17. ON THE THEORY OF CONTINUOUS RANDOM PROCESSES*

Let $\mathcal{S}$ be a physical system with $n$ degrees of freedom; this means that the admissible states $x$ of $\mathcal{S}$ form a Riemannian manifold $\mathcal{R}$ of dimension $n$. The process of variation of $\mathcal{S}$ is said to be stochastically determined if under an arbitrary choice of $x$, the region $\mathcal{E}$ (in $\mathcal{R}$) and times $t'$ and $t''$ ($t' < t''$), the probability $P(t', x, t'', \mathcal{E})$ that the system in state $x$ at time $t'$ will be in one of the states of $\mathcal{E}$ at time $t''$ is defined. It is further assumed that the probability $P(t', x, t'', \mathcal{E})$ can be given by the formula

$$P(t', x, t'', \mathcal{E}) = \int_{\mathcal{E}} f(t', x, t'', y) dV_y,$$

where $dV_y$ denotes the volume element. Here $f(t', x, t'', y)$ has to satisfy the following fundamental equations:

$$\int_{\mathcal{R}} f(t', x, t'', y) dV_y = 1,$$

$$f(t_1, x, t_3, y) = \int_{\mathcal{R}} f(t_1, x, t_2, y) f(t_2, z, t_3, y) dV_z, \quad t_1 < t_2 < t_3.$$

The integral equation (3) was studied by Smolukhovski and then by other authors. In the paper 'Über die analytischen Methoden in der Wahrscheinlichkeitsrechnung' I have proved that, under certain additional conditions, $f(t', x, t'', y)$ satisfies certain differential equations of parabolic type. But in A.M. there was no answer to the question as to what extent $f(t', x, t'', y)$ is uniquely determined by the coefficients $A(t, x)$ and $B(t, x)$. In this paper the theory is developed in the general case of a Riemannian manifold $\mathcal{R}$ and the question of uniqueness is answered affirmatively for a closed manifold $\mathcal{R}$.

§1. The first differential equation

Let $\mathcal{R}$ be a Riemannian manifold of dimension $n$. Because of the assumptions made, $f(t', x, t'', y)$ is defined for $t' < t''$ and any pair of points $x, y$. Moreover,

---

3 These differential equations were introduced by Fokker and Planck independently of Smolukhovski's integral equation. See: A. Fokker, Ann. Phys. 43 (1914), 812; M. Planck, Sitzungsber. Preuss. Acad. Wiss. (1917) 10 May.
4 See A.M. §15.
we assume that \( f(t', x, t'', y) \) has continuous derivatives up to the third order with respect to all the arguments \((t', t'' \text{ and the coordinates } x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \text{ of the points } x \text{ and } y)\) and satisfies the continuity condition

\[
\frac{\int_{\mathbb{R}} f(t, x, t + \Delta, z) \rho^3(z, z) dV_z}{\int_{\mathbb{R}} f(t, x, t + \Delta, z) \rho^2(z, z) dV_z} \rightarrow 0 \quad \text{as } \Delta \rightarrow 0,
\]

(4)

where \( \rho(x, z) \) denotes the geodesic distance\(^5\) between \( x \) and \( z \).

We choose a coordinate system \( z = (z_1, \ldots, z_n) \) in a neighbourhood \( \mathfrak{A} \) of \( x \). Then we set

\[
\int_{\mathfrak{A}} f(s, x, s + \Delta, z) (z_i - x_i) dV_z = a_i(s, x, \Delta),
\]

(5)

\[
\int_{\mathfrak{A}} f(s, x, s + \Delta, z) (z_i - x_i)(z_j - x_j) dV_z = b_{ij}(s, x, \Delta),
\]

(6)

\[
\int_{\mathfrak{A}} f(s, x, s + \Delta, z) \rho^2(z, z) dV_z = \beta(s, x, \Delta),
\]

(7)

\[
\int_{\mathfrak{A}} f(s, x, s + \Delta, z) \rho^3(z, z) dV_z = \nu(s, x, \Delta).
\]

(8)

