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Growth rate

investment in the stock market
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d assets
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d assets
$S_n^{(j)}$ price of asset $j$ at the end of trading period (day) $n$
initial price $S_0^{(j)} = 1$, $j = 1, \ldots, d$
investment in the stock market

$d$ assets

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\[ S_{n}^{(j)} = e^{nW_{n}^{(j)}} \approx e^{nW^{(j)}} \]
investment in the stock market
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$$S_n^{(j)} = e^{nW_n^{(j)}} \approx e^{nW^{(j)}}$$

average growth rate

$$W_n^{(j)} = \frac{1}{n} \ln S_n^{(j)}$$
investment in the stock market

\(d\) assets

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average growth rate

\[
W_n^{(j)} = \frac{1}{n} \ln S_n^{(j)}
\]

asymptotic average growth rate

\[
W^{(j)} = \lim_{n \to \infty} \frac{1}{n} \ln S_n^{(j)}
\]
the aim is to achieve \( \max_j W^{(j)} \)
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static portfolio selection
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a portfolio vector $\mathbf{b} = (b^{(1)}, \ldots b^{(d)})$
$b^{(j)} \geq 0, \sum_j b^{(j)} = 1$
the aim is to achieve \( \max_j W^{(j)} \)

static portfolio selection

a portfolio vector \( \mathbf{b} = (b^{(1)}, \ldots, b^{(d)}) \)

\( b^{(j)} \geq 0, \quad \sum_j b^{(j)} = 1 \)

\( b^{(j)} \) gives the proportion of the investor’s capital invested in stock \( j \)
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initial capital \( S_0 \)

\[
S_n = S_0 \sum_j b^{(j)} S_n^{(j)}
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the aim is to achieve $\max_j W^{(j)}$

static portfolio selection

a portfolio vector $\mathbf{b} = (b^{(1)}, \ldots, b^{(d)})$

$b^{(j)} \geq 0$, $\sum_j b^{(j)} = 1$

$b^{(j)}$ gives the proportion of the investor’s capital invested in stock $j$

initial capital $S_0$

$$S_n = S_0 \sum_j b^{(j)} S_n^{(j)}$$

$$S_0 \max_j b^{(j)} S_n^{(j)} \leq S_n \leq dS_0 \max_j b^{(j)} S_n^{(j)}$$
assume that $b^{(j)} > 0$

$$\frac{1}{n} \ln \left( \max_j S_0 b^{(j)} S_n^{(j)} \right) \leq \frac{1}{n} \ln S_n \leq \frac{1}{n} \ln \left( \max_j S_0 d b^{(j)} S_n^{(j)} \right)$$
assume that $b^{(j)} > 0$

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\frac{1}{n} \ln \left( \max_j S_0 b^{(j)} S_n^{(j)} \right) \leq \frac{1}{n} \ln S_n \leq \frac{1}{n} \ln \left( \max_j S_0 d b^{(j)} S_n^{(j)} \right)
\]

\[
\max_j \left( \frac{1}{n} \ln (S_0 b^{(j)}) + \frac{1}{n} \ln S_n^{(j)} \right) \leq \frac{1}{n} \ln S_n
\]

\[
\leq \max_j \left( \frac{1}{n} \ln (S_0 d b^{(j)}) + \frac{1}{n} \ln S_n^{(j)} \right)
\]
assume that $b^{(j)} > 0$

$$\frac{1}{n} \ln \left( \max_j S_0 b^{(j)} S_n^{(j)} \right) \leq \frac{1}{n} \ln S_n \leq \frac{1}{n} \ln \left( \max_j S_0 db^{(j)} S_n^{(j)} \right)$$

$$\max_j \left( \frac{1}{n} \ln(S_0 b^{(j)}) + \frac{1}{n} \ln S_n^{(j)} \right) \leq \frac{1}{n} \ln S_n$$

$$\leq \max_j \left( \frac{1}{n} \ln(S_0 db^{(j)}) + \frac{1}{n} \ln S_n^{(j)} \right)$$

