THREE PAPERS ON THE HISTORY OF BRANCHING PROCESSES

Translated from Danish by Peter Guttorp

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ABSTRACT

The first complete proof of the criticality theorem for Bienaymê-Galton-Watson branching processes was published in Danish in 1930 by J. F. Steffensen. In this report his paper is translated into English, together with another proof of the same result, obtained by C. M. Christensen at the same time. These papers are prefaced by a historical review by K. Albertsen on the problem of extinction of families, also originally published in Danish.

Introduction

It is generally agreed that the so-called criticality theorem for Bienaymé-Galton-Watson branching processes was first completely stated and proved by J. F. Steffensen in 1930. Steffensen's paper, being in Danish, is not accessible to a large number of scholars (although an expanded version in French was published in 1933), whence I found it desirable to have an English translation available. In this report I present both Steffensen's original paper, and another solution to the problem, received by the editors of *Matematisk Tidsskrift* only a few months after Steffensen's. In addition, a thorough review of the history of the problem of surname extinction, written by K. Albertsen, has been translated and prepended to the other two papers.

I have added some footnotes (appearing with numbers) to the review by Albertsen, mainly adding information that was not available to the original author at the time of writing. In addition some biographic data have been added in square brackets. The paper by Steffensen and the solution by Christensen (published by Albertsen and Kristensen, 1976) are direct translations, and have been left as far as possible in their original version. Thus, footnotes and editing remarks in those papers are due to their authors.

I am grateful to Niels Keiding for sending me copies of the papers by Albertsen (1976) and Albertsen and Kristensen (1976), and to Klaus Krøier for helping me with the translation. Any translation inaccuracies are, of course, my own fault.

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THE EXTINCTION OF FAMILIES

K. ALBERTSEN

The idea that families die out originated in antiquity, and much has since been written about this problem. These works have been concerned partly with the statistical evidence of extinction of families, partly with considerations regarding the reasons for it.

The interest was always in the extinction of the highest social classes. At the same time it was thought that the extinct families were continually replaced by families from lower social layers. The Latin word proles means (many) children, whence the word proletariat is derived.

Later it became clear that the phenomenon occurred also in other strata of the population. In the 18th and 19th centuries this was demonstrated for towns. Apparently no investigation was ever made of the countryside population, but already in 1762 the Danish country vicar Westenholtz realized that "many farm families die out. But since they live in the dark without being known, yes, without having a family name, this is not noted or felt."

The reasons for extinction of families were sought in biological and social conditions or in combinations thereof. Foremost among the biological reasons was "degeneration," and among the social the main emphasis was on the effects of wars, celibate, and refined life styles.

However, the problem of extinction of families has also a mathematical-statistical side. This can briefly be expressed as follows. If we know the probability that a man has 0, 1, 2 ... sons, is it possible to compute the probability that his family dies out? This idea is so simple that it has occurred independently to a series of researchers, and in what follows a description will be given of how the problem has at various times been formulated, treated mathematically, and forgotten, only to be taken up later by others.

In the subsequent paper by K. Albertsen and E. Kristensen a previously unpublished solution to the problem of extinction of families is mentioned.

THE HISTORY OF THE PROBLEM

Bienaymé (1845). In 1962, Heyde and Seneta discovered a paper from 1845 by the French statistician I. J. Bienaymé (1796-1878), in which he considered the problem of extinction of families. Apparently, Bienaymé had undertaken a complete investigation of the problem and knew the solution which was first published by Steffensen in 1930. Judging from the quotations Heyde and Seneta cite from Bienaymé's work, there can be little doubt that he in 1845 knew the so called "criticality theorem" (cf. Steffensen below). His solution was, however, not given in mathematical form, and the advertised work that was to expound on his thoughts never seems to have been published.[1] Bienaymé's work remained obscure. None of the researchers discussed in this paper even mention Bienaymé, and it would thus take 129 years before someone noticed it again.

de Candolle (1873). In the book "Histoire des sciences et des savants" (1873), Adolphe de Candolle (1806-1893) undertook a review of the current literature on extinction of noble and bourgoeis families, presenting the available numerical data. He continued: "... je n'ai pas rencontré la réflexion bien importante qu'ils auraient dû faire de l'extinction *inevitable* des noms de famille. Evidemment tous les noms doivent s'éteindre,... Un mathématicien pourrait calculer comment la réduction des noms ou titres aurait leu, d'après la probabilité des naissances toutes fèminines ou toutes masculines ou mélangées et la probabilité des naissances dans un couple quelconque[2]" (pp. 388-389).

