Exercise 1 (Nature of an integral). Set \( d = 1 \). Let us consider the following integral, for \( t \geq 0 \),
\[
I_t = \int_0^t B_s \, ds.
\]
2. Show that \( d(tB_t) = B_t \, dt + t \, dB_t \);
3. Deduce from the preceding question that \( I_t = \int_0^t (t - s) \, dB_s \) for all \( t \geq 0 \);
4. Deduce from the preceding question that \( I_t \sim \mathcal{N}(0, \frac{1}{2} t^2) \) for all \( t \geq 0 \);
5. For all \( t \geq 0, n \geq 1, 0 \leq k \leq n \), let us define \( t_k = \frac{k}{n} t \). Show that
\[
\sum_{k=0}^{n-1} B_{t_k} (t_{k+1} - t_k) = \frac{t}{n} \sum_{j=0}^{n-2} (n - j - 1)(B_{t_{j+1}} - B_{t_j}).
\]
6. Deduce from the preceding question another proof that \( I_t \sim \mathcal{N}(0, \frac{1}{3} t^3) \) for all \( t \geq 0 \);
7. Is the process \((I_t)_{t \geq 0}\) a martingale?

Exercise 2 (Study of a special process). Set \( d = 2 \). For all \( t \geq 0 \), we write \( B_t = (X_t, Y_t) \) and
\[
A_t = \int_0^t X_s \, dY_s - \int_0^t Y_s \, dX_s.
\]
1. Show that \( \langle A \rangle = \int_0^t (X_s^2 + Y_s^2) \, ds \) and that the process \( A \) is a square integrable martingale;
2. From now on let \( \lambda > 0 \). Show that for all \( t \geq 0 \),
\[
E e^{i \lambda A_t} = E \cos(\lambda A_t).
\]
3. From now on, let \( f : \mathbb{R} \to \mathbb{R} \) be \( \sigma^2 \), and let us define the continuous semi-martingales
\[
(Z_t)_{t \geq 0} = (\cos(\lambda A_t))_{t \geq 0} \quad \text{and} \quad (W_t)_{t \geq 0} = \left( -\frac{f'(t)}{2} (X_t^2 + Y_t^2) + f(t) \right)_{t \geq 0}.
\]
Show that for all \( t \geq 0 \),
\[
Z_t = 1 - \lambda \int_0^t \sin(\lambda A_s) \, dA_s - \frac{\lambda^2}{2} \int_0^t (X_s^2 + Y_s^2) \, Z_s \, ds.
\]
and
\[
W_t = f(0) - \int_0^t f'(s) X_s \, dX_s - \int_0^t f'(s) Y_s \, dY_s - \frac{1}{2} \int_0^t f''(s) (X_s^2 + Y_s^2) \, ds,
\]
and deduce that
\[
\langle Z, W \rangle = 0.
\]
4. Show that if \( f \) solves \( f'' = f'^2 - \lambda^2 \) then \( Ze^W \) is a continuous local martingale and
\[
Z_t e^{W_t} = e^{f(0)} - \lambda \int_0^t \sin(\lambda A_s) e^{W_s} dA_s - \int_0^t f'(s) Z_s e^{W_s} X_s dX_s - \int_0^t f'(s) Z_s e^{W_s} Y_s dY_s.
\]

5. Let \( r > 0 \). By using \( f(t) = -\log \cosh(\lambda(r-t)) \) deduce from the previous question that
\[
\mathbb{E} e^{\lambda A_t} = \frac{1}{\cosh(\lambda r)}.
\]

**Exercise 3** (Criterion for a stochastic differential equation). Set \( d = 1 \). Let \( \sigma, b \) be two functions \( \mathbb{R} \to \mathbb{R} \) such that for some finite constant \( C < \infty \) and for all \( x, y \in \mathbb{R} \),
\[
|\sigma(x) - \sigma(y)| \leq C \sqrt{x - y} \quad \text{and} \quad |b(x) - b(y)| \leq C|x - y|.
\]
The goal of this exercise is to prove pathwise uniqueness for the stochastic differential equation
\[
dX_t = \sigma(X_t) dB_t + b(X_t) dt.
\]
A solution \( X \) is a continuous semi-martingale with canonical decomposition \( X = X_0 + M + V \) with \( X_0 \in L^2 \),
local martingale part \( M = \int_0^t \sigma(X_s) dB_s \), and finite variation part \( V = \int_0^t b(X_s) ds \). Note that the continuity of \( \sigma, X, b \) gives that almost surely, for all \( t \geq 0, s = \sigma(X_s) + b(X_s) \) is locally bounded.

