Exam 2018/2019

January 9, 2019, from 09:00 to 12:00
Documents allowed, Internet not allowed
Do what you can, and do not worry

\((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\) is a filtered probability space, with complete and right continuous filtration. 
\(B := (B_t)_{t \geq 0}\) is a \(d\)-dimensional Brownian motion issued from the origin, \(d \geq 1\).
If \(Z\) is a semi-martingale, we denote by \(\langle Z \rangle\) the increasing process of its local martingale part.
If \(Z = Z_0 + M + V\), do not confuse \(\langle Z \rangle = \langle M \rangle\) with the finite variation part \(V\) of \(Z\).

**Exercise 1** (Nature of an integral). Set \(d = 1\). Let us consider the following integral, for \(t \geq 0\),

\[
I_t := \int_0^t B_s \, ds.
\]

2. Show that \(d(I_t) = B_t \, dt + tdB_t\);
3. Deduce from the preceding question that \(I_t = \int_0^t (t - s) \, dB_s\) for all \(t \geq 0\);
4. Deduce from the preceding question that \(I_t \sim \mathcal{N}(0, \frac{1}{3} t^3)\) for all \(t \geq 0\);
5. For all \(t \geq 0\), \(n \geq 1\), \(0 \leq k \leq n\), let us define \(t_k := \frac{k}{n} t\). Show that

\[
\sum_{k=0}^{n-1} B_{t_k}(t_{k+1} - t_k) = \frac{t}{n} \sum_{j=0}^{n-2} (n - j - 1)(B_{t_{j+1}} - B_{t_j}).
\]
6. Deduce from the preceding question another proof that \(I_t \sim \mathcal{N}(0, \frac{1}{3} t^3)\) for all \(t \geq 0\);
7. Is the process \((I_t)_{t \geq 0}\) a martingale?

**Exercise 2** (Study of a special process). Set \(d = 2\). For all \(t \geq 0\), we write \(B_t = (X_t, Y_t)\) and

\[
A_t := \int_0^t X_s \, dY_s - \int_0^t Y_s \, dX_s.
\]

1. Show that \(\langle A \rangle = \int_0^t (X_s^2 + Y_s^2) \, ds\) and that the process \(A\) is a square integrable martingale;
2. From now on let \(\lambda > 0\). Show that for all \(t \geq 0\),

\[
\mathbb{E}e^{\lambda A_t} = \mathbb{E} \cos(\lambda A_t).
\]
3. From now on, let \(f: \mathbb{R}_+ \rightarrow \mathbb{R}\) be \(\mathcal{C}_2\), and let us define the continuous semi-martingales

\[
(Z_t)_{t \geq 0} := (\cos(\lambda A_t))_{t \geq 0}\quad \text{and}\quad (W_t)_{t \geq 0} := \left\{ -\frac{f'(t)}{2} (X_t^2 + Y_t^2) + f(t) \right\}_{t \geq 0}.
\]

Show that for all \(t \geq 0\),

\[
Z_t = 1 - \lambda \int_0^t \sin(\lambda A_s) \, dA_s - \frac{\lambda^2}{2} \int_0^t (X_s^2 + Y_s^2) Z_s \, ds.
\]
and

\[
W_t = f(0) - \int_0^t f'(s) X_s \, dX_s - \int_0^t f'(s) Y_s \, dY_s - \frac{1}{2} \int_0^t f''(s)(X_s^2 + Y_s^2) \, ds,
\]
and deduce that

\[
\langle Z, W \rangle = 0.
\]
4. Show that if \( f \) solves \( f'' = f'^2 - \lambda^2 \) then \( Ze^W \) is a continuous local martingale and

\[
Z_t e^{W_t} = e^{f(0) - \lambda t} \int_0^t \sin(\lambda A_s) e^{W_s} dA_s - \int_0^t f'(s) Z_s e^{W_s} X_s dX_s - \int_0^t f''(s) Z_s e^{W_s} Y_s dY_s.
\]

5. Let \( r > 0 \). By using \( f(t) = -\log \cosh(\lambda(r-t)) \) deduce from the previous question that

\[
\mathbb{E} e^{\lambda A_r} = \frac{1}{\cosh(\lambda r)}.
\]

**Exercise 3** (Criterion for a stochastic differential equation). Set \( d = 1 \). Let \( \sigma, b \) be two functions \( \mathbb{R} \to \mathbb{R} \) such that for some finite constant \( C < \infty \) and for all \( x, y \in \mathbb{R} \),

\[
|\sigma(x) - \sigma(y)| \leq C \sqrt{x-y} \quad \text{and} \quad |b(x) - b(y)| \leq C|x - y|
\]