[1] In a recent paper (Bru et al., Intern. Statist. Rev. **60**: 177-183, 1992) a mathematical proof, which may be due to Bienaymé, was found in a book by Cournot from 1847.

[2] "I have not encountered the very important thought that the extinction of family names must be *inevitable*. Apparently all names have to die out,... A mathematician could compute in what order the reduction in names or titles should occur, according

This is a clear statement of the idea that it should be possible to compute the probability that a family becomes extinct if its fertility circumstances are known. de Candolle does not treat the problem in his book, and his formulation rules out knowledge of the work of Bienaymé mentioned above, since Bienaymé *had* shown that it is possible to compute such a probability, and that it is not always 1, i.e., the family does not always die out.

Galton (1873), Galton and Watson (1874). Francis Galton (1822-1911) published in 1869 the book "Hereditary genius", in which he treated the extinction of different social groups (nobility, judges, etc.), which he ascribed to biological reasons, causing reduced fertility. But in 1873 he published the following problem in Educational Times:

"Problem 4001. A large nation, of whom we will only concern ourselves with the adult males, N in number, and who each bear separate surnames, colonise a district. Their law of population is such that, in each generation, a_0 per cent of the adult males have no male children who reach adult life; a_1 have one such male child; a_2 have two; and so on up to a_5, who have five. Find (1) what proportion of the surnames will have become extinct after r generations; and (2) how many instances there will be of the same surname being held by m persons."

Already before the publication of de Candolle's book, Galton had realized the statistical nature of the problem, and had tried to work out some numerical examples, but the computations quickly became so complex that he had to give up (Galton and Watson (1874)). The appearance of de Candolle's book, however, caused Galton to revisit the problem of extinction of families, and to formulate it as given above. The disappearance of family names and the extinction of families are, of course, just two different ways to describe the same thing. An indirect indication that it was de Candolle's book that made Galton publish his problem is in Pearson (1924, p.341).

Concurrently with this he encouraged his friend Watson, who had earlier solved some mathematical problems for him, to try to solve also this problem.

H. W. Watson (1827-1913), who in one individual combined a creative mathematician, a country vicar, and an eager mountaineer, published his solution in Educational times a few months later. The same solution was presented by Galton and Watson at a meeting in May of 1874 of The Anthropological Institute in London (Galton and Watson (1874)), and was published in nearly the same form (and with the same misprints) as an appendix to Galton's book "Natural inheritance" (1889).

Watson's solution was, however, not complete, in that he thought that his results showed that *all* families sooner or later must die out. Even Pearson (1924), who in his biography of Galton works through Watson's proof, appears, in spite of his vast mathematical knowledge, not to have caught the mistake.

Galton thought himself that one could compute "using the laws of probability" the percentage of families that die out, and that, consequently, one should be able to determine whether a given group of families died out because of reduced fertility (Galton and Watson (1874)). This is wrong, since every computation must start from empirically determined fertility data. It is not possible for a group of families to divide the extinction probability into a "pure" probability and another probability that only depends on fertility. Thus the computations can not be used to prove "degeneration!"

Fahlbeck (1898-1902), Westergaard (1900). In his great work on Swedish nobility, the first part of which was published in 1898, P. E. Fahlbeck (1850-1923) showed that a large number of families died out, and supposed that this was due to "degeneration of reproduction." He knew about Galton and Watson's work (1889), but completely rejected their results, probably due to lack of understanding of probability theory. The Danish statistician and economist Harald Westergaard (1853-1936) showed in his review (1900) of the first part of Fahlbeck's work that it was not necessary to assume that there is any

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to the probability of births of all females or all males or a mixture, and the probability of births for an arbitrary couple."

form of degeneration, but that the results "completely correspond to the laws of probability theory." In the second part of his work (1902), Fahlbeck made certain amendements under the influence of Westergaards's criticism, but his statements show that he still had not understood the mathematical-statistical nature of the problem. A more detailed study of Fahlbeck and Westergaard is given in Albertsen (1973).

