Exam 2019/2020

December 4, 2019, from 13:45 to 16:45
Documents allowed, Internet not allowed
Do what you can, and do not worry

We use the notations of the lecture notes.
$B = (B_t)_{t\geq 0}$ is a d-dimensional Brownian motion issued from the origin, $d \geq 1$.

Exercise 1. Assume that $d = 1$. Let $\sigma > 0$, $\rho \in \mathbb{R}$, and $x \in \mathbb{R}$ be fixed parameters.

1. Solve the ODE $X_0 = x$ and $X'(t) = \rho X(t)$ and discuss its sign depending on x.
2. Solve the SDE $X_0 = x$ and $\mathrm{d}X_t = \rho X_t \mathrm{d}t + \sigma X_t \mathrm{d}B_t$ (existence, uniqueness, explicit formula).

Exercise 2. Let $\theta > 0$, $\rho \in \mathbb{R}$, $z \in \mathbb{R}^d$ be parameters, and let Z^z be the solution of

$$Z_0^z = z, \quad \mathrm{d}Z_t^z = \theta \mathrm{d}B_t - \rho Z_t^z \mathrm{d}t$$

1. Why this SDE has a pathwise unique solution? What is the name of the process Z^z?
2. Show that the process $W_t = \int_0^t \frac{Z_s^z}{|Z_s^z|} \mathrm{d}B_s$ with the convention $0/0 = 1$ is a Brownian motion.
3. Let us define $x = |z|^2$. Show that the process $X_t^x = |Z_t^z|^2$ solves the stochastic differential equation

$$X_0^x = x, \quad \mathrm{d}X_t^x = \sigma\sqrt{X_t^x}\mathrm{d}W_t + (a - bX_t^x)\mathrm{d}t \quad \text{where} \quad \sigma = 2\theta,\ a = \theta^2 d,\ b = 2\rho.$$

4. Show that if $\rho > 0$ then[1] $X_t^x \xrightarrow[t\to\infty]{\text{law}} \mathrm{Gamma}(d/2, 2b/\sigma^2)$. What happens when $b \leq 0$?
5. **From now on**, we assume that X^x solves the SDE above for $x \geq 0$ and an arbitrary **real parameter** $d > 0$, without relation to Z^z. Our goal is to evaluate $\mathbb{P}(T_0^x < \infty)$, $T_c^x = \inf\{t \geq 0 : X_t^x = c\}$. Show that

$$u \in (0, +\infty) \mapsto \varphi(u) = \int_1^u v^{-\frac{2a}{\sigma^2}} \mathrm{e}^{\frac{2b}{\sigma^2}v} \mathrm{d}v \quad \text{satisfies} \quad \frac{\sigma^2}{2} u\varphi''(u) + (a - bu)\varphi'(u) = 0.$$

6. **From now on**, we take $x > 0$ and $0 < \varepsilon < x < R$. Let us define $T_{\varepsilon,R}^x = T_\varepsilon^x \wedge T_R^x$. Show that for all $t > 0$,

$$\varphi(X_{t\wedge T_{\varepsilon,R}^x}^x) = \varphi(x) + \int_0^{t\wedge T_{\varepsilon,R}^x} \varphi'(X_s^x)\sigma\sqrt{X_s^x}\mathrm{d}W_s.$$

7. Show that $\mathbb{E}(T_{\varepsilon,R}^x) < \infty$, which gives $T_{\varepsilon,R} < \infty$ a.s. (hint: use an isometry, and a lower bound on φ').
8. Show that

$$\varphi(x) = \varphi(\varepsilon)\mathbb{P}(T_\varepsilon^x < T_R^x) + \varphi(R)\mathbb{P}(T_\varepsilon^x > T_R^x).$$

9. Show that if $a \geq \frac{\sigma^2}{2}$ then $\mathbb{P}(T_0^x < \infty) = 0$ (hint: use $\lim_{u\to 0}\varphi(u) = -\infty$).
10. Show that if $0 \leq a < \frac{\sigma^2}{2}$ and $b \geq 0$ then $\mathbb{P}(T_0^x < \infty) = 1$ (hint: use $\lim_{R\to+\infty}\varphi(R) = +\infty$).
11. Show that if $0 \leq a < \frac{\sigma^2}{2}$ and $b < 0$ then $\mathbb{P}(T_0^x < \infty) = (\varphi(\infty) - \varphi(x))/(\varphi(\infty) - \varphi(0)) \in (0,1)$.