Our purpose is to prove that the ratios

\[
a_i(s, x, \Delta)/\Delta, \quad b_{ij}(s, x, \Delta)/\Delta
\]

tend to limits \( A_i(s, x) \) and \( B_{ij}(s, x) \) as \( \Delta \rightarrow 0 \), independently of \( \mathfrak{A} \). Below this is proved under the assumption that all \( N = n + n(n + 1)/2 \) functions

\[
\frac{\partial}{\partial x_i} f(s, x, t, y), \quad \frac{\partial^2}{\partial x_i \partial x_j} f(s, x, t, y)
\]

of \( y \) and \( t \) (for fixed \( s \) and \( x \)) are linearly independent, that is, that \( t_1, y_1, t_2, y_2, \ldots, t_k, y_k, \ldots, t_N, y_N \) can be chosen so that the determinant

\[
D^N(s, x) = \begin{vmatrix}
\frac{\partial}{\partial x_i} f(s, x, t_k, y_k) \\
\frac{\partial^2}{\partial x_i \partial x_j} f(s, x, t_k, y_k)
\end{vmatrix}
\]

(9)

is non-zero.\(^6\)

---

\(^5\) See A.M., §13, formula (112).

\(^6\) See A.M., §13, determinant (119).
In $\mathfrak{X}$ we have
\[ \rho^3(x, z) = \sum g_{ij}(z_i - x_i)(z_j - x_j) + \Theta \rho^3(x, z), \quad |\Theta| \leq C, \]
while outside $\mathfrak{X}$ we clearly have
\[ \rho^3(x, z) = \Theta' \rho^3(x, z), \quad |\Theta'| \leq C', \]
where $C'$ and $C$ are constants independent of $z$. Hence
\begin{align*}
\beta(s, x, \Delta) &= \int_{\mathfrak{X}} f(s, x, s + \Delta, z) \rho^3(x, z) dV_z = \\
&= \sum g_{ij} \int_{\mathfrak{X}} f(s, x, s + \Delta, z)(z_i - x_i)(z_j - x_j) dV_z + \\
&\quad + \int_{\mathfrak{X}} f(s, x, s + \Delta, z) \Theta \rho^3(x, z) dV_z + \\
&\quad + \int_{\mathfrak{X}} f(s, x, s + \Delta, z) \Theta' \rho^3(x, z) dV_z = \\
&= \sum g_{ij}b_{ij}(s, x, \Delta) + \Theta'' \nu(s, x, \Delta), \quad |\Theta''| \leq C''.
\end{align*}
(10)

But since, by the continuity condition (4),
\[ \frac{\beta(s, x, \Delta)}{\nu(s, x, \Delta)} \to +\infty \text{ as } \Delta \to 0, \]
(11)
formula (10) implies that
\[ \frac{\sum g_{ij}b_{ij}(s, x, \Delta)}{\nu(s, x, \Delta)} \to +\infty \text{ as } \Delta \to 0. \]
(12)

Now, for fixed $x, y, s, \tau, t$, $s < \tau < t$, we consider only $\Delta$ so small that $s + \Delta < \tau$. Then $f(s + \Delta, z, t, y)$ and its derivatives with respect to $z$ up to the fourth order are uniformly bounded and continuous in $\mathfrak{X}$ (we assume that $\mathfrak{X}$ is compact). Hence, for every point $z$ in $\mathfrak{X}$ we have
\begin{align*}
f(s + \Delta, z, t, y) - f(s + \Delta, z, t, y) &= \sum (z_i - x_i) \frac{\partial}{\partial z_i} f(s + \Delta, z, t, y) + \\
&\quad + \frac{1}{2} \sum (z_i - x_i)(z_j - x_j) \frac{\partial^2}{\partial z_i \partial z_j} f(s + \Delta, z, t, y) + \Theta \rho^3(x, z), \quad |\Theta| \leq C,
\end{align*}
(13)
where \( C \) does not depend on \( \Delta \) or \( z \). On the other hand, the fundamental equation (3) implies that

\[
f(s, x, t, y) = \int_{\mathcal{M}} f(s, x, s + \Delta, z)f(s + \Delta, x, t, y)dV_z =
\]

\[
= \int_{\mathcal{M}} f(s, x, s + \Delta, z)f(s + \Delta, x, t, y)dV_z +
\]

\[
+ \int_{\mathcal{M}} f(s, x, s + \Delta, z)\{ f(s + \Delta, z, t, y) - f(s + \Delta, x, t, y) \}dV_z +
\]

\[
+ \int_{\mathcal{M}} f(s, x, s + \Delta, z)\{ f(s + \Delta, z, t, y) - f(s + \Delta, x, t, y) \}dV_z =
\]

\[
= I_1 + I_2 + I_3. \tag{14}
\]