Weinberg (1914). The German medial statistician Weinberg investigated in 1914 the problem of extinction of families in a long paper with extensive mathematical analysis. Fahlbeck had noted that in the extinct Swedish noble families the ratio between male and female births was decreasing, and he assumed that this was a so-called degenerative phenomenon. Weinberg showed, however, that the phenomenon noted by Fahlbeck *must* occur in all families whose male line becomes extinct, without needing to make assumptions of degeneration.

Weinberg used the German translation of Fahlbeck's book (1903), and added that his own work built on that of Westergaard, Galton and Watson (mentioned in that order). He emphasized, however, that he had worked through the problem himself before gaining knowledge of these other works.

Fisher (1922, 1930), Haldane (1927,1932). In 1922, the English statistician R. A. Fisher [1890-1962] published a paper in which he investigated, among other things, how a trait in which a change (a mutation) has taken place spreads through a population, and whether it will eventually disappear. This problem corresponds closely to investigating the extinction of a family or a family name. Fisher's mathematical treatment is fairly similar to that of Galton and Watson, but the latter authors are never mentioned. It is hard to believe that Fisher would not have known their work, which was reproduced in Galton's book from 1889, but one cannot exclude that it is an independent development. The latter was argued by Good (1949).

The English geneticist J. B. S. Haldane treated the same problem in 1927, referring to Fisher's work, but did not mention Galton and Watson.

Later work on the same subject by Fisher (1930 a and b) and by Haldane (1932) also contain no reference to Galton and Watson. Neither of these two authors have given any clear formulation of the criticality theorem[1].

Erlang (1929), Steffensen (1930, 1933). The Danish statistician and mathematician Agner Krarup Erlang (1878-1929) posed in 1929 the following problem, published as an exercise for the readers of Matematisk Tidsskrift shortly after his death:

"Question nr. 15. When the probability that an individual has n children is a_n, where $a_0+a_1+a_2+...=1$, find the probability that his family dies out."

The similarity to Galton's problem is striking. Nothing indicates, however, that Erlang knew of Galton's formulation (Steffensen (1933), Kendall (1966)). The similarity extends to the circumstances of publication. In both cases the problem was posed as an exercise for the readers of a journal.

Erlang's interest in the problem of extinctions of families was not only statistical, but was related to considering the possible extinction of a certain Danish family (Kendall (1966)). He realized, however, that the problem is of a statistical nature, and gave it the above formulation.

Erlang did not publish a solution, but he apparently left behind an attempt at a solution, which was used by J. F. Steffensen (1873-1961) in a paper published in Matematisk Tidsskrift. This work is,

[1] Haldane (1927) shows that the extinction probability is the root of $f(x)=x$ (in the notation of the next paper) in the neighborhood of zero, and shows that it is between 0 and 1 whenever the offspring mean $m=1+k$ for a genetic advantage k. The case of an offspring mean smaller than 1 is of little interest in the genetic context of Haldane's paper.

however, Steffensen's own treatment of the problem, and contains the first published complete solution to the problem posed by Galton which Watson tried to solve.

Steffensen showed that the desired probability of family extinction depends on the quantity

$$m = a_1 + 2a_2 + 3a_3 + \cdots ,$$

which can be thought of as the average number of children to an individual. If $m \leq 1$ the probability that the family dies out is 1, i.e., the family will always die out. If, on the other hand, $m > 1$, there will be a probability of family extinction that lies between 0 and 1, i.e., the family has a chance to survive. These conditions were for the first time clearly indicated by Steffensen and was later named the "criticality theorem."

In 1933 Steffensen published in French a more detailed treatment of the problem.

After the publication of Steffensen's solution (1930), the English statistician W. P. Elderton made him aware of the work of Galton and Watson, which he thus far had not known about (and which can therefore not be assumed to have been known by Erlang) (Steffensen (1933)). On the other hand, neither Elderton nor Steffensen knew at this time of the papers by Fisher and Haldane mentioned above.

Erlang's exercise was, independently of Steffensen, also solved by C. M. Christensen (1898-1973). The solution has, however, not been published, but was mentioned in a footnote to Steffensen's paper (1930).

