(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}) is a filtered probability space, with complete and right continuous filtration. 

If \( Z \) is a semi-martingale, we denote by \( \langle Z \rangle \) the increasing process of its local martingale part.

1. Let us consider the following integral, for \( t \geq 0 \),

\[
I_t := \int_0^t B_s \, ds.
\]

2. Show that \( d(I_t) = B_t \, dt + t \, dB_t; \)
3. Deduce from the preceding question that \( I_t = \int_0^t (t - s) \, dB_s \) for all \( t \geq 0; \)
4. Deduce from the preceding question that \( I_t \sim \mathcal{N}(0, \frac{1}{2} t^2) \) for all \( t \geq 0; \)
5. For all \( t \geq 0, n \geq 1, 0 \leq k \leq n \), let us define \( t_k := \frac{k}{n} t. \) Show that

\[
\sum_{k=0}^{n-1} B_{t_{k+1}}(t_{k+1} - t_k) = \frac{t}{n} \sum_{j=0}^{n-2} (n - j - 1)(B_{t_{j+1}} - B_{t_j}).
\]

6. Deduce from the preceding question another proof that \( I_t \sim \mathcal{N}(0, \frac{1}{3} t^3) \) for all \( t \geq 0; \)
7. Is the process \( (I_t)_{t \geq 0} \) a martingale?

**Exercise 2** (Study of a special process). Set \( d = 2. \) For all \( t \geq 0, \) we write \( B_t = (X_t, Y_t) \) and

\[
A_t := \int_0^t X_s \, dY_s - \int_0^t Y_s \, dX_s.
\]

1. Show that \( \langle A \rangle = \int_0^t (X_s^2 + Y_s^2) \, ds \) and that the process \( A \) is a square integrable martingale;
2. From now on let \( \lambda > 0. \) Show that for all \( t \geq 0, \)

\[
\mathbb{E} e^{\lambda A_t} = \mathbb{E} \cos(\lambda A_t).
\]

3. From now on, let \( f : \mathbb{R}_+ \to \mathbb{R} \) be \( \mathcal{C}^2, \) and let us define the continuous semi-martingales

\[
(Z_t)_{t \geq 0} := (\cos(\lambda A_t))_{t \geq 0} \quad \text{and} \quad (W_t)_{t \geq 0} := \left(-\frac{f'(t)}{2}(X_t^2 + Y_t^2) + f(t)\right)_{t \geq 0}.
\]

Show that for all \( t \geq 0, \)

\[
Z_t = 1 - \lambda \int_0^t \sin(\lambda A_s) \, dA_s - \frac{\lambda^2}{2} \int_0^t (X_s^2 + Y_s^2) \, Z_s \, ds.
\]

and

\[
W_t = f(0) - \int_0^t f'(s) X_s \, dX_s - \int_0^t f'(s) Y_s \, dY_s - \frac{1}{2} \int_0^t f''(s)(X_s^2 + Y_s^2) \, ds,
\]

and deduce that

\[
\langle Z, W \rangle = 0.
\]
4. Show that if \( f \) solves \( f'' = f'^{2} - \lambda^{2} \) then \( Ze^{W} \) is a continuous local martingale and
\[
Ze^{W} = e^{f(0)} - \lambda \int_{0}^{t} \sin(\lambda s) e^{W_{s+}}dA_{s} - \int_{0}^{t} f'(s) Z_{s} e^{W_{s}} X_{s} dW_{s} - \int_{0}^{t} f'(s) Z_{s} e^{W_{s}} Y_{s} dY_{s}.
\]

5. Let \( r > 0 \). By using \( f(t) = -\log \cosh(\lambda(t - r)) \) deduce from the previous question that
\[
\mathbb{E} e^{\lambda A_{t}} = \frac{1}{\cosh(\lambda r)}.
\]

**Exercise 3** (Criterion for a stochastic differential equation). Set \( d = 1 \). Let \( \sigma, b \) be two functions \( \mathbb{R} \to \mathbb{R} \) such that for some finite constant \( C < \infty \) and for all \( x, y \in \mathbb{R} \),
\[
|\sigma(x) - \sigma(y)| \leq C \sqrt{x - y} \quad \text{and} \quad |b(x) - b(y)| \leq C|x - y|
\]
The goal of this exercise is to prove pathwise uniqueness for the stochastic differential equation
\[
dX_{t} = \sigma(X_{t}) dB_{t} + b(X_{t}) dt.
\]
A solution \( X \) is a continuous semi-martingale with canonical decomposition \( X = X_{0} + M + V \) with \( X_{0} \in L^{2} \), local martingale part \( M := \int_{0}^{\cdot} \sigma(X_{s}) dB_{s} \), and finite variation part \( V := \int_{0}^{\cdot} b(X_{s}) ds \). Note that the continuity of \( \sigma, X, b \) gives that almost surely, for all \( t \geq 0 \), \( s \to \sigma(X_{s}) + b(X_{s}) \) is locally bounded.