By (2),

\[
I_1 = \int_{\mathcal{M}} f(s, x, s + \Delta, z)f(s + \Delta, x, t, y)dV_z =
\]

\[
= f(s + \Delta, x, t, y)\int_{\mathcal{M}} f(s, x, s + \Delta, z)dV_z = f(s + \Delta, x, t, y). \tag{15}
\]

Then (13), (5) and (6) imply that

\[
I_2 = \int_{\mathcal{M}} f(s, x, s + \Delta, z)\{ f(s + \Delta, z, t, y) - f(s + \Delta, x, t, y) \}dV_z =
\]

\[
= \int_{\mathcal{M}} f(s, x, s + \Delta, z)\left\{ \sum (z_i - x_i) \frac{\partial}{\partial z_i} f(s + \Delta, x, t, y) +
\right.
\]

\[
+ \frac{1}{2} \sum (z_i - x_i)(z_j - x_j) \frac{\partial^2}{\partial z_i \partial z_j} f(s + \Delta, x, t, y) +
\]

\[
+ \Theta \rho^3(x, z) \right\}dV_z = \sum a_i(s, x, \Delta) \frac{\partial}{\partial z_i} f(s + \Delta, x, t, y) +
\]

\[
+ \frac{1}{2} \sum b_{ij}(s, x, \Delta) \frac{\partial^2}{\partial z_i \partial z_j} f(s + \Delta, x, t, y) +
\]

\[
+ \int_{\mathcal{M}} f(s, x, s + \Delta, z)\Theta \rho^3(x, z)dV_z. \tag{16}
\]

Finally, since throughout \( \mathcal{M} - \mathfrak{A} \) we have

\[
\rho^3(x, z) > K > 0,
\]

where \( K \) does not depend on \( z \), in \( \mathcal{M} - \mathfrak{A} \) we can set

\[
f(s + \Delta, x, t, y) - f(s + \Delta, x, t, y) = \Theta' \rho^3(x, z).
\]
Then

\[ I_3 = \int_{\mathbb{R}^n} f(s, x, s + \Delta, z) \{ f(s + \Delta, z, t, y) - f(s + \Delta, x, t, y) \} dv_z = \]

\[ = \int_{\mathbb{R}^n} f(s, x, s + \Delta, z) \Theta' \rho^3(x, z) dv_z, \quad |\Theta'| \leq C' = \frac{1}{K}. \]  

(17)

Substituting (15)–(17) into (14) we finally obtain

\[ f(s, x, t, y) = f(s + \Delta, x, t, y) + \sum a_i(s, x, \Delta) \frac{\partial}{\partial x_i} f(s + \Delta, x, t, y) + \]

\[ + \frac{1}{2} \sum b_{ij}(s, x, \Delta) \frac{\partial^2}{\partial x_i \partial x_j} f(s + \Delta, x, t, y) + \]

\[ + \int_{\mathbb{R}} f(s, x, s + \Delta, z) \Theta'' \rho^3(x, z) dv_z, \quad |\Theta''| \leq C''. \]  

(18)

If we also take into account the obvious equality

\[ \int_{\mathbb{R}} f(s, x, s + \Delta, z) \Theta'' \rho^3(x, z) dv_z = \Theta'' \int_{\mathbb{R}} f(s, x, s + \Delta, z) \rho^3(x, z) dv_z = \]

\[ = \Theta'' \nu(s, x, \Delta), \quad |\Theta''| \leq C''', \]

then (18) can be rewritten as follows:

\[ f(s + \Delta, x, t, y) - f(s, x, t, y) \]

\[ \Delta \]

\[ = - \sum a_i(s, x, \Delta) \frac{\partial}{\partial x_i} f(s + \Delta, x, t, y) - \]

\[ - \sum b_{ij}(s, x, \Delta) \frac{\partial^2}{\partial x_i \partial x_j} f(s + \Delta, x, t, y) - \Theta'' \nu(s, x, \Delta) \]  

(19)

The left-hand side in (19) tends to \( \frac{\partial}{\partial \Delta} f(s, x, t, y) \) as \( \Delta \to 0 \).

