Exam 2018/2019

January 9, 2019, from 09:00 to 12:00
Documents allowed, Internet not allowed
Do what you can, and do not worry

\((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\) is a filtered probability space, with complete and right continuous filtration. \(B := (B_t)_{t \geq 0}\) is a \(d\)-dimensional Brownian motion issued from the origin, \(d \geq 1\).

If \(Z\) is a semi-martingale, we denote by \((Z)\) the increasing process of its local martingale part. If \(Z = Z_0 + M + V\), do not confuse \((Z) = (M)\) with the finite variation part \(V\) of \(Z\).

**Exercise 1** (Nature of an integral). Set \(d = 1\). Let us consider the following integral, for \(t \geq 0\),

\[
I_t := \int_0^t B_s \, ds.
\]


2. Show that \(\frac{d(B_t)}{t} = B_t \, dt + t \, dB_t\);

3. Deduce from the preceding question that \(I_t = \int_0^t (t - s) \, dB_s\) for all \(t \geq 0\);

4. Deduce from the preceding question that \(I_t \sim \mathcal{N}(0, \frac{1}{3} t^3)\) for all \(t \geq 0\);

5. For all \(t \geq 0, n \geq 1, 0 \leq k \leq n\), let us define \(t_k := \frac{k}{n} t\). Show that

\[
\frac{1}{n} \sum_{k=0}^{n-1} B_{t_k} (t_{k+1} - t_k) = \frac{1}{n} \sum_{j=0}^{n-2} (n - j - 1) (B_{t_{j+1}} - B_{t_j}).
\]

6. Deduce from the preceding question another proof that \(I_t \sim \mathcal{N}(0, \frac{1}{3} t^3)\) for all \(t \geq 0\);

7. Is the process \((I_t)_{t \geq 0}\) a martingale?

**Exercise 2** (Study of a special process). Set \(d = 2\). For all \(t \geq 0\), we write \(B_t = (X_t, Y_t)\) and

\[
A_t := \int_0^t X_s \, dY_s - \int_0^t Y_s \, dX_s.
\]

1. Show that \((A) = \int_0^t (X_s^2 + Y_s^2) \, ds\) and that the process \(A\) is a square integrable martingale;

2. From now on let \(\lambda > 0\). Show that for all \(t \geq 0\),

\[
E e^{i \lambda A_t} = E \cos(\lambda A_t).
\]

3. From now on, let \(f : \mathbb{R}_+ \rightarrow \mathbb{R}\) be \(\mathcal{C}^2\), and let us define the continuous semi-martingales

\[
(Z_t)_{t \geq 0} := (\cos(\lambda A_t))_{t \geq 0} \quad \text{and} \quad (W_t)_{t \geq 0} := \left( - \frac{f'(t)}{2} (X_t^2 + Y_t^2) + f(t) \right)_{t \geq 0}.
\]

Show that for all \(t \geq 0\),

\[
Z_t = 1 - \lambda \int_0^t \sin(\lambda A_s) \, dA_s - \frac{\lambda^2}{2} \int_0^t (X_s^2 + Y_s^2) \, dA_s.
\]

and

\[
W_t = f(0) - \int_0^t f'(s) X_s \, dX_s - \int_0^t f'(s) Y_s \, dY_s - \frac{1}{2} \int_0^t f''(s) (X_s^2 + Y_s^2) \, ds,
\]

and deduce that

\[
(Z, W) = 0.
\]
4. Show that if $f$ solves $f'' = f'^2 - \lambda^2$ then $Ze^{W_t}$ is a continuous local martingale and

$$Z_t e^{W_t} = e^{f(0) - \lambda t} \int_0^t \sin(\lambda A_s) e^{W_s} \, dA_s - \int_0^t f'(s) Z_s e^{W_s} X_s \, dX_s - \int_0^t f'(s) Z_s e^{W_s} Y_s \, dY_s.$$ 

5. Let $r > 0$. By using $f(t) = -\log \cosh(\lambda (r - t))$ deduce from the previous question that

$$\mathbb{E} e^{\lambda A_t} = \frac{1}{\cosh(\lambda r)}.$$ 

**Exercise 3** (Criterion for a stochastic differential equation). Set $d = 1$. Let $\sigma, b$ be two functions $\mathbb{R} \to \mathbb{R}$ such that for some finite constant $C < \infty$ and for all $x, y \in \mathbb{R}$,

$$|\sigma(x) - \sigma(y)| \leq C \sqrt{x - y} \quad \text{and} \quad |b(x) - b(y)| \leq C |x - y|$$