1. Let \( Z \) be a continuous semi-martingale such that \( \langle Z \rangle = \int_0^t \varphi_s ds \) for a progressive process \( \varphi \) such that \( 0 \leq \varphi \leq C|Z| \) for some constant \( C < \infty \). Prove that for all \( t \geq 0 \) and all \( a > 0 \),
\[
\mathbb{E} \int_0^t 1_{|Z_s| \leq a} d\langle Z \rangle_s \leq Ct.
\]

2. Deduce from the preceding question that for all \( t \geq 0 \),
\[
\lim_{n \to \infty} n \mathbb{E} \int_0^t 1_{|Z_s| \leq \frac{a}{n}} d\langle Z \rangle_s = 0.
\]

3. For all \( n \geq 1, x \in \mathbb{R} \), let us define \( g_n(x) = 2n(1 + nx) 1_{x \in [-\frac{1}{n},0]} + 2n 1_{x=0} + 2n(1 - nx) 1_{x \in (0, \frac{1}{n})}. \)
Let \( f_n : \mathbb{R} \to \mathbb{R} \) be the twice differentiable function such that \( f''_n = g_n \) and \( f'_n(0) = f''_n(0) = 0 \).
Show that for all \( x \in \mathbb{R} \), the following properties hold true:
(a) \( f''_n(x) \in [-1,1] \) and \( \lim_{n \to \infty} f''_n(x) = \text{sign}(x) = 1_{x>0} - 1_{x<0} \);
(b) \( |f'_n(x)| \leq |x| \) and \( \lim_{n \to \infty} f'_n(x) = |x| \).

4. By using Itô formula, prove that for all continuous semi-martingale \( Z = (Z_t)_{t \geq 0}, \) all \( t \geq 0, \)
\[
\int_0^t 1_{Z_t = 0} d\langle Z \rangle_s = 0.
\]

5. From now on, let \( X \) and \( X' \) be two solutions of (SDE) on \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\) and with respect to the Brownian motion \( B \). Show that for all \( t \geq 0, \)
\[
\langle X - X' \rangle_t = \int_0^t (\sigma(X_s) - \sigma(X'_s))^2 ds.
\]

6. By using the assumption on \( \sigma \), deduce from the preceding questions that for all \( t \geq 0, \)
\[
\lim_{n \to \infty} \mathbb{E} \int_0^t g_n(X_s - X'_s) d\langle X - X' \rangle_s = 0.
\]
7. Set $Z = X - X'$. From now on, let $T$ be a stopping time such that the semi-martingale $(Z_{t∧T})_{t≥0}$ is bounded. By using notably the assumption on $\sigma$, prove that for all $t ≥ 0$, $n ≥ 1$,

$$
E(f_n(Z_{t∧T})) = E(f_n(Z_0)) + E\int_0^{t∧T} f'_n(Z_s)(b(X_s) - b(X'_s))ds + \frac{1}{2} E\int_0^{t∧T} f''_n(Z_s)d\langle Z \rangle_s.
$$

8. Deduce from the preceding questions and the assumption on $b$ that for all $t ≥ 0$,

$$
E(|X_{t∧T} - X'_{t∧T}|) = E(|X_0 - X'_0|) + E\int_0^{t∧T} (b(X_s) - b(X'_s))\text{sign}(X_s - X'_s)ds.
$$

9. By using the Grönwall lemma, deduce that if $X_0 = X'_0$ then $X_t = X'_t$ for all $t ≥ 0$. 