Suppose that the determinant \( D_N(s, x) \) is non-zero for \( t_1, y_1, t_2, y_2, \ldots, t_N, y_N \). Then \( D_N(s + \Delta, x) \neq 0 \) for sufficiently small \( \Delta \). Hence, there exist \( \lambda_k(\Delta) \), \( k = 1, 2, \ldots, N \), such that

\[ \sum_k \lambda_k(\Delta) \frac{\partial}{\partial x_i} f(s + \Delta, x, t_k, y_k) = \alpha_i, \]

\[ \sum_k \lambda_k(\Delta) \frac{\partial^2}{\partial x_i \partial x_j} f(s + \Delta, x, t_k, y_k) = \alpha_{ij}. \]  

(20)

If we multiply (19) by \( \lambda_k(\Delta) \) with \( t = t_k \) and \( y = y_k \) and sum all the \( N \) equalities thus obtained, then we have

\[ \sum_k \lambda_k(\Delta) f(s + \Delta, x, t_k, y_k) - f(s, x, t_k, y_k) \]

\[ = - \sum_i \frac{a_i(s, x, \Delta)}{\Delta} \alpha_i - \sum_{i,j} \frac{b_{ij}(s, x, \Delta)}{2\Delta} \alpha_{ij} - \sum_k \lambda_k(\Delta) \Theta'' \nu(s, x, \Delta) \]  

(21)
If $\Delta$ tends to zero, then the $\lambda_k(\Delta)$, as solutions of (20), tend to the solution $\lambda_k(0)$ of the equations

\[
\sum_k \lambda_k(0) \frac{\partial}{\partial x_i} f(s, x, t_k, y_k) = \alpha_i,
\]

\[
\sum_k \lambda_k(0) \frac{\partial^2}{\partial x_i \partial x_j} f(s, x, t_k, y_k) = \alpha_{ij}.
\]

(22)

Hence, the left-hand side of (21) has a finite limit

\[
\Lambda_0 = \sum_k \lambda_k(0) \frac{\partial}{\partial s} f(s, x, t_k, y_k)
\]

(23)

as $\Delta \to 0$.

In particular, if we set $\alpha_i = 0$, $\alpha_{ij} = g_{ij}$, then

\[
\frac{\sum g_{ij} b_{ij}(s, x, \Delta)}{2\Delta} + \sum \lambda_k(\Delta) \Theta_k^{\nu} \frac{\nu(s, x, \Delta)}{\Delta} \to \Lambda_0 \text{ as } \Delta \to 0.
\]

(24)

By (12), the second term in (24) is infinitesimally small as compared with the first one (since the $\lambda_k(\Delta)$ are bounded). Hence we have

\[
\sum g_{ij} b_{ij}(s, x, \Delta)/2\Delta \to \Lambda_0 \text{ as } \Delta \to 0.
\]

(25)

But (25) and (12) imply

\[
\nu(s, x, \Delta)/\Delta \to 0 \text{ as } \Delta \to 0.
\]

(26)

If we now equate all but one of the coefficients $\alpha_i$ and $\alpha_{ij}$ in (21) to zero, then a similar passage to the limit using (26) shows that all the limits

\[
A_i(s, x) = \lim_{\Delta \to 0} \frac{\alpha_i(s, x, \Delta)}{\Delta}
\]

(27)

\[
B_{ij}(s, x) = \lim_{\Delta \to 0} \frac{b_{ij}(s, x, \Delta)}{2\Delta}
\]

(28)

exist and do not depend on the choice\footnote{See A.M.,§13, formulas (122)-(124).} of $\Theta$. Then (27), (28), (26) and (19) immediately imply the first differential equation

\[
\frac{\partial}{\partial s} f(s, x, t, y) = -\sum A_i(s, x) \frac{\partial}{\partial x_i} f(s, x, t, y) -
\]

\[-\sum B_{ij}(s, x) \frac{\partial^2}{\partial x_i \partial x_j} f(s, x, t, y).
\]

(29)
Certainly the condition that \( D_N(s, z) \) does not vanish identically can be replaced by the direct requirement that the limits (27) and (28) exist, since (28) implies the existence of a finite limit (25) and therefore of (26).