$$\lim_{n \to \infty} \frac{1}{n} \ln S_n = \lim_{n \to \infty} \max_j \frac{1}{n} \ln S_n^{(j)} = \max_j W^{(j)}$$
assume that \( b^{(j)} > 0 \)

\[
\frac{1}{n} \ln \left( \max_j S_0 b^{(j)} S_n^{(j)} \right) \leq \frac{1}{n} \ln S_n \leq \frac{1}{n} \ln \left( \max_j S_0 db^{(j)} S_n^{(j)} \right)
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\[
\max_j \left( \frac{1}{n} \ln(S_0 b^{(j)}) + \frac{1}{n} \ln S_n^{(j)} \right) \leq \frac{1}{n} \ln S_n \leq \max_j \left( \frac{1}{n} \ln(S_0 db^{(j)}) + \frac{1}{n} \ln S_n^{(j)} \right)
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\lim_{n \to \infty} \frac{1}{n} \ln S_n = \lim_{n \to \infty} \max_j \frac{1}{n} \ln S_n^{(j)} = \max_j W^{(j)}
\]

we can do much better using multi-period investment
Dynamic portfolio selection: multi-period investment

relative prices

\[ x_i^{(j)} = \frac{S_i^{(j)}}{S_i^{(j-1)}} \]
Dynamic portfolio selection: multi-period investment

Relative prices

\[ x_i^{(j)} = \frac{S_i^{(j)}}{S_{i-1}^{(j)}} \]

\[ x_i = (x_i^{(1)}, \ldots x_i^{(d)}) \] the return vector on trading period \( i \)
relative prices

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multi-period investment
relative prices

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\( x_i = (x_i^{(1)}, \ldots, x_i^{(d)}) \) the return vector on trading period \( i \)

multi-period investment

\( x_i^{(j)} \) is the factor by which capital invested in stock \( j \) grows during the market period \( i \)
relative prices

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Constantly Re-balanced Portfolio (CRP)
relative prices

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Constantly Re-balanced Portfolio (CRP)
a portfolio vector \( b = (b^{(1)}, \ldots b^{(d)}) \)
relative prices

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Constantly Re-balanced Portfolio (CRP)

a portfolio vector \( \mathbf{b} = (b^{(1)}, \ldots b^{(d)}) \)
\( b^{(j)} \) gives the proportion of the investor’s capital invested in stock \( j \)
relative prices

$$x_i^{(j)} = \frac{S_i^{(j)}}{S_{i-1}^{(j)}}$$

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multi-period investment

$x_i^{(j)}$ is the factor by which capital invested in stock $j$ grows during the market period $i$

Constantly Re-balanced Portfolio (CRP)

a portfolio vector $\mathbf{b} = (b^{(1)}, \ldots b^{(d)})$

$b^{(j)}$ gives the proportion of the investor’s capital invested in stock $j$

$\mathbf{b}$ is the portfolio vector for each trading day
for the first trading period $S_0$ denotes the initial capital

$$S_1 = S_0 \sum_{j=1}^{d} b^{(j)} x_1^{(j)} = S_0 \langle b, x_1 \rangle$$
for the first trading period $S_0$ denotes the initial capital

$$S_1 = S_0 \sum_{j=1}^{d} b^{(j)} x^{(j)}_1 = S_0 \langle \mathbf{b}, \mathbf{x}_1 \rangle$$

for the second trading period, $S_1$ new initial capital

$$S_2 = S_1 \cdot \langle \mathbf{b}, \mathbf{x}_2 \rangle = S_0 \cdot \langle \mathbf{b}, \mathbf{x}_1 \rangle \cdot \langle \mathbf{b}, \mathbf{x}_2 \rangle.$$
for the first trading period $S_0$ denotes the initial capital

$$S_1 = S_0 \sum_{j=1}^{d} b^{(j)} x^{(j)}_1 = S_0 \langle \mathbf{b} , \mathbf{x}_1 \rangle$$