In the following paper this solution is reproduced.[1]

Kolmogoroff (1938). The Russian mathematician Kolmogoroff [1903-1987] computed the probability that a family still exists after a large, but finite, number of generations. This work was published in Russian, but from the abstract available it is not clear whether it is an independent formulation and treatment of the problem[2].

von Schelling (1944). In an investigation of the average kinship between individuals in a population, von Schelling also dealt with the extinction of families. Given the distribution of the number of children, he investigated the distribution of the number of grandchildren and great grandchildren. He went through several numerical examples, but did not reach any general conclusions. He did not discover the criticality theorem. He did, however, realize that families would either become extinct or else have numerous descendants. He appears to have worked completely without knowledge of any of the authors mentioned above. He reproduces an excerpt of a letter from the German statistician H. Koller, in which Koller makes statements regarding the distribution of the number of children. This quotation also seems to exclude that Koller would have known about any of the predecessors, even if it is nearly unthinkable that tone of the authors to the book "Erbmathematik" (1938) would have been unaware of the work of Fisher and Haldane in this area.

In a different paper from the same year (1944 b) von Schelling again mentions the same problem, but presents now the criticality theorem. Here is thus another example of an independent solution.

The mathematical theory that was developed on the basis of Watson's solution to Galton's problem on the extinction of family names is now commonly referred to as "the Galton-Watson branching process" (Harris (1963)). This theory has found use in such disparate areas as chemical chain reactions, the theory of electron amplifiers, and queueing theory. Major expositions of the theory and examples of applications can be found in Bartlett (1949), Feller (1958), Kendall (1949, 1966), and Keyfitz (1968). A survey of the history of the problem and its importance to geneology is given in Albertsen (1973).

[1] This is the last paper in this report, starting on p. 12.

[2] It is not. The paper refers to Fisher and Steffensen, and says that "References to previous works can be found in Steffensen's paper."

COMPUTATIONS ON THE BASIS OF THE THEORY OF GALTON AND WATSON

Already in 1874, Galton and Watson clearly stated that further progress with this problem would be impossible until there were statistical data on the probability that a man has 0, 1, 2, ... sons. Only 57 years later the results of a computation based on empirical data was published (Lotka (1931)). The official demographic statistics have never collected this type of data, but on the basis of the US Census Bureau data for the white population of 1920 on the number of births ordered by age of mother or father, Lotka was able to compute the probability of male line extinction to be 0.8797.

Lotka knew in October of 1931, when his first paper was published, both Galton and Watson's work and that of Steffensen (1930). Kendall (1966) claims that the occurrence of Steffensen's paper prompted Lotka's work. Later (1939) Lotka treated the same problem more comprehensively. The probability mentioned above was, using more refined methods, recomputed to 0.819, but is still confined to the white US population in 1920, and he made no efforts to extend the computations to other years or other populations.

Only after the emergence of computers did it become possible on a large scale to perform the massive calculations needed in order to apply Galton and Watson's theory to statistical data.

Keyfitz and Tyree (1967) and Keyfitz (1968) use the same starting point as did Lotka, namely the official statistics tables of the distribution of women according to number of children. This type of data are available for many populations. The computations are based on information about women whose fertile period is over, namely the age class 45-49 years. The number of children is recalculated to the number of daughters, and after various corrections the probability that the female line becomes extinct is computed. While Lotka used the male line, Keyfitz used only the female line, but the numbers should not differ much.

The probability that the female line dies out is computed for the following countries (data from 1960-61):

USA	0.8206
Hungary	0.7130
Israel	0.5144
Mexico	0.4066
Japan	0.3242

Apart from the examples mentioned above, there are no other computations in the literature based on empirical data from populations or groups of families.

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On the Probability That the Offspring Dies Out

By J. F. Steffensen

1. In Matematisk Tidsskrift B 1929, p. 36, the following problem is given as **Exercise no. 15** by the late mathematician A. K. Erlang:

When the probability that a person has n children is a_n, where $a_0+a_1+a_2+\cdots=1$, find the probability that his family dies out.