The goal of this exercise is to prove pathwise uniqueness for the stochastic differential equation

\[
dX_t = \sigma(X_t)dB_t + b(X_t)dt.
\]

A solution \( X \) is a continuous semi-martingale with canonical decomposition \( X = X_0 + M + V \) with \( X_0 \in L^2 \), local martingale part \( M := \int_0^\cdot \sigma(X_s)dB_s \), and finite variation part \( V := \int_0^\cdot b(X_s)ds \). Note that the continuity of \( \sigma, X, b \) gives that almost surely, for all \( t \geq 0 \), \( s \mapsto \sigma(X_s) + b(X_s) \) is locally bounded.

1. Let \( Z \) be a continuous semi-martingale such that \( \langle Z \rangle = \int_0^\cdot \varphi_t ds \) for a progressive process \( \varphi \) such that \( 0 \leq \varphi \leq C|Z| \) for some constant \( C < \infty \). Prove that for all \( t \geq 0 \) and all \( a > 0 \),

\[
\mathbb{E} \int_0^t 1_{|Z_s| \leq a} \, d\langle Z \rangle_s \leq Ct.
\]

2. Deduce from the preceding question that for all \( t \geq 0 \),

\[
\lim_{n \to \infty} n \mathbb{E} \int_0^t 1_{|Z_s| \leq \frac{a}{n}} \, d\langle Z \rangle_s = 0.
\]

3. For all \( n \geq 1 \), \( x \in \mathbb{R} \), let us define \( g_n(x) := 2n(1 + nx)1_{x \in [-\frac{1}{n}, 0]} + 2n1_{x=0} + 2n1_{x=1}1_{x \in (0, \frac{1}{n}]} \).

Let \( f_n : \mathbb{R} \to \mathbb{R} \) be the twice differentiable function such that \( f''_n = g_n \) and \( f_n(0) = f'_n(0) = 0 \).

Show that for all \( x \in \mathbb{R} \), the following properties hold true:

a) \( f'_n(x) \in [-1, 1] \) and \( \lim_{n \to \infty} f'_n(x) = \text{sign}(x) := 1_{x>0} - 1_{x<0} \);

b) \( |f_n(x)| \leq |x| \) and \( \lim_{n \to \infty} f_n(x) = |x| \).

4. By using Itô's formula, prove that for all continuous semi-martingale \( Z = (Z_t)_{t \geq 0} \), all \( t \geq 0 \),

\[
\int_0^t 1_{Z_s = 0} \, d\langle Z \rangle_s = 0.
\]

5. From now on, let \( X \) and \( X' \) be two solutions of (SDE) on \( (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}) \) and with respect to the Brownian motion \( B \). Show that for all \( t \geq 0 \),

\[
\langle X - X' \rangle_t = \int_0^t (\sigma(X_s) - \sigma(X'_s))^2 \, ds.
\]

6. By using the assumption on \( \sigma \), deduce from the preceding questions that for all \( t \geq 0 \),

\[
\lim_{n \to \infty} \mathbb{E} \int_0^t g_n(X_s - X'_s) \, d\langle X - X' \rangle_s = 0.
\]

7. Set \( Z := X - X' \). From now on, let \( T \) be a stopping time such that the semi-martingale \( (Z_{t \wedge T})_{t \geq 0} \) is bounded. By using notably the assumption on \( \sigma \), prove that for all \( t \geq 0 \), \( n \geq 1 \),

\[
\mathbb{E}(f_n(Z_{t \wedge T})) = \mathbb{E}(f_n(Z_0)) + \mathbb{E} \int_0^{t \wedge T} f''_n(Z_s)(b(X_s) - b(X'_s))ds + \frac{1}{2} \mathbb{E} \int_0^{t \wedge T} f''_n(Z_s) \, d\langle Z \rangle_s.
\]

8. Deduce from the preceding questions and the assumption on \( b \) that for all \( t \geq 0 \),

\[
\mathbb{E}(|X_{t \wedge T} - X'_{t \wedge T}|) = \mathbb{E}(|X_0 - X'_0|) + \mathbb{E} \int_0^{t \wedge T} (b(X_s) - b(X'_s)) \text{sign}(X_s - X'_s) \, ds.
\]

9. By using the Grönwall lemma, deduce that if \( X_0 = X'_0 \) then \( X_t = X'_t \) for all \( t \geq 0 \).