1. Let \( Z \) be a continuous semi-martingale such that \( \langle Z \rangle = \int_{0}^{\cdot} \varphi_{s} ds \) for a progressive process \( \varphi \) such that \( 0 \leq \varphi \leq C|Z| \) for some constant \( C < \infty \). Prove that for all \( t \geq 0 \) and all \( a > 0 \),
\[
\mathbb{E} \int_{0}^{t} \frac{1_{0 < |Z_{s}| < a}}{|Z_{s}|} d\langle Z \rangle_{s} \leq Ct.
\]

2. Deduce from the preceding question that for all \( t \geq 0 \),
\[
\lim_{n \to \infty} n \mathbb{E} \int_{0}^{t} 1_{0 < |Z_{s}| < \frac{1}{n}} d\langle Z \rangle_{s} = 0.
\]

3. For all \( n \geq 1 \), \( x \in \mathbb{R} \), let us define \( g_{n}(x) := 2n(1 + nx) 1_{x \in [-1, 0)} + 2n 1_{x = 0} + 2n(1 - nx) 1_{x \in (0, 1]} \). Let \( f_{n} : \mathbb{R} \to \mathbb{R} \) be the twice differentiable function such that \( f''_{n} = g_{n} \) and \( f_{n}(0) = f'_{n}(0) = 0 \). Show that for all \( x \in \mathbb{R} \), the following properties hold true:
   (a) \( f'_{n}(x) \in [-1, 1] \) and \( \lim_{n \to \infty} f'_{n}(x) = \text{sign}(x) := 1_{x > 0} - 1_{x < 0} \);
   (b) \( |f_{n}(x)| \leq |x| \) and \( \lim_{n \to \infty} f_{n}(x) = |x| \).

4. By using Itô formula, prove that for all continuous semi-martingale \( Z = (Z_{t})_{t \geq 0} \), all \( t \geq 0 \),
\[
\int_{0}^{t} 1_{Z_{s} = 0} d\langle Z \rangle_{s} = 0.
\]

5. From now on, let \( X \) and \( X' \) be two solutions of (SDE) on \( (\Omega, \mathcal{F}, (\mathcal{F}_{t})_{t \geq 0}, \mathbb{P}) \) and with respect to the Brownian motion \( B \). Show that for all \( t \geq 0 \),
\[
\langle X - X' \rangle_{t} = \int_{0}^{t} (\sigma(X_{s}) - \sigma(X'_{s}))^{2} ds.
\]

6. By using the assumption on \( \sigma \), deduce from the preceding questions that for all \( t \geq 0 \),
\[
\lim_{n \to \infty} \mathbb{E} \int_{0}^{t} g_{n}(X_{s} - X'_{s}) d\langle X - X' \rangle_{s} = 0.
\]

2/3
7. Set $Z := X - X'$. From now on, let $T$ be a stopping time such that the semi-martingale $(Z_{t∧T})_{t≥0}$ is bounded. By using notably the assumption on $σ$, prove that for all $t ≥ 0$, $n ≥ 1$,

$$E(f_n(Z_{t∧T})) = E(f_n(Z_0)) + E \int_0^{t∧T} f''_n(Z_s)(b(X_s) - b(X'_s))ds + \frac{1}{2} E \int_0^{t∧T} f'''_n(Z_s)d⟨Z⟩_s.$$

8. Deduce from the preceding questions and the assumption on $b$ that for all $t ≥ 0$,

$$E(|X_{t∧T} - X'_{t∧T}|) = E(|X_0 - X'_0|) + E \int_0^{t∧T} (b(X_s) - b(X'_s))\text{sign}(X_s - X'_s)ds.$$

9. By using the Grönwall lemma, deduce that if $X_0 = X'_0$ then $X_t = X'_t$ for all $t ≥ 0$. 