At certain exceptional points the limits (27) and (28) need not exist. This was illustrated in A.M.\(^8\) by the following example: \( \mathcal{A} \) is the ordinary number axis and

\[
f(s, x, t, y) = \frac{3y^2}{2\sqrt{\pi(t-s)}} \exp\left[ -\frac{(y^3-x^3)^2}{4(t-s)} \right];
\]

for \( x = 0 \) we easily obtain

\[
b(s, x, \Delta)/2\Delta \to +\infty \quad \text{as} \quad \Delta \to 0.
\]

Hence there is no finite limit \( B(s, x) \).

§2. The second differential equation

Assume now that in a neighbourhood \( \mathcal{A} \) of the point \( y_0 \) for a given \( t \) the limits \( A_i(t, y) \) and \( B_{ij}(t, y) \) exist uniformly and that \( \nu(t, y, \Delta)/\Delta \) tends uniformly to 0 in \( \mathcal{A} \). Suppose further that \( R(y) \) is a non-negative function vanishing outside \( \mathcal{A} \) with bounded derivatives up to the third order. Then for \( y \in \mathcal{A}, \; z \in \mathcal{A} \) we have

\[
R(y) = R(z) + \sum (y_i - z_i) \frac{\partial}{\partial z_i} R(z) + \frac{1}{2} \sum (y_i - z_i)(y_j - z_j) \frac{\partial^2}{\partial z_i \partial z_j} R(z) + \Theta \rho^3(y, z), \quad |\Theta| \leq C', \quad (31)
\]

whereas for \( y \in \mathcal{A} - \mathcal{A} \) and \( z \in \mathcal{A} \),

\[
R(y) = R(z) + \Theta'' \rho^3(y, z), \quad |\Theta''| \leq C''. \quad (32)
\]

Finally, for \( y \in \mathcal{A} - \mathcal{A}, \; z \in \mathcal{A} - \mathcal{A} \)

\[
R(y) = 0. \quad (33)
\]

\(^8\) See A.M., §13, formula (126).
If in the corresponding regions $R(y)$ is replaced by (31)--(33), we obtain

\[
\int_{\mathbb{R}} R(y) \frac{\partial}{\partial t} f(s, x, t, y) dV_y =
\]

\[
= \frac{\partial}{\partial t} \int_{\mathbb{R}} R(y) f(s, x, t, y) dV_y = \frac{\partial}{\partial t} \int_{\mathbb{R}} R(y) f(s, x, t, y) dV_y =
\]

\[
= \lim_{\Delta} \frac{1}{\Delta} \int_{\mathbb{R}} R(y) \left[ f(s, x, t + \Delta, y) - f(s, x, t, y) \right] dV_y =
\]

\[
= \lim_{\Delta} \frac{1}{\Delta} \left\{ \int_{\mathbb{R}} R(y) \int_{\mathbb{R}} f(s, x, t, z) f(t, z, t + \Delta, y) dV_z dV_y - \int_{\mathbb{R}} R(y) f(s, x, t, y) dV_y \right\} =
\]

\[
= \lim_{\Delta} \frac{1}{\Delta} \left\{ \int_{\mathbb{R}} f(s, x, t, z) \int_{\mathbb{R}} R(y) f(t, z, t + \Delta, y) dV_y dV_z - \int_{\mathbb{R}} R(z) f(s, x, t, z) dV_z \right\} =
\]

\[
= \lim_{\Delta} \frac{1}{\Delta} \left\{ \int_{\mathbb{R}} f(s, x, t, z) \int_{\mathbb{R}} R(z) f(t, z, t + \Delta, y) dV_y dV_z + \int_{\mathbb{R}} f(s, x, t, z) \int_{\mathbb{R}} \left[ \sum (y_i - z_i) \frac{\partial}{\partial z_i} R(z) \right] f(t, z, t + \Delta, y) dV_y dV_z + \right.
\]

\[
+ \frac{1}{2} \sum (y_i - z_i)(y_j - z_j) \frac{\partial^2}{\partial z_i \partial z_j} R(z) \right] f(t, z, t + \Delta, y) dV_y dV_z +
\]