for the second trading period, $S_1$ new initial capital

$$S_2 = S_1 \cdot \langle \mathbf{b} , \mathbf{x}_2 \rangle = S_0 \cdot \langle \mathbf{b} , \mathbf{x}_1 \rangle \cdot \langle \mathbf{b} , \mathbf{x}_2 \rangle .$$

for the $n$th trading period:

$$S_n = S_{n-1} \langle \mathbf{b} , \mathbf{x}_n \rangle = S_0 \prod_{i=1}^{n} \langle \mathbf{b} , \mathbf{x}_i \rangle = S_0 e^{n W_n(b)}$$

with the average growth rate

$$W_n(b) = \frac{1}{n} \sum_{i=1}^{n} \ln \langle \mathbf{b} , \mathbf{x}_i \rangle .$$
Special market process: \( X_1, X_2, \ldots \) is independent and identically distributed (i.i.d.)
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log-optimum portfolio $b^*$
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log-optimum portfolio \(b^*\)

\[
E\{\ln \langle b^* , X_1 \rangle \} = \max_b E\{\ln \langle b , X_1 \rangle \}
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Special market process: \( X_1, X_2, \ldots \) is independent and identically distributed (i.i.d.)

log-optimum portfolio \( b^* \)

\[
\mathbb{E}\{\ln \langle b^*, X_1 \rangle\} = \max_b \mathbb{E}\{\ln \langle b, X_1 \rangle\}
\]

Best Constantly Re-balanced Portfolio (BCRP)
If $S_n^* = S_n(b^*)$ denotes the capital after trading period $n$ achieved by a log-optimum portfolio strategy $b^*$,
If $S^*_n = S_n(b^*)$ denotes the capital after trading period $n$ achieved by a log-optimum portfolio strategy $b^*$, then for any portfolio strategy $b$ with capital $S_n = S_n(b)$ and for any i.i.d. process $\{X_n\}_{-\infty}^{\infty}$,
If $S^*_n = S_n(b^*)$ denotes the capital after trading period $n$ achieved by a log-optimum portfolio strategy $b^*$, then for any portfolio strategy $b$ with capital $S_n = S_n(b)$ and for any i.i.d. process $\{X_n\}_{-\infty}^{\infty}$,

$$\lim_{n \to \infty} \frac{1}{n} \ln S_n \leq \lim_{n \to \infty} \frac{1}{n} \ln S^*_n \quad \text{almost surely}$$

$$W^* = \mathbb{E}\{\ln \langle b^*, X_1 \rangle\}$$

is the maximal growth rate of any portfolio.
If \( S^*_n = S_n(b^*) \) denotes the capital after trading period \( n \) achieved by a log-optimum portfolio strategy \( b^* \), then for any portfolio strategy \( b \) with capital \( S_n = S_n(b) \) and for any i.i.d. process \( \{X_n\}_{-\infty}^{\infty} \),

\[
\lim_{n \to \infty} \frac{1}{n} \ln S_n \leq \lim_{n \to \infty} \frac{1}{n} \ln S^*_n \quad \text{almost surely}
\]

and

\[
\lim_{n \to \infty} \frac{1}{n} \ln S^*_n = W^* \quad \text{almost surely},
\]

where

\[
W^* = \mathbb{E}\{\ln \langle b^*, X_1 \rangle\}
\]

is the maximal growth rate of any portfolio.
Proof

\[
\frac{1}{n} \ln S_n = \frac{1}{n} \sum_{i=1}^{n} \ln \langle b, X_i \rangle
\]
\[
\frac{1}{n} \ln S_n = \frac{1}{n} \sum_{i=1}^{n} \ln \langle b, X_i \rangle \\
= \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}\{\ln \langle b, X_i \rangle\} \\
+ \frac{1}{n} \sum_{i=1}^{n} (\ln \langle b, X_i \rangle - \mathbb{E}\{\ln \langle b, X_i \rangle\})
\]
Proof

