Exercise 1 (Nature of an integral). Set $d = 1$. Let us consider the following integral, for $t \geq 0$,

$$I_t := \int_0^t B_s \, ds.$$

2. Show that $d(tB_t) = B_t \, dt + t dB_t$;
3. Deduce from the preceding question that $I_t = \int_0^t (t - s) dB_s$ for all $t \geq 0$;
4. Deduce from the preceding question that $I_t \sim \mathcal{N}(0, \frac{1}{2} t^2)$ for all $t \geq 0$;
5. For all $t \geq 0$, $n \geq 1$, $0 \leq k \leq n$, let us define $t_k := \frac{k}{n} t$. Show that

$$\sum_{k=0}^{n-1} B_t^k (t_{k+1} - t_k) = \frac{t}{n} \sum_{j=0}^{n-2} (n - j - 1)(B_{t_{j+1}} - B_{t_j}).$$

6. Deduce from the preceding question another proof that $I_t \sim \mathcal{N}(0, \frac{1}{3} t^3)$ for all $t \geq 0$;
7. Is the process $(I_t)_{t \geq 0}$ a martingale?

Exercise 2 (Study of a special process). Set $d = 2$. For all $t \geq 0$, we write $B_t = (X_t, Y_t)$ and

$$A_t := \int_0^t X_s \, dY_s - \int_0^t Y_s \, dX_s.$$

1. Show that $(A) = \int_0^t (X_s^2 + Y_s^2) \, ds$ and that the process $A$ is a square integrable martingale;
2. From now on let $\lambda > 0$. Show that for all $t \geq 0$,

$$\mathbb{E}e^{\lambda A_t} = \mathbb{E}\cos(\lambda A_t).$$

3. From now on, let $f : \mathbb{R}^+ \to \mathbb{R}$ be 'c²', and let us define the continuous semi-martingales

$$(Z_t)_{t \geq 0} := (\cos(\lambda A_t))_{t \geq 0} \quad \text{and} \quad (W_t)_{t \geq 0} := \left(-\frac{f'(t)}{2} (X_t^2 + Y_t^2) + f(t)\right)_{t \geq 0}.$$ 

Show that for all $t \geq 0$,

$$Z_t = 1 - \lambda \int_0^t \sin(\lambda A_s) \, dA_s - \frac{\lambda^2}{2} \int_0^t (X_s^2 + Y_s^2) \, Z_s \, ds,$$

and

$$W_t = f(0) - \int_0^t f'(s) X_s \, dX_s - \int_0^t f'(s) Y_s \, dY_s - \frac{1}{2} \int_0^t f''(s) (X_s^2 + Y_s^2) \, ds,$$

and deduce that

$$\langle Z, W \rangle = 0.$$
4. Show that if \( f \) solves \( f'' = f'^2 - \lambda^2 \) then \( Z e^W \) is a continuous local martingale and
\[
Z e^W = e^{f(0)} - \lambda \int_0^t \sin(\lambda A_s) e^W dA_s - \int_0^t f'(s) Z e^W X_s dX_s - \int_0^t f''(s) Z e^W Y_s dY_s.
\]

5. Let \( r > 0 \). By using \( f(t) = -\log \cosh(\lambda(t-r)) \) deduce from the previous question that
\[
\mathbb{E} e^{\lambda A_t} = \frac{1}{\cosh(\lambda r)}.
\]

**Exercise 3** (Criterion for a stochastic differential equation). Set \( d = 1 \). Let \( \sigma, b \) be two functions \( \mathbb{R} \to \mathbb{R} \) such that for some finite constant \( C < \infty \) and for all \( x, y \in \mathbb{R} \),
\[
|\sigma(x) - \sigma(y)| \leq C \sqrt{x - y} \quad \text{and} \quad |b(x) - b(y)| \leq C|x - y|
\]
The goal of this exercise is to prove pathwise uniqueness for the stochastic differential equation
\[
dX_t = \sigma(X_t)dB_t + b(X_t)dt.
\]
(A SDE)

A solution \( X \) is a continuous semi-martingale with canonical decomposition \( X = X_0 + M + V \) with \( X_0 \in L^2 \), local martingale part \( M := \int_0^\cdot \sigma(X_s)dB_s \), and finite variation part \( V := \int_0^\cdot b(X_s)ds \). Note that the continuity of \( \sigma, X, b \) gives that almost surely, for all \( t \geq 0 \), \( s \mapsto \sigma(X_s) + b(X_s) \) is locally bounded.