– The third and last exercise is on the opposite side of this page –

1. If $G \sim \mathcal{N}(0, I_d)$ then $|G|^2 \sim \chi^2(d) = \mathrm{Gamma}(d/2, 1/2)$. The law $\mathrm{Gamma}(a,\lambda)$ has density $u \mapsto \frac{\lambda^a}{\Gamma(a)} u^{a-1}\mathrm{e}^{-\lambda u}\mathbf{1}_{u\geq 0}$.

Exercise 3. Let $U \in \mathscr{C}^2(\mathbb{R}^d, \mathbb{R})$. In particular $-\nabla U$ is locally Lipschitz but is not globally Lipschitz in general. Let us fix $x \in \mathbb{R}^d$. From the lecture notes, we recall and admit that there exists an adapted process X with values in $\mathbb{R}^d \cup \{\infty\}$ and a stopping time T with values in $(0, +\infty]$ such that

— $X_t \in \mathbb{R}^d$ if $t < T$ while $X_t = \infty$ if $t \geq T$, and $\lim_{t \to_< T} |X_t| = \infty$ on $\{T < \infty\}$

— $t \in [0, T) \mapsto X_t \in \mathbb{R}^d$ is continuous

— $X_t = x + B_t - \int_0^t \nabla U(X_s) \mathrm{d}s$ on the (maximal) time interval $[0, T)$

We study now a couple of sufficient criteria on U in order to get $\mathbb{P}(T < \infty) = 0$ (no explosion in finite time).

1. Suppose that
$$\lim_{|x| \to \infty} U(x) = +\infty \quad \text{and} \quad C_2 = \sup_{x \in \mathbb{R}^d} \Big(\frac{1}{2} \Delta U - |\nabla U|^2 \Big) < \infty.$$
   (a) Show that $T_R = \inf\{t \geq 0 : U(X_t) > R\} \underset{R \to \infty}{\nearrow} T$.

   (b) Show that $Y = X^{T_R} = (X_{t \wedge T_R})_{t \geq 0}$ solves the following SDE
$$Y_t = x + \int_0^t \mathbf{1}_{s \leq T_R} \mathrm{d}B_s - \int_0^t \mathbf{1}_{s \leq T_R} \nabla U(X_s) \mathrm{d}s, \quad t \geq 0.$$
   (c) Show that for all $R > 0$ and $t > 0$,
$$\mathbb{E}(U(X_{t \wedge T_R})) = U(x) + \mathbb{E}\Big(\int_0^{t \wedge T_R} \Big(\frac{1}{2} \Delta U - |\nabla U|^2 \Big)(X_s) \mathrm{d}s \Big).$$
   (d) Show that $C_1 = \inf_{\mathbb{R}^d} U > -\infty$ and, for all $R > 0$ and $t > 0$,
$$R \mathbf{1}_{T_R \leq t} - |C_1| \leq U(X_{t \wedge T_R}).$$
   (e) Show that for all $R > 0$ and $t > 0$,
$$\mathbb{E}(R \mathbf{1}_{T_R \leq t} - |C_1| - U(x)) \leq \mathbb{E}(C_2 (t \wedge T_R)) \leq t.$$
   (f) Show that $\mathbb{P}(T < \infty) = 0$.

2. Suppose that for some $a, b \in \mathbb{R}$ and all $x \in \mathbb{R}^d$,
$$\langle x, \nabla U(x) \rangle \geq -a|x|^2 - b.$$
   (a) Show that
$$T_n = \inf\{t \geq 0 : |X_t|^2 > n\} \underset{n \to \infty}{\nearrow} T.$$
   (b) Show that $Y = X^{T_n} = (X_{t \wedge T_n})_{t \geq 0}$ solves the following SDE
$$Y_t = x + \int_0^t \mathbf{1}_{s \leq T_n} \mathrm{d}B_s - \int_0^t \mathbf{1}_{s \leq T_n} \nabla U(X_s) \mathrm{d}s, \quad t \geq 0.$$
   (c) Show that for all $t \geq 0$ and $n \geq 1$,
$$\mathbb{E}(|X_{t \wedge T_n}|^2) \leq |x|^2 + (1 + 2|b|)t + 2|a| \int_0^t \mathbb{E}(|X_{s \wedge T_n}|^2) \mathrm{d}s.$$
   (d) Show that for all $t \geq 0$ and $n \geq 1$,
$$\mathbb{E}(|X_{t \wedge T_n}|^2) \leq (|x|^2 + (1 + 2|b|)t) \mathrm{e}^{2|a|t}.$$
   (e) Show that $\mathbb{P}(T < \infty) = 0$.

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