\[
\frac{1}{n} \ln S_n = \frac{1}{n} \sum_{i=1}^{n} \ln \langle b, X_i \rangle \\
= \frac{1}{n} \sum_{i=1}^{n} E\{\ln \langle b, X_i \rangle\} \\
+ \frac{1}{n} \sum_{i=1}^{n} (\ln \langle b, X_i \rangle - E\{\ln \langle b, X_i \rangle\})
\]

and

\[
\frac{1}{n} \ln S_n^* = \frac{1}{n} \sum_{i=1}^{n} E\{\ln \langle b^*, X_i \rangle\} \\
= \frac{1}{n} \sum_{i=1}^{n} (\ln \langle b^*, X_i \rangle - E\{\ln \langle b^*, X_i \rangle\})
\]
gambling, horse racing, information theory

Kelly (1956)
Latané (1959)
Breiman (1961)
Finkelstein and Whitley (1981)
Barron and Cover (1988)
gambling, horse racing, information theory

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Example 1: 1 stock + cash

\[ d = 2, \quad \mathbf{X} = (X^{(1)}, X^{(2)}) \]

Stock:

\[ X^{(1)} = \begin{cases} 
2 & \text{with probability } 1/2, \\
1/2 & \text{with probability } 1/2.
\end{cases} \]
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\[ \mathbb{E}\{X^{(1)}\} = 1/2 \cdot (2 + 1/2) = 5/4 > 1 \]
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\[ \mathbb{E}\{S_{n}^{(1)}\} = \mathbb{E}\left\{ \prod_{i=1}^{n} X_{i}^{(1)} \right\} = (5/4)^{n} \]
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What about \( S_n^{(1)} \) or \( W^{(1)} \)?
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What about \( S_n^{(1)} \) or \( W^{(1)} \)?

\[ W^{(1)} = \lim_{n \to \infty} \frac{1}{n} \ln S_n^{(1)} = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \ln X_i^{(1)} = \mathbb{E}\{\ln X^{(1)}\} \]

\[ = 1/2 \ln 2 + 1/2 \ln(1/2) = 0 \]

zero growth rate
Cash:

\[ X^{(2)} = 1 \]

zero growth rate
Cash:

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zero growth rate portfolio

\[ \mathbf{b} = (b, 1 - b) \]
Cash:

\[ X^{(2)} = 1 \]

zero growth rate
portfolio

\[
\mathbf{b} = (b, 1 - b)
\]

\[
\mathbb{E}\{\ln \langle \mathbf{b}, \mathbf{X} \rangle\} = \frac{1}{2} (\ln(2b + (1 - b)) + \ln(b/2 + (1 - b))
\]

\[
= \frac{1}{2} \ln[(1 + b)(1 - b/2)]
\]
Cash:

\[ X^{(2)} = 1 \]

zero growth rate portfolio

\[ b = (b, 1 - b) \]

\[
\mathbb{E}\{\ln \langle b, X \rangle\} = \frac{1}{2} \left( \ln(2b + (1 - b)) + \ln\left(\frac{b}{2} + (1 - b)\right) \right) = \frac{1}{2} \ln[(1 + b)(1 - b/2)]
\]

log-optimal portfolio

\[ b^* = (1/2, 1/2) \]
Cash:

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zero growth rate portfolio

\[ b = (b, 1 - b) \]

\[
\mathbb{E}\{\ln \langle b, X \rangle\} = \frac{1}{2} \ln(2b + (1 - b)) + \ln(b/2 + (1 - b)) \\
= \frac{1}{2} \ln[(1 + b)(1 - b/2)]
\]

log-optimal portfolio

\[ b^* = (1/2, 1/2) \]

asymptotic average growth rate

\[
\mathbb{E}\{\ln \langle b^*, X \rangle\} = \frac{1}{2} \ln(9/8) = 0.059 = W^*
\]

positive growth rate
Example 2: 2 stocks + cash

\[ d = 3, \quad \mathbf{X} = (X^{(1)}, X^{(2)}, X^{(3)}) \]
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1/2 & \text{with probability } 1/2.
\end{cases} \]