There is an obvious assumption that this probability exists, which we will assume in what follows. Furthermore it is assumed that the probabilities a_n are the same for all of the offspring, which cannot be correct, since one has to take into account combinations of more or less fertile lines. It is, however, difficult to express this difference in a way that corresponds more or less to reality, and one may anyway be entitled to start by finding a solution that is valid in a completely homogeneous society, which we therefore shall do. In this context we should also mention the objection that the probabilities with which we are going to operate are not mutually independent, so that the rules for multiplication of probabilities can only be used with reservations; we will ignore this as well in the first analysis.

If we call the desired probability x, we obviously have

$$x = a_0 + a_1x + a_2x^2 + \cdots \qquad (\sum a_\nu = 1) \tag{1}$$

since the desired probability consists of the probability of having no children; the probability that there is 1 child whose offspring dies out; 2 children whose offspring die out, etc.

The relation (1), which has been derived by several workers, first and foremost by Erlang himself, is not always sufficient to determine the desired probability, since there, in addition to the obvious root $x=1$, may be another*) root in the interval between 0 and 1. Before we proceed to investigate whether the question can be phrased analytically in such a fashion that there is only one possible answer, we shall investigate the circumstances under which (1) has two solutions that both can be interpreted as probabilities. Note first that even if the sequence $a_0,a_1,a_2,\cdots$ is continued indefinitely, there is no convergence problem as long as we are only interested in values of x in the interval from 0 to 1, since $\sum a_\nu=1$ and $a_\nu\geq 0$.

If in (1) we write $a_0=1-a_1-a_2-\cdots$, then

$$1-x = a_1(1-x) + a_2(1-x^2) + \cdots$$

whence, in order to investigate the possible roots different from 1, dividing by $1-x$ on both sides we obtain

$$1 = a_1 + a_2(1+x) + a_3(1+x+x^2) + \cdots \tag{2}$$

If we introduce the probability that there are **at least** ν children,

$$s_\nu = a_\nu + a_{\nu+1} + a_{\nu+2} + \cdots, \tag{3}$$

then equation (2), using that $\sum a_\nu=1$, can be written

$$a_0 = xs_2 + x^2s_3 + x^3s_4 + \cdots \tag{4}$$

Imagining that x on the right-hand side of (2) goes from 0 to 1, one sees immediately that whenever

$$1 \leq a_1 + 2_2 + 3a_3 + \cdots, \tag{5}$$

*) We ignore the trivial case where $a_1=1$, $0=a_0=a_2=a_3=\ldots$, in which case (1) does not help in determining the desired probability, which then is 0.

then (2) has precisely 1 root in the interval from 0 to 1, and otherwise none. This root must by $\geq a_0$, as can be seen from (1)*).

The condition (5) can be expressed in a way that yields a better understanding of, if I may say so, of the biological interpretation. If we define

$$m = a_1 + 2a_2 + 3a_3 + \cdots, \tag{6}$$

then m is clearly the average, or in the mathematical sense "expected" number of children. It is not surprising that this number has something to do with the question of the persistence of families. As far as we know, Erlang therefore claimed that whenever $m>1$ the smaller root should be preferred. If, on the other hand, $m<1$, the only possible solution is $x=1$; the same holds in the case $m=1$, when the solution $x=1$ is a double root.

2. It is, however, clear that these considerations, which as we pointed out are not novel, does not contain a proof, and we shall therefore attack the question in a different fashion, starting by computing the probability that the sth generation has n members. Hereby we shall consider the children the first generation, grand children the second, etc.

The probability that the sth generation has n members we shall denote $a_n^{(s)}$, so that $a_n^{(1)}=a_n$. Furthermore we introduce the power series, convergent in the interval form 0 to 1,

$$f_s(x) = a_0^{(s)} + a_1^{(s)}x + a_2^{(s)}x^2 + \cdots \tag{7}$$

and set $f_1(x) \equiv f(x)$, so that

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots \tag{8}$$

The distribution of the $(s=1)$th generation can now be expressed in terms of the distribution of the sth generation, in that, using the polynomial formula

$$a_r^{(s+1)} = \sum_{n=0}^{\infty} a_n^{(s)} \sum_{(\mathrm{v})} \frac{n!}{\mathrm{v}_0!\mathrm{v}_1!\cdots\mathrm{v}_r!} a_0^{\mathrm{v}_0} a_1^{\mathrm{v}_1} \cdots a_r^{\mathrm{v}_r}, \tag{9}$$

where the inner sum extends over all v for which the two conditions

$$\left.\begin{array}{c} \mathrm{v}_0 + \mathrm{v}_1 + \cdots + \mathrm{v}_r + n \\ \mathrm{v}_1 + 2\mathrm{v}_2 + \cdots + r\mathrm{v}_r + r \end{array}\right\} \tag{10}$$

are satisfied. In this case the inner sum is the probability that n individuals together have r children, where v_0 of them die childless, v_1 have one child, v_2 have 2 children, etc.