1. Let \( Z \) be a continuous semi-martingale such that \( \langle Z \rangle = \int_0^\cdot \varphi, ds \) for a progressive process \( \varphi \) such that \( 0 \leq \varphi \leq C|Z| \) for some constant \( C < \infty \). Prove that for all \( t \geq 0 \) and all \( a > 0 \),
\[
\mathbb{E} \int_0^t 1_{0 < \langle Z \rangle \leq a} d\langle Z \rangle_s \leq Ct.
\]

2. Deduce from the preceding question that for all \( t \geq 0 \),
\[
\lim_{n \to \infty} n\mathbb{E} \int_0^t 1_{0 < \langle Z \rangle \leq \frac{1}{n}} d\langle Z \rangle_s = 0.
\]

3. For all \( n \geq 1, x \in \mathbb{R} \), let us define \( g_n(x) := 2n(1 + nx)1_{x \in [-\frac{1}{n}, 0]} + 2n1_{x=0} + 2n(1 - nx)1_{x \in (0, \frac{1}{n}]} \)
Let \( f_n : \mathbb{R} \to \mathbb{R} \) be the twice differentiable function such that \( f_n'' = g_n \) and \( f_n(0) = f_n'(0) = 0 \).
Show that for all \( x \in \mathbb{R} \), the following properties hold true:
(a) \( f_n'(x) \in [-1, 1] \) and \( \lim_{n \to \infty} f_n'(x) = \text{sign}(x) := 1_{x>0} - 1_{x<0} \)
(b) \( |f_n(x)| \leq |x| \) and \( \lim_{n \to \infty} f_n(x) = |x| \).

4. By using Itô formula, prove that for all continuous semi-martingale \( Z = (Z_t)_{t \geq 0} \), all \( t \geq 0 \),
\[
\int_0^t 1_{Z_s = 0} d\langle Z \rangle_s = 0.
\]

5. From now on, let \( X \) and \( X' \) be two solutions of (SDE) on \( (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}) \) and with respect to the Brownian motion \( B \). Show that for all \( t \geq 0 \),
\[
\langle X - X' \rangle_t = \int_0^t (\sigma(X_s) - \sigma(X'_s))^2 ds.
\]

6. By using the assumption on \( \sigma \), deduce from the preceding questions that for all \( t \geq 0 \),
\[
\lim_{n \to \infty} \mathbb{E} \int_0^t g_n(X_s - X'_s) d\langle X - X' \rangle_s = 0.
\]
7. Set $Z := X - X'$. From now on, let $T$ be a stopping time such that the semi-martingale $(Z_{t∧T})_{t ≥ 0}$ is bounded. By using notably the assumption on $σ$, prove that for all $t ≥ 0$, $n ≥ 1$,

$$E(f_n(Z_{t∧T})) = E(f_n(Z_0)) + E\int_0^{t∧T} f'_n(Z_s)(b(X_s) - b(X'_s))ds + \frac{1}{2} E\int_0^{t∧T} f''_n(Z_s)d\langle Z \rangle_s.$$ 

8. Deduce from the preceding questions and the assumption on $b$ that for all $t ≥ 0$,

$$E(|X_{t∧T} - X'_{t∧T}|) = E(|X_0 - X'_0|) + E\int_0^{t∧T} (b(X_s) - b(X'_s))\text{sign}(X_s - X'_s)ds.$$ 

9. By using the Grönwall lemma, deduce that if $X_0 = X'_0$ then $X_t = X'_t$ for all $t ≥ 0$. 