Cash:

\[ X^{(3)} = 1 \]
Example 2: 2 stocks + cash

\[ d = 3, \quad X = (X^{(1)}, X^{(2)}, X^{(3)}) \]

Stocks:

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\end{cases} \]

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2 & \text{with probability } 1/2, \\
1/2 & \text{with probability } 1/2.
\end{cases} \]

Cash:

\[ X^{(3)} = 1 \]

log-optimal portfolio

\[ b^* = (0.46, 0.46, 0.08) \]
Example 2: 2 stocks + cash

\[ d = 3, \quad \mathbf{X} = (X^{(1)}, X^{(2)}, X^{(3)}) \]

Stocks:

\[ X^{(1)} = \begin{cases} 2 & \text{with probability } 1/2, \\ 1/2 & \text{with probability } 1/2. \end{cases} \]

\[ X^{(2)} = \begin{cases} 2 & \text{with probability } 1/2, \\ 1/2 & \text{with probability } 1/2. \end{cases} \]

Cash:

\[ X^{(3)} = 1 \]

log-optimal portfolio

\[ \mathbf{b}^* = (0.46, 0.46, 0.08) \]

asymptotic average growth rate

\[ \mathbb{E}\{\ln \langle \mathbf{b}^*, \mathbf{X} \rangle\} = 0.112 = W^* \]
Example 3: 3 stocks + cash

\[ d = 4, \quad \mathbf{X} = (X^{(1)}, X^{(2)}, X^{(3)}, X^{(4)}) \]
Example 3: 3 stocks + cash

\[ d = 4, \quad X = (X^{(1)}, X^{(2)}, X^{(3)}, X^{(4)}) \]

log-optimal portfolio

\[ b^* = (1/3, 1/3, 1/3, 0) \]
Example 3: 3 stocks + cash

d = 4, \quad \mathbf{X} = (X^{(1)}, X^{(2)}, X^{(3)}, X^{(4)})

log-optimal portfolio

\quad \mathbf{b}^* = (1/3, 1/3, 1/3, 0)

the cash has zero weight
Example 3: 3 stocks + cash

\[ d = 4, \quad \mathbf{X} = (X^{(1)}, X^{(2)}, X^{(3)}, X^{(4)}) \]

log-optimal portfolio

\[ \mathbf{b}^* = (1/3, 1/3, 1/3, 0) \]

the cash has zero weight

asymptotic average growth rate

\[ \mathbb{E}\{\ln \langle \mathbf{b}^*, \mathbf{X} \rangle \} = 0.152 = W^* \]
Example 4: many stocks

\( d \) is large
Example 4: many stocks

d is large
log-optimal portfolio

\[ b^* = \left( \frac{1}{d}, \ldots, \frac{1}{d} \right) \]
Example 4: many stocks

$d$ is large
log-optimal portfolio

\[ b^* = (1/d, \ldots, 1/d) \]

asymptotic average growth rate

\[ E\{\ln \langle b^*, X \rangle \} = 0.223 = W^* \]
Example 5: horse racing

\(d\) horses in a race
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d horses in a race
horse $j$ wins with probability $p_j$
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horse \(j\) wins with probability \(p_j\)

payoff \(o_j\): investing 1$ on horse \(j\) results in \(o_j\) if it wins, otherwise 0$
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\(d\) horses in a race

horse \(j\) wins with probability \(p_j\)