In order to write (9) in a clearer fashion we first expand $f_s(f(x))$ in powers of x. We have immediately that

$$f_s(f(x)) = \sum_{n=0}^{\infty} a_n^{(s)}(a_0 + a_1 x + a_2 x^2 + \cdots)^n.$$

In order to find $\frac{1}{r!}D_{x=0}^r f_s(f(x))$ it suffices to find the coefficient of x^r in the expansion of

$$\sum_{n=0}^{\infty} a_n^{(s)}(a_0 + a_1 x + \cdots + a_r x^r)^n.$$

The general term in the expansion of

$$(a_0 + a_1 x + \cdots + a_r x^r)^n$$

is, according to the polynomial formula,

*) The problem was solved correctly up to this point by Mr. *H. P. Hansen*, Færøernes Mellem- og Realskole.

$$\frac{n!}{\nu_0!\nu_1!\cdots\nu_r!}(a_0)^{\nu_0}(a_1x)^{\nu_1}\cdots(a_rx^r)^{\nu_r},$$

where $\nu_0+\nu_1+\cdots+\nu_r=n$. From these terms we only need those that have $\nu_1+2\nu_2+\cdots+r\nu_r=r$, whereby x gets the exponent r.

We therefore see that the desired coefficient is precisely the expression on the right-hand side of (9) under the conditions (1), thus that

$$a_r^{(s+1)} = \frac{1}{r!}D_{x=0}^r f_s(f(x)). \tag{11}$$

It follows that

$$f_s(f(x) = \sum_{r=0}^{\infty} a_r(s+1)x^r$$

or

$$f_s(f(x) = f_{s+1}(x), \tag{12}$$

so that $f_s(x)$ is the sth iterate of $f(x)\equiv f_1(x)$,

Combining this equation with (11) we wee that

$$a_n^{(s)} = \frac{1}{n!}f_s^{(n)}(0), \tag{13}$$

which in words can be expressed as:

The probability that the sth generation has n individuals is equal to the coefficient of x^n in the sth iterate of $f(x)$.

3. In solving Erlang's problem we particularly need the probability that the sth generation has no members; using (13) this is

$$a_0^{(s)} = f_s(0). \tag{14}$$

We shall use the abridged notation

$$x_s = a_0^{(s)},$$

using which (14) yields

$$x_s = f_s(0) = f(f_{s-1}(0)) = f(a_0^{(s-1)}) = f(x_{s-1}),$$

so that

$$x_{s+1} = f(x_s). \tag{15}$$

Using (15), and starting from $x_1=a_0$, we can now successively calculate $x_2, x_3, \cdots$. If this process converges, it yields a root to the solution

$$x = f(x) \tag{16}$$

or (1).

In order to prove the convergence we first remark that x_s, which is a probability, is bounded. Thus the sequence $x_1, x_2, x_3, \cdots$ would converge were it monotone. But the latter property is easy to prove, since $f(x)$ is increasing in x; if $x_s<x_{s+1}$ is $f(x_s)<f(x_{s+1}$, i.e., $x_{s+1}<x_{s+2}$. Since $x_1<x_2$ *) we see that x_s is monotonically **increasing** towards its limit.

*) Apart from the trivial case $a_0=1$, when $x_1=x_2$ and $x=1$.

Provided that we can also show that there cannot be a root between x_s and x_{s+1}, x_s will tend towards the smallest root in the interval between a_0 and 1. In order to prove this, we only need to show that if x is a root, and if for a given x_s we have $x_s < x$, then we also have $x_{s+1} < x$. But this follows immediately when we subtract (15) from (16), which, taking into account that $f(x)$ increases with x, yields

$$x - x_{s+1} = f(x) - f(x_s) > 0,$$

whenever $x_s < x$.