The goal of this exercise is to prove pathwise uniqueness for the stochastic differential equation

$$dX_t = \sigma(X_t) \, dB_t + b(X_t) \, dt. \quad \text{(SDE)}$$

A solution $X$ is a continuous semi-martingale with canonical decomposition $X = X_0 + M + V$ with $X_0 \in L^2$, local martingale part $M := \int_0^\cdot \sigma(X_s) \, dB_s$, and finite variation part $V := \int_0^\cdot b(X_s) \, ds$. Note that the continuity of $\sigma, X, b$ gives that almost surely, for all $t \geq 0$, $s \mapsto \sigma(X_s) + b(X_s)$ is locally bounded.

1. Let $Z$ be a continuous semi-martingale such that $\langle Z \rangle = \int_0^\cdot \varphi_s \, ds$ for a progressive process $\varphi$ such that $0 \leq \varphi \leq C |Z|$ for some constant $C < \infty$. Prove that for all $t \geq 0$ and all $a > 0$,

$$\mathbb{E} \int_0^t \frac{1_{0 < \langle Z \rangle \leq a}}{|Z_s|} \, d\langle Z \rangle_s \leq Ct.$$ 

2. Deduce from the preceding question that for all $t \geq 0$,

$$\lim_{n \to \infty} n \mathbb{E} \int_0^t 1_{0 < \langle Z \rangle \leq \frac{1}{n}} \, d\langle Z \rangle_s = 0.$$ 

3. For all $n \geq 1, x \in \mathbb{R}$, let us define $g_n(x) := 2n(1 + nx) 1_{x \in [-\frac{1}{n}, 0]} + 2n 1_{x = 0} + 2n(1 - nx) 1_{x \in (0, \frac{1}{n})}$. Let $f_n : \mathbb{R} \to \mathbb{R}$ be the twice differentiable function such that $f''_n = g_n$ and $f_n(0) = f'_n(0) = 0$. Show that for all $x \in \mathbb{R}$, the following properties hold true:

   (a) $f'_n(x) \in [-1, 1]$ and $\lim_{n \to \infty} f'_n(x) = \text{sign}(x) := 1_{x > 0} - 1_{x < 0};$

   (b) $|f_n(x)| \leq |x|$ and $\lim_{n \to \infty} f_n(x) = |x|.$

4. By using Itô formula, prove that for all continuous semi-martingale $Z = (Z_t)_{t \geq 0}$, all $t \geq 0$,

$$\int_0^t 1_{Z_s = 0} \, d\langle Z \rangle_s = 0.$$ 

5. From now on, let $X$ and $X'$ be two solutions of (SDE) on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ and with respect to the Brownian motion $B$. Show that for all $t \geq 0$,

$$\langle X - X' \rangle_t = \int_0^t (\sigma(X_s) - \sigma(X'_s))^2 \, ds.$$ 

6. By using the assumption on $\sigma$, deduce from the preceding questions that for all $t \geq 0$,

$$\lim_{n \to \infty} \mathbb{E} \int_0^t g_n(X_s - X'_s) \, d\langle X - X' \rangle_s = 0.$$
7. Set $Z := X - X'$. From now on, let $T$ be a stopping time such that the semi-martingale $(Z_{t \wedge T})_{t \geq 0}$ is bounded. By using notably the assumption on $\sigma$, prove that for all $t \geq 0$, $n \geq 1$,

$$\mathbb{E}(f_n(Z_{t \wedge T})) = \mathbb{E}(f_n(Z_0)) + \mathbb{E} \int_0^{t \wedge T} f_n'(Z_s)(b(X_s) - b(X'_s))ds + \frac{1}{2} \mathbb{E} \int_0^{t \wedge T} f_n''(Z_s)d\langle Z_s \rangle.$$

8. Deduce from the preceding questions and the assumption on $b$ that for all $t \geq 0$,

$$\mathbb{E}(|X_{t \wedge T} - X'_{t \wedge T}|) = \mathbb{E}(|X_0 - X'_0|) + \mathbb{E} \int_0^{t \wedge T} (b(X_s) - b(X'_s))\text{sign}(X_s - X'_s)ds.$$

9. By using the Grönwall lemma, deduce that if $X_0 = X'_0$ then $X_t = X'_t$ for all $t \geq 0$. 