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\[ X = (0, \ldots, 0, o_j, 0, \ldots, 0) \]

if horse \(j\) wins
Example 5: horse racing

$d$ horses in a race
horse $j$ wins with probability $p_j$
payoff $o_j$: investing 1$ on horse $j$ results in $o_j$ if it wins, otherwise 0$

\[ X = (0, \ldots, 0, o_j, 0, \ldots, 0) \]
if horse $j$ wins
repeated races

\[ E\{\ln \langle b, X \rangle \} = \sum_{j=1}^{d} p_j \ln (b^{(j)} o_j) = \sum_{j=1}^{d} p_j \ln b^{(j)} + \sum_{j=1}^{d} p_j \ln o_j \]
Example 5: horse racing

d horses in a race
horse j wins with probability $p_j$
payoff $o_j$: investing 1$ on horse j results in $o_j$ if it wins, otherwise 0$

\begin{align*}
X &= (0, \ldots, 0, o_j, 0, \ldots, 0) \\
\text{if horse } j \text{ wins} \\
\text{repeated races}
\end{align*}

\begin{align*}
\mathbb{E}\{\ln \langle b, X \rangle \} &= \sum_{j=1}^{d} p_j \ln (b(j) o_j) \\
&= \sum_{j=1}^{d} p_j \ln b(j) + \sum_{j=1}^{d} p_j \ln o_j
\end{align*}

therefore

\[ \arg \max_b \mathbb{E}\{\ln \langle b, X \rangle \} = \arg \max_b \sum_{j=1}^{d} p_j \ln b(j) \]
arg max \( b \) \( \sum_{j=1}^{d} p_j \ln b^{(j)} \)
arg max \( b \sum_{j=1}^{d} p_j \ln b(j) \)

Kullback-Leibler divergence:

\[
KL(p, b) = \sum_{j=1}^{d} p_j \ln \frac{p_j}{b(j)}
\]
\[ \arg \max_b \sum_{j=1}^d p_j \ln b^{(j)} \]

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basic property:

\[ KL(p, b) \geq 0 \]
\[
\arg \max_b \sum_{j=1}^{d} p_j \ln b^{(j)}
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KL(p, b) = \sum_{j=1}^{d} p_j \ln \frac{p_j}{b^{(j)}}
\]

basic property:

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KL(p, b) \geq 0
\]

Proof:

\[
KL(p, b) = - \sum_{j=1}^{d} p_j \ln \frac{b^{(j)}}{p_j}
\]
arg max \( \sum_{j=1}^{d} p_j \ln b^{(j)} \)

Kullback-Leibler divergence:

\[
KL(p, b) = \sum_{j=1}^{d} p_j \ln \frac{p_j}{b^{(j)}}
\]

basic property:

\[
KL(p, b) \geq 0
\]

Proof:

\[
KL(p, b) = - \sum_{j=1}^{d} p_j \ln \frac{b^{(j)}}{p_j} \geq - \sum_{j=1}^{d} p_j \left( \frac{b^{(j)}}{p_j} - 1 \right)
\]

\[
= - \sum_{j=1}^{d} b^{(j)} + \sum_{j=1}^{d} p_j = 0
\]
\arg \max_b \sum_{j=1}^d p_j \ln b^{(j)} = p

usual choice of payoffs: \( o_j = \frac{1}{p_j} \)

any gambling strategy has negative growth rate
\[
\arg\max_b \sum_{j=1}^{d} p_j \ln b^{(j)} = p
\]

independent of the payoffs

\[
W^* = \sum_{j=1}^{d} p_j \ln(p_j o_j)
\]
\[
\arg \max_b \sum_{j=1}^{d} p_j \ln b^{(j)} = \mathbf{p}
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\]

usual choice of payoffs:

\[
o_j = \frac{1}{p_j}
\]

\[
W^* = 0
\]

any gambling strategy has negative growth rate
The image contains a graph representing the performance of a portfolio selection strategy labeled "coke" over time. The graph plots the logarithmic returns (ln) against the number of days. The x-axis represents the days ranging from 0 to 12,000, while the y-axis shows the logarithmic returns ranging from -0.006 to 0.008. The graph shows a trend where the logarithmic returns stabilize over time, indicating a potential stationary or predictable pattern in the portfolio's performance.
ibm