\[
+ \int_{\mathbb{R}} f(s, x, t, z) \int_{\mathbb{R}} \Theta'' \rho^3(y, z) f(t, z, t + \Delta, y) dV_y dV_z - \int_{\mathbb{R}} R(z) f(s, x, t, z) dV_z \right\} = \lim_{\Delta} \frac{1}{\Delta} \left\{ \int_{\mathbb{R}} f(s, x, t, z) R(z) dV_z + \right.
\]

\[
+ \int_{\mathbb{R}} f(s, x, t, z) \left[ \sum a_i(t, z, \Delta) \frac{\partial}{\partial z_i} R(z) + \right.
\]

\[
+ \frac{1}{2} \sum b_{ij}(t, z, \Delta) \frac{\partial^2}{\partial z_i \partial z_j} R(z) \right] dV_z + \left. \right\}
\]

\[
+ \Theta \int_{\mathbb{R}} f(s, x, t, z) \nu(t, z, \Delta) dV_z - \int_{\mathbb{R}} f(s, x, t, z) R(z) dV_z \right\} =
\]
\begin{align*}
= \int_{\mathcal{A}} f(s, x, t, z) & \left[ \sum A_i(t, z) \frac{\partial}{\partial z_i} R(z) + \\
+ \sum B_{ij}(t, z) \frac{\partial^2}{\partial z_i \partial z_j} R(z) \right] dV_z.
\end{align*}

Replacing \( z \) by \( y \) in the right-hand side of the equation we obtain

\begin{align*}
\int_{\mathcal{A}} R(y) \frac{\partial}{\partial t} f(s, x, t, y) dV_y = \int_{\mathcal{A}} f(s, x, t, y) & \left[ \sum A_i(t, y) \frac{\partial}{\partial y_i} R(y) + \\
+ \sum B_{ij}(t, y) \frac{\partial^2}{\partial y_i \partial y_j} R(y) \right] dV_y.
\end{align*}

(34)

Now assume that \( A_i(t, z) \) and \( B_{ij}(t, z) \) are twice continuously differentiable in \( \mathcal{A} \). Then we set

\[ Q(t, y) = |g_{ij}(t, y)| \]

and after integration by parts, we obtain

\begin{align*}
\int_{\mathcal{A}} f(s, x, t, y) A_i(t, y) \frac{\partial}{\partial y_i} R(y) dV_y = \\
= \int_{\mathcal{A}} f(s, x, t, y) A_i(t, y) Q(t, y) \frac{\partial}{\partial y_i} R(y) dy_1 dy_2 \ldots dy_n = \\
= - \int_{\mathcal{A}} \frac{\partial}{\partial y_i} [f(s, x, t, y) A_i(t, y) Q(t, y)] R(y) dy_1 dy_2 \ldots dy_n. 
\end{align*}

(35)

Double integration by parts (since all the derivatives vanish on the boundary of \( \mathcal{A} \)) yields

\begin{align*}
\int_{\mathcal{A}} f(s, x, t, y) B_{ij}(t, y) & \frac{\partial^2}{\partial y_i \partial y_j} R(y) dV_y = \\
= \int_{\mathcal{A}} \frac{\partial^2}{\partial y_i \partial y_j} [f(s, x, t, y) B_{ij}(t, y) Q(t, y)] R(y) dy_1 dy_2 \ldots dy_n. 
\end{align*}

(36)

Formulas (34)–(36) immediately imply that

\begin{align*}
\int_{\mathcal{A}} R(y) Q(t, y) \frac{\partial}{\partial t} f(s, x, t, y) dy_1 dy_2 \ldots dy_n = \\
= \int_{\mathcal{A}} R(y) \{ - \sum \frac{\partial}{\partial y_i} [A_i(t, y) Q(t, y) f(s, x, t, y)] + \\
+ \sum \frac{\partial^2}{\partial y_i \partial y_j} [B_{ij}(t, y) Q(t, y) f(s, x, t, y)] \} dy_1 dy_2 \ldots dy_n.
\end{align*}
Since \( R(y) \) is arbitrary, apart from the above conditions, it is easy to conclude that at interior points of \( \mathfrak{A} \) the second differential equation

\[
Q(t, y) \frac{\partial}{\partial t} f(s, x, t, y) = -\sum \frac{\partial}{\partial y_i} [A_i(t, y) Q(t, y) f(s, x, t, y)] + \\
+ \sum \frac{\partial^2}{\partial y_i \partial y_j} [B_{ij}(t, y) Q(t, y) f(s, x, t, y)]
\]  

(37)

also holds.