If we interpret the quantity

$$\lim_{s \to \infty} a_0^{(s)}$$

as the probability that the family dies out, we have proved both that this quantity exists, and that it is the smallest root to equation (1) in the interval between a_0 and 1.

After Professor Steffensens paper had been sent to the printer, the editors received a complete solution to the problem from Mr. *C. M. Christensen*, teacher, Odense.

A PREVIOUSLY UNPUBLISHED SOLUTION TO ERLANG'S PROBLEM OF EXTINCTION OF FAMILIES BY CARL MARIUS CHRISTENSEN

K. ALBERTSEN and E. KRISTENSEN

In 1929 A. K. Erlang (1878-1929) posed the following problem for the readers of Matematisk Tidsskrift [4]:

"When the probability that an individual has n children is a_n, where $a_0+a_1+a_2+...=1$, find the probability that his family dies out."

The following year J. F. Steffensen [6] published in the same journal a paper "On the probability that the offspring dies out", which includes a general solution to this problem.

In a footnote to Steffensen's paper the editors note that they have received "a complete solution to the problem" from C. M. Christensen, a teacher.

In a paper [1] in 1973 the manuscript to this answer was solicited. It has now been found by C. M. Christensen's brother-in-law, H. Busk-Jensen, a principal from Charlottenlund [2], and is reproduced below.

C. M. Christensen's solution is dated March 3, 1930. It is a completely independent effort, apparently only inspired by Erlang's problem. None of the workers who had previously attacked this problem are mentioned.

It is clear from the manuscript that Christensen has found the same general solution as did Steffensen, and that he has clearly formulated what has later been named the "criticality theorem." If his work had been published as a paper, and not just been sent to the editors as a solution to a problem, he would presumably today have been put on equal footing with Steffensen, whose name presently is solely attached to the solution to this problem.

Professor Harald Bohr returned on behalf of the editors the manuscript to the author. In his accompanying letter, dated March 21, 1930, he wrote:

"Your answer appears excellent and complete. Unfortunately we had already before we received your answer sent to the printer a small article by professor Steffensen on the problem, where he, in addition to a general discussion of the questions, gives a solution which is essentially the same as yours. We are very sorry that under these circumstances we cannot publish your solution—which otherwise would have given us great pleasure—but have to be satisfied by adding to professor Steffensen's paper a remark that we later received a complete solution to the problem from you."

C. M. Christensen's manuscript is reproduced in its original form below. Only minor stylistic changes have been made. To clarify the text there are simple additions, which are marked with square brackets. The preceding paper [3] contains a historical survey of the treatment of the problem of extinction of families.

C. M. Christiansen's solution to Erlang's problem. We let x_p denote the probability that the family of an arbitrarily chosen individual becomes extinct at or before the pth generation (the 0th generation is the individual himself, the 1st generation his children, etc.). If we can apply simple probability rules, the probability that the families of n individuals all become extinct at or before the pth generation becomes x_p^n.

The family of a person A can now become extinct at or before generation p if A has

either 0 children
or 1 child whose family becomes extinct at or before the $(p-1)$th generation
or 2 children, both of whose families become extinct at or before the $(p-1)$th generation,
etc.

Since the probability that A has n children is given to be a_n, the probability that he has n children and all their families die out at or before the $(p-1)$th generation [equals] $a_n x_{p-1}^n$. The probabilities for the mutually exclusive events given above consequently are

$$a_0,\ a_1 x_{p-1},\ a_2 x_{p-1}^2,\ a_3 x_{p-1}^3,\ \ldots,$$

and the probability that one of these happens [becomes]

$$a_0 + a_1 x_{p-1} + a_2 x_{p-1}^2 + a_3 x_{p-1}^3 + \cdots.$$

This must, however, precisely be the probability that the family dies out at or before the pth generation. We thus obtain the following equation for successive determination of the probabilities x_p:

$$x_p = a_0 + a_1 x_{p-1} + a_2 x_{p-1}^2 + a_3 x_{p-1}^3 + \cdots.$$

Here the function

$$\phi(x) = a_0 + a_1 x_{p-1} + a_2 x_{p-1}^2 + a_3 x_{p-1}^3 + \cdots$$

plays a role; it can with a natural and rather unimportant restriction be regarded as a polynomial of finite degree.