Days

0 2000 4000 6000 8000 10000 12000

0 0.006 0.008

-0.004 -0.006
Mean = 1,0004707
Std. Dev. = 0,01611594
N = 11,177
Györfi

Machine learning and portfolio selections. I.
Consequences

Corollary: with large probability

\[ S_n(b) \text{ is not close to } \mathbb{E}\{S_n(b)\} \]
Corollary: with large probability

\[ S_n(b) \text{ is not close to } \mathbb{E}\{S_n(b)\} \]

Proof:

\[
\left\{ -\delta < \frac{1}{n} \ln S_n(b) - \mathbb{E}\{\ln \langle b, X_1 \rangle\} < \delta \right\}
\]
Consequences

Corollary: with large probability

\[ S_n(b) \text{ is not close to } E\{S_n(b)\} \]

Proof:

\[ \left\{ -\delta < \frac{1}{n} \ln S_n(b) - E\{\ln \langle b, X_1 \rangle\} < \delta \right\} \]

\[ \left\{ -\delta + E\{\ln \langle b, X_1 \rangle\} < \frac{1}{n} \ln S_n(b) < \delta + E\{\ln \langle b, X_1 \rangle\} \right\} \]
Corollary: with large probability

\[ S_n(b) \text{ is not close to } E\{S_n(b)\} \]

Proof:

\[
\left\{-\delta < \frac{1}{n} \ln S_n(b) - E\{\ln \langle b, X_1 \rangle\} < \delta \right\}
\]

\[
\left\{-\delta + E\{\ln \langle b, X_1 \rangle\} < \frac{1}{n} \ln S_n(b) < \delta + E\{\ln \langle b, X_1 \rangle\} \right\}
\]

\[
\left\{e^{n(-\delta+E\{\ln \langle b, X_1 \rangle\})} < S_n(b) < e^{n(\delta+E\{\ln \langle b, X_1 \rangle\})} \right\}
\]
$S_n(b)$ is close to $e^{nE\{\ln\langle b, X_1 \rangle\}}$
\( S_n(b) \) is close to \( e^{n\mathbb{E}\{\ln \langle b, X_1 \rangle\}} \)

\[
\mathbb{E}\{S_n(b)\} = \mathbb{E}\left\{ \prod_{i=1}^{n} \langle b, X_i \rangle \right\} = \prod_{i=1}^{n} \langle b, \mathbb{E}\{X_i\} \rangle = e^{n\ln \langle b, \mathbb{E}\{X_1\} \rangle}
\]
$S_n(b)$ is close to $e^{nE\{\ln \langle b, X_1 \rangle \}}$

$$E\{S_n(b)\} = E\{\prod_{i=1}^{n} \langle b, X_i \rangle \} = \prod_{i=1}^{n} \langle b, E\{X_i\} \rangle = e^{n\ln \langle b, E\{X_1\} \rangle}$$

by Jensen inequality

$$\ln \langle b, E\{X_1\} \rangle > E\{\ln \langle b, X_1 \rangle \}$$
$$S_n(b)$$ is close to $$e^{nE\{\ln\langle b, X_1 \rangle\}}$$

$$E\{S_n(b)\} = E\{\prod_{i=1}^{n} \langle b, X_i \rangle\} = \prod_{i=1}^{n} \langle b, E\{X_i\}\rangle = e^{n \ln \langle b, E\{X_1\}\rangle}$$

by Jensen inequality

$$\ln \langle b, E\{X_1\}\rangle > E\{\ln \langle b, X_1 \rangle\}$$

therefore

$$S_n(b)$$ is much less than $$E\{S_n(b)\}$$
Naive approach

$$\text{arg max}_b E\{ S_n(b) \}$$
Naive approach

$$\arg \max_b \mathbb{E}\{S_n(b)\}$$

because of

$$\mathbb{E}\{S_n(b)\} = \langle b, \mathbb{E}\{X_1\} \rangle^n$$
Naive approach

\[
\arg \max_b \mathbb{E}\{S_n(b)\}
\]

because of

\[
\mathbb{E}\{S_n(b)\} = \langle b, \mathbb{E}\{X_1\} \rangle^n
\]

\[
\arg \max_b \mathbb{E}\{S_n(b)\} = \arg \max_b \langle b, \mathbb{E}\{X_1\} \rangle
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$$\arg \max_b \mathbb{E}\{S_n(b)\}$$