If at time \( t_0 \) the differential function of the probability distribution is given, that is, a non-negative function \( g(t_0, y) \) of \( y \) satisfying the condition

\[
\int_{\mathfrak{A}} g(t_0, y) dV_y = 1,
\]

(38)

then for arbitrary \( t > t_0 \) the distribution function \( g(t, y) \) is given by the formula

\[
g(t, y) = \int_{\mathfrak{A}} g(t_0, x) f(t_0, x, t, y) dV_x.
\]

(39)

The function \( g(t, y) \) satisfies the equation\(^9\)

\[
Q \frac{\partial g}{\partial t} = -\sum \frac{\partial}{\partial y_i} (A_i Q g) + \sum \frac{\partial^2}{\partial y_i \partial y_j} (B_{ij} Q g).
\]

(40)

§3. Uniqueness

Under a change of the coordinate system the coefficients \( A_i(s, x) \) and \( B_{ij}(s, x) \) are transformed in the following way:

\[
A_i' = \sum \frac{\partial x_i'}{\partial x_k} A_k + \sum \frac{\partial^2 x_i'}{\partial x_k \partial x_l} B_{kl},
\]

(41)

\[
B_{ij}' = \sum \frac{\partial x_i'}{\partial x_k} \frac{\partial x_j'}{\partial x_l} B_{kl}.
\]

(42)

Here we always have

\[
B_{ii} = \lim_{2\Delta \to 0} \frac{b_{ii}(s, x, \Delta)}{2\Delta} = \lim \frac{1}{2\Delta} \int_{\mathfrak{A}} f(s, x, s + \Delta, x)(x_i - x_i)^2 dV_x \geq 0.
\]

(43)

Hence the quadratic form

\[
\sum B_{ij} \xi_i \xi_j
\]

(44)

\(^9\) See A.M., §18, formulas (169) and (170).
is non-negative. This is crucial in the proof of the following theorem.\textsuperscript{10}

**Uniqueness Theorem 1.** If \( R \) is closed, then (40) has at most one solution \( g(t, y) \) with given continuous initial condition \( g(t_0, y) = g(y) \).

**Proof.** Clearly it suffices to consider the initial condition \( g(t_0, y) = 0 \) and prove that \( g(t, y) = 0 \) also for \( t > t_0 \). We can transform (40) into the form

\[
\frac{\partial g}{\partial t} = \sum B_{ij} \frac{\partial^2 g}{\partial y_i \partial y_j} + \sum S_i \frac{\partial g}{\partial y_i} + Tg. \tag{45}
\]

Now set

\[
v(t, y) = g(t, y)e^{-ct}.
\]

The function \( v(t, y) \) satisfies the equation

\[
\frac{\partial v}{\partial t} = \sum B_{ij} \frac{\partial^2 v}{\partial y_i \partial y_j} + \sum S_i \frac{\partial v}{\partial y_i} + Tv - cv. \tag{46}
\]

For fixed \( t_0 \) and \( t_1 \) the constant \( c \) can be chosen so large that

\[
T(t, y) - c < 0
\]

for all \( y \) and \( t, t_0 \leq t \leq t_1 \). Under these conditions \( v(t, y) \) cannot have a positive maximum at any point \((t, y), t_0 < t < t_1\), since at such a maximum

\[
\frac{\partial v}{\partial t} = 0, \quad \frac{\partial v}{\partial y_i} = 0, \quad \sum B_{ij} \frac{\partial^2 v}{\partial y_i \partial y_j} \leq 0, \quad (T - c)v < 0,
\]

which contradicts (46). Neither can there be a negative minimum of \( v(t, y) \) within these limits. Since \( v(t_0, y) = 0 \) at \( t = t_0 \), we obtain for \( t_0 < t < t_1 \),

\[
v(t, y) < \max v(t_1, y) = e^{-ct_1} \max g(t_1, y)
\]

\[
g(t, y) < e^{-(t_1 - t)} \max g(t_1, y).
\]

Since \( c \) was arbitrarily large, it follows that

\[
g(t, y) = 0.
\]

Uniqueness Theorem 2. Let $\mathcal{R}$ be closed. Then there is at most one non-negative continuous solution $f(s, x, t, y)$ for (2) and (3) that satisfies (29) with given twice continuously differentiable coefficients $A_i(t, y)$ and $B_{ij}(t, y)$, and the continuity condition (4).