[When $a_0=0$, then $\phi(0)=0$ and

$$x_0 = x_1 = x_2 = \cdots = 0.$$

When $a_0=1$, then $\phi(0)=1$ and

$$x_0 = x_1 = x_2 = \cdots = 1.$$

In what follows we shall therefore assume that

$$a_0 \neq 0 \text{ and } a_0 \neq 1.$$

From this we obtain that the function $\phi(x)$ is increasing for $x \geq 0$ and that $\phi(0)>0$.]

The equation obtained can be written

$$x_p = \phi(x_{p-1}) \tag{1}$$

and it shows us (apart from the trivial cases $a_0=0$ and $a_0=1$) that, as one would expect, the numbers

$$x_0 = a_0 = \phi(0),\quad x_1=\phi(x_0),\quad x_2=\phi(x_1), \ldots,$$

form a monotonically increasing sequence of positive numbers that are all smaller than 1. [That the sequence is increasing can be proved by induction using equation (1), and it therefore has a limit.] If we then set

$$\lim_{p=\infty} x_p = \xi$$

we have that $a_0 < \xi \leq 1$.

This limit ξ will be interpreted as the probability that the family of an individual becomes extinct. If we in equation (1) let p tend to ∞, we get

$$\xi = \phi(\xi),$$

whence again

$$\frac{\phi(\xi)-\phi(x_m)}{\xi - x_m} = \frac{\xi - x_{m+1}}{\xi - x_m} < 1,$$

which for $m=\infty$ yields the result

$$\phi'(\xi) \leq 1. \tag{2}$$

Since

$$\phi'(x) = a_1 + 2a_2x + 3a_3x^2 + \cdots$$

is increasing for $X \geq 0$, we see that [the following holds:]

If $\phi'(1) = a_1 + 2a_2 + 3a_3 + \cdots$ is greater than 1, it follows from equation (2) that ξ is smaller than 1. It is easy to see that in this case ξ and 1 are the only positive roots to the equality $\phi(x)=x$.

If, on the other hand, $\phi'(1) = a_1 + 2a_2 + 3a_3 + \cdots$ is smaller than or equal to 1, we have for $0 \leq x < 1$ [that]

$$\phi'(x) - 1 < 0,$$

so that the smallest positive zero of the function $\phi(x)-x$ must be 1, thus $\xi=1$.

The answer to the given question must be the following:

The desired probability ξ is the smallest positive root to the equation $x=\phi(x)$. If the "average number of children to one individual" $a_1+2a_2+3a_3+\cdots$ is ≤ 1, then ξ simply is 1, otherwise ξ is smaller than 1; although $\xi=0$ when $a_0=0$, and also when $a_1=1$. [The number] ξ can be determined by iteration from $x=0$:

$$\xi = \cdots \phi(\phi(\phi(0))) \cdots .$$

A more explicit convergent expression for ξ can, at least in certain cases, be computed from, e.g., determining the radius of convergence for the power series expansion of the function

$$\frac{1}{\phi(x)-x} \quad \text{or} \quad \frac{f(x)}{\phi(x)-x},$$

where $f(x)$ is a suitably chosen function. Here the question of numerical values of negative and complex roots to the equation $\phi(x)=x$ becomes important.

In order for the result obtained to be reliable, one assumption must be that the families from two children to the same individual are independent of each other with respect to extinction. Among the many problems that can easily be seen to impede the application of this kind of problem to the real world are that this type of assumption is invalid. Offspring may "marry" each other. From the given basis—the probabilities a_n—the problem would have been better posed if instead of talking about an individual, children, and the extinction of families, it had mentioned a man, his sons, and the extinction of all pure male lines.

Biographic information. Carl Marius Christiansen was born in 1898 in Odense. He graduated from Odense Katedralskole in 1916 and started studies at the University of Copenhagen, but worked thereafter 1923-39 as a teacher at the Giersing Realskole in Odense. In 1927 he was accepted to the University of Copenhagen by answering a prize question in mathematics, and in 1936 he obtained a M. Sc. in mathematics. He has written several papers on mathematical subjects. During 1939-64 he was first appointed *adjunct* and later *lector* at Sønderborg Statsskole. He died in 1973.

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