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$$\mathbb{E}\{S_n(b)\} = \langle b, \mathbb{E}\{X_1\} \rangle^n$$

$$\arg \max_b \mathbb{E}\{S_n(b)\} = \arg \max_b \langle b, \mathbb{E}\{X_1\} \rangle$$

$$\arg \max_b \langle b, \mathbb{E}\{X_1\} \rangle$$ is a portfolio vector having 1 at the position, where $$\mathbb{E}\{X_1\}$$ has the largest component.
Naive approach

\[
\arg \max_b \mathbf{E}\{S_n(b)\}
\]

because of

\[
\mathbf{E}\{S_n(b)\} = \langle b, \mathbf{E}\{X_1\} \rangle^n
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\[
\arg \max_b \mathbf{E}\{S_n(b)\} = \arg \max_b \langle b, \mathbf{E}\{X_1\} \rangle
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arg max \_b \langle b, \mathbf{E}\{X_1\} \rangle is a portfolio vector having 1 at the position,
where \(\mathbf{E}\{X_1\}\) has the largest component
it is a dangerous portfolio
Naive approach

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\arg \max_b \mathbb{E}\{S_n(b)\}
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because of

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\mathbb{E}\{S_n(b)\} = \langle b, \mathbb{E}\{X_1\} \rangle^n
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\arg \max_b \mathbb{E}\{S_n(b)\} = \arg \max_b \langle b, \mathbb{E}\{X_1\} \rangle
\]

\[
\arg \max_b \langle b, \mathbb{E}\{X_1\} \rangle
\]

is a portfolio vector having 1 at the position, where \( \mathbb{E}\{X_1\} \) has the largest component. It is a dangerous portfolio.

Markowitz:

\[
\arg \max_{b: \text{Var}(\langle b, X_1 \rangle) \leq \lambda} \langle b, \mathbb{E}\{X_1\} \rangle
\]
log-optimal:

$$\arg \max_b E\{\ln \langle b, X_1 \rangle\}$$
log-optimal:

$$\arg \max_b E \{ \ln \left< b, X_1 \right> \}$$

Taylor expansion: $\ln z \approx h(z) = z - 1 - \frac{1}{2}(z - 1)^2$
Semi-log-optimal portfolio

log-optimal:

$$\arg \max_b \mathbf{E}\{ \ln \langle b, X_1 \rangle \}$$

Taylor expansion: $\ln z \approx h(z) = z - 1 - \frac{1}{2}(z - 1)^2$

semi-log-optimal:

$$\arg \max_b \mathbf{E}\{ h(\langle b, X_1 \rangle) \}$$

Connection to the Markowitz theory.


http://www.szit.bme.hu/~oti/portfolio/articles/marko.pdf
Semi-log-optimal portfolio

log-optimal:

\[ \arg \max_b \mathbb{E}\{ \ln \langle b, X_1 \rangle \} \]

Taylor expansion: \[ \ln z \approx h(z) = z - 1 - \frac{1}{2}(z - 1)^2 \]

semi-log-optimal:

\[ \arg \max_b \mathbb{E}\{ h(\langle b, X_1 \rangle) \} = \arg \max_b \{ \langle b, m \rangle - \langle b, Cb \rangle \} \]
log-optimal:

$$\arg \max_b \mathbb{E}\{\ln \langle b, X_1 \rangle\}$$

Taylor expansion: $$\ln z \approx h(z) = z - 1 - \frac{1}{2}(z - 1)^2$$

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Connection to the Markowitz theory.
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Connection to the Markowitz theory.
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