On Random Matrices
Related to Quantum Statistical Mechanics
and Informatics

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Variations on the theme of "sample" (or "empirical") covariance matrices $XX^T$, where $X = \{X_{jk}\}_{j,k=1}^n$ are random square matrices. The subject is rather old with a lot of versions and motivations (e.g. a "typical" positive definite operator in spectral theory). Recent ones are from

$$(Quantum\ Statistical\ Mechanics \cap (Quantum\ Informatics)).$$

Key words: quantum phase transitions, entanglement entropy, area law.
Let $A$ be $n \times n$ real symmetric and $B$ be $n \times n$ real anti-symmetric. Set

$$X = A + B,$$

assume a certain distribution for $A$ and $B$, and study the Normalized Counting Measure (NCM)

$$N_n = n^{-1} \sum_{l=1}^{n} \delta_{\lambda_l^{(n)}}$$

of $XX^T$ as $n \to \infty$, and also rate of convergence, extreme eigenvalues, fluctuations of $N_n$, local statistics, eigenvectors, etc.

If the entries of $A$ and $B$ are i.i.d. Gaussian (modulo symmetry), then $XX^T$ is asymptotically Wishart, the historically first random matrix.
Product of Triangular Matrices

Generalities

Recall that in the standard RMT setting $X = n^{-1/2} Y$, where $\{ Y_{jk} \}_{j,k=1}^n$ are independent standard Gaussian ($\mathbb{E} Y_{jk} = 0$, $\mathbb{E} Y_{jk}^2 = 1$) and then $N_n$ tends weakly with probability 1 to the "quarter-circle" law

$$\rho(\lambda) := N'(\lambda) = \frac{1}{4\pi} \sqrt{\frac{4 - \lambda}{\lambda}} \mathbf{1}_{[0,4]}(\lambda)$$

in which $\lambda = 4$ ($\lambda = 0$) is known as the standard soft (hard) edge. This is an old result of Marchenko-P. 68

Write

$$X = (X + X^T)/2 + (X - X^T)/2 := A + B$$

and obtain the simplest example of the above setting.
A bit more: replace $X \to X + yI_n$. This is a particular case of Silverstein-Dozier 04. Here the limiting DOS is:

$y^2 < 1$: similar to quarter-circle law (standard soft and hard edges, the latter at 0);
$y^2 = 1$: upper edge is standard soft, lower edge is at zero and non standard hard
$\rho(\lambda) \sim \text{Const } \lambda^{-1/3}, \lambda \downarrow 0$;
$y^2 > 1$: both edges are strictly positive and standard soft.
Product of Triangular Matrices

Motivations

Quasi-free Fermions

\[ H_\Lambda = \sum_{x,y \in \Lambda} A_{xy} c_x^+ c_y + \frac{1}{2} \sum_{x,y \in \Lambda} B_{xy} c_x^+ c_y^+ + h.c. \]

\( A \) is real symmetric, \( B \) is real antisymmetric. For \( d = 1 \) and n.n. interaction follows from quantum spin chains by Jordan-Wigner transformation.

QSM: Spectrum of \( H_\Lambda \) as \( \Lambda \rightarrow \mathbb{Z}^d \). By Bogolyubov transformation reduces to the spectrum of

\[ A_\Lambda = \begin{pmatrix} A & B \\ -B & -A \end{pmatrix}. \]

QI: Spectrum of \( K_\Lambda|_{\Lambda_1}, \Lambda_1 \subset \Lambda \), where \( K_\Lambda = (I_{2n} + e^{-\beta A_\Lambda})^{-1} \) and \( 1 << |\Lambda_1| << |\Lambda| \).
Product of Triangular Matrices

Motivations

We have

\[ \det(A_{\Lambda} - \lambda I_{2n}) = \det \left( (A + B)(A - B) - \lambda^2 I_n \right) \]

Write

\[ A = \frac{1}{2} A^+ + \frac{1}{2} (A^+)^T + A^0, \quad B = \frