Exercise 1 (Nature of an integral). Set $d = 1$. Let us consider the following integral, for $t \geq 0$,

$$I_t := \int_0^t B_s \, ds.$$ 

2. Show that $d(I_t) = B_t \, dt + t \, dB_t$;
3. Deduce from the preceding question that $I_t = \int_0^t (t-s) \, dB_s$ for all $t \geq 0$;
4. Deduce from the preceding question that $I_t \sim \mathcal{N}(0, \frac{1}{3} t^3)$ for all $t \geq 0$;
5. For all $t \geq 0$, $n \geq 1$, $0 \leq k \leq n$, let us define $t_k := \frac{k}{n} t$. Show that

$$\sum_{k=0}^{n-1} B_{t_{k+1}} - B_{t_k} = \frac{t}{n} \sum_{j=0}^{n-2} (n - j - 1)(B_{t_{j+1}} - B_{t_j}).$$

6. Deduce from the preceding question another proof that $I_t \sim \mathcal{N}(0, \frac{1}{3} t^3)$ for all $t \geq 0$;
7. Is the process $(I_t)_{t \geq 0}$ a martingale?

Exercise 2 (Study of a special process). Set $d = 2$. For all $t \geq 0$, we write $B_t = (X_t, Y_t)$ and

$$A_t := \int_0^t X_s \, dY_s - \int_0^t Y_s \, dX_s.$$ 

1. Show that $\langle A \rangle = \int_0^t (X_s^2 + Y_s^2) \, ds$ and that the process $A$ is a square integrable martingale;
2. From now on let $\lambda > 0$. Show that for all $t \geq 0$,

$$\mathbb{E}e^{\lambda A_t} = \mathbb{E}\cos(\lambda A_t).$$

3. From now on, let $f : \mathbb{R}_+ \to \mathbb{R}$ be $C^2$, and let us define the continuous semi-martingales

$$(Z_t)_{t \geq 0} := (\cos(\lambda A_t))_{t \geq 0} \quad \text{and} \quad (W_t)_{t \geq 0} := \left(-\frac{f'(t)}{2}(X_t^2 + Y_t^2) + f(t) \right)_{t \geq 0}.$$ 

Show that for all $t \geq 0$,

$$Z_t = 1 - \lambda \int_0^t \sin(\lambda A_s) \, dA_s - \frac{\lambda^2}{2} \int_0^t (X_s^2 + Y_s^2) \, Z_s \, ds,$$

and

$$W_t = f(0) - \int_0^t f'(s) X_s \, dX_s - \int_0^t f'(s) Y_s \, dY_s - \frac{1}{2} \int_0^t f''(s) (X_s^2 + Y_s^2) \, ds,$$

and deduce that

$$\langle Z, W \rangle = 0.$$
Exercise 3

4. Show that if $f$ solves $f'' = f'^2 - \lambda^2$ then $Ze^W$ is a continuous local martingale and
\[ Ze^W_t = e^{f(0)} - \lambda \int_0^t \sin(\lambda As) e^{W_s} dA_s - \int_0^t f'(s) Ze^W_s X_s dX_s - \int_0^t f'(s) Ze^W_s Y_s dY_s. \]

5. Let $r > 0$. By using $f(t) = -\log \cosh(\lambda (r - t))$ deduce from the previous question that
\[ \mathbb{E} e^{i\lambda A_t} = \frac{1}{\cosh(\lambda r)}. \]

Exercise 3 (Criterion for a stochastic differential equation). Set $d = 1$. Let $\sigma, b$ be two functions $\mathbb{R} \to \mathbb{R}$ such that for some finite constant $C < \infty$ and for all $x, y \in \mathbb{R}$,
\[ |\sigma(x) - \sigma(y)| \leq C \sqrt{|x - y|} \quad \text{and} \quad |b(x) - b(y)| \leq C|x - y| \]

The goal of this exercise is to prove pathwise uniqueness for the stochastic differential equation
\[ dX_t = \sigma(X_t) dB_t + b(X_t) dt. \]

A solution $X$ is a continuous semi-martingale with canonical decomposition $X = X_0 + M + V$ with $X_0 \in L^2$, local martingale part $M := \int_0^t \sigma(s) dB_s$, and finite variation part $V := \int_0^t b(X_s) ds$. Note that the continuity of $\sigma, X, b$ gives that almost surely, for all $t \geq 0$, $s \mapsto \sigma(X_s) + b(X_s)$ is locally bounded.

1. Let $Z$ be a continuous semi-martingale such that $(Z) = \int_0^t \varphi_s ds$ for a progressive process $\varphi$ such that $0 \leq \varphi \leq C|Z|$ for some constant $C < \infty$. Prove that for all $t \geq 0$ and all $a > 0$,
\[ \mathbb{E} \int_0^t 1_{0 < |Z| < a} d(Z)_s \leq Ct. \]

2. Deduce from the preceding question that for all $t \geq 0$,
\[ \lim_{n \to \infty} n \mathbb{E} \int_0^t 1_{0 < |Z| < \frac{a}{n}} d(Z)_s = 0. \]

3. For all $n \geq 1, x \in \mathbb{R}$, let us define $g_n(x) := 2n(1 + nx)1_{x \in (0, \frac{1}{n})} + 2n1_{x=0} + 2n(1 - nx)1_{x \in (0, \frac{1}{n})}$. Let $f_n : \mathbb{R} \to \mathbb{R}$ be the twice differentiable function such that $f_n'' = g_n$ and $f_n(0) = f''_n(0) = 0$.

Show that for all $x \in \mathbb{R}$, the following properties hold true:
(a) $f_n'(x) \in [-1, 1]$ and $\lim_{n \to \infty} f_n(x) = \text{sgn}(x) := 1_{x > 0} - 1_{x < 0};$
(b) $|f_n(x)| \leq |x|$ and $\lim_{n \to \infty} f_n(x) = |x|$.

4. By using Itô formula, prove that for all continuous semi-martingale $Z = (Z_t)_{t \geq 0}$, all $t \geq 0$,
\[ \int_0^t 1_{Z_s = 0} d(Z)_s = 0. \]

5. From now on, let $X$ and $X'$ be two solutions of (SDE) on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ and with respect to the Brownian motion $B$. Show that for all $t \geq 0$,
\[ \langle X - X' \rangle_t = \int_0^t (\sigma(X_s) - \sigma(X'_s))^2 ds. \]

6. By using the assumption on $\sigma$, deduce from the preceding questions that for all $t \geq 0$,
\[ \lim_{n \to \infty} \mathbb{E} \int_0^t g_n(X_s - X'_s) d(X - X')_s = 0. \]

7. Set $Z := X - X'$. From now on, let $T$ be a stopping time such that the semi-martingale $(Z_{t \wedge T})_{t \geq 0}$ is bounded. By using notably the assumption on $\sigma$, prove that for all $t \geq 0$, $n \geq 1$,
\[ \mathbb{E}(f_n(Z_{t \wedge T})) = \mathbb{E}(f_n(Z_0)) + \mathbb{E} \int_0^{t \wedge T} f_n'(Z_s)(b(X_s) - b(X'_s)) ds + \frac{1}{2} \mathbb{E} \int_0^{t \wedge T} f_n''(Z_s) d(Z)_s. \]

8. Deduce from the preceding questions and the assumption on $b$ that for all $t \geq 0$,
\[ \mathbb{E}(|X_{t \wedge T} - X'_{t \wedge T}|) = \mathbb{E}(|X_0 - X'_0|) + \mathbb{E} \int_0^{t \wedge T} (b(X_s) - b(X'_s)) \text{sgn}(X_s - X'_s) ds. \]

9. By using the Grönwall lemma, deduce that if $X_0 = X'_0$ then $X_t = X'_t$ for all $t \geq 0$. 

2/2