The continuity condition (4) can be replaced by the following, weaker one:

$$\int_{\mathcal{R}} f(s, x, t, y) \rho^2(x, y) dV_y \to 0 \quad \text{as} \quad t \to s. \quad (47)$$

Proof. Assume that two different functions $f_1(s, x, t, y)$ and $f_2(s, x, t, y)$ satisfy all our conditions. Then we can choose $s$ and a continuous function $g(x)$ such that

$$g_1(t, y) = \int_{\mathcal{R}} g(x) f_1(s, x, t, y) dV_x,$$

$$g_2(t, y) = \int_{\mathcal{R}} g(x) f_2(s, x, t, y) dV_x$$

are also different. By (2) and (47), $g_1(t, y)$ and $g_2(t, y)$ tend to $g(y)$ as $t \to s$. Since the functions $g_1(t, y)$ and $g_2(t, y)$ satisfy (40), this contradicts Uniqueness Theorem 1.

§4. An example

The following example, which is interesting also for applications, demonstrates that the quadratic form (44) need not be positive definite: let $\mathcal{R}$ be the usual Euclidean plane and let

$$f(s, x_1, x_2, t, y_1, y_2) = \frac{2\sqrt{3}}{\pi(t-s)^2} \exp\left\{ - \frac{(y_1 - x_1)^2}{4(t-s)} - \frac{3[y_2 - x_2 - (t-s)(y_1 + x_2)/2]^2}{(t-s)^3} \right\}. \quad (48)$$

A simple computation shows that

$$B_{11} = 1, \quad B_{12} = 0, \quad B_{22} = 0, \quad A_1 = 0, \quad A_2(s, x) = x_1.$$

§5. The limit solution

Let $\mathcal{R}$ be closed and $f(s, x, t, y)$ everywhere positive and dependent only on the difference $t-s$:

$$f(s, x, t, y) = \phi(t-s, x, y). \quad (49)$$
Then general ergodic theorems$^{11}$ imply the existence of the limit probability distribution. In other words, for any distribution $g(t,y)$ determined by (38) and (39) and any region $E$ the relation

$$\int_E g(t,y) dV_y \rightarrow P(E) \quad \text{as} \ t \rightarrow +\infty, \quad (50)$$

holds, where $P(E)$ does not depend on $g(t_0,y)$. It can easily be proved that $g(t,y)$ is uniformly continuous for large $t$. From this we deduce that$^{12}$

$$P(E) = -\int_E g(y) dV_y, \quad (51)$$

$$g(t,y) \rightarrow g(y) \quad \text{as} \ t \rightarrow +\infty. \quad (52)$$

Clearly, $g(y)$ and $P(E)$ do not depend on $g(t_0,y)$.

Now, let $g(y)$ be the solution of the equations

$$-\sum \frac{\partial}{\partial y_i} [A_i(y)Q(y)g(y)] + \sum \frac{\partial^2}{\partial y_i \partial y_j} [B_{ij}(y)Q(y)g(y)] = 0, \quad (53)$$

$$\int_{\mathbb{R}} g(y) dV_y = 1. \quad (53a)$$

Setting $g(t_0,y) = g(y)$ it can easily be seen that $g(t,y) = g(y)$ also for $t > t_0$ (see (40) and Uniqueness Theorem 1). From this we deduce that the solution of (53) and (53a) (if it exists) is uniquely determined and coincides with the limit function $g(y)$.

As a particular case, (52) implies

$$f(s,x,t,y) \rightarrow g(y) \quad \text{as} \ t \rightarrow +\infty. \quad (54)$$

Klyazma, near Moscow, 12 April 1932

$^{11}$ See A.M., §4, Theorem IV.

$^{12}$ See footnote 1.