and also the final results.

## Real Unitary Representation of the Many-Channel $S$ Matrix for Complex $l$ and $E$

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A representation is established for the multichannel  $S$  matrix in terms of a matrix function  $A(l, E)$  and a scalar function  $B(l, E)$ , both holomorphic in the domain formed by the product of the whole finite  $l$  plane with the finite  $E$  plane, cut only along the left-hand dynamical cut. The representation satisfies the known analytic properties of  $S(l, E)$ , and also all the generalized unitarity conditions of Peierls, LeCouteur, and Newton. The reality condition on  $S$  for complex  $l$  and  $E$  is guaranteed by a simple condition on  $A$  and  $B$ .

### 1. INTRODUCTION

THE purpose of this paper is to write down a simple representation of the Schrödinger multichannel  $S$  matrix which will automatically satisfy all the unitarity conditions, the reality condition, and the analytic properties of  $S$  as a function of the two complex variables  $l$  and  $E$ . The discussion is mainly kinematical in the sense that it involves only the general characteristics in the formulation of the Schrödinger scattering problem, and does not depend on the details of the interaction. Most of the results are thus expected to hold even in the case of relativistic scattering of particles. For defi-

nitens, however, we may restrict ourselves to the class of potentials studied, for example, by Mandelstam.<sup>1</sup> It was shown<sup>1,2</sup> that the single-channel  $S$  matrix is a meromorphic function of the variables  $l$  and  $k$  in the domain formed by the product of the whole finite  $l$  plane, with the  $k$  plane cut along the imaginary axis from  $k = \frac{1}{2}i\mu$  to  $i\infty$ , and again

<sup>1</sup> S. Mandelstam, *Ann. Phys. (NY)*, **19**, 254, (1962). The potentials studied by Mandelstam are very restrictive. For the more general class studied by Bottino *et al.*, all the discussion in this paper is unchanged except for the domain of analyticity on the  $l$  plane, namely the right half- $l$  plane ( $\text{Re } l > -\frac{1}{2}$ ) instead of the whole finite  $l$  plane.

<sup>2</sup> A. Bottino, A. M. Longoni, and T. Regge, *Nuovo Cimento* **23**, 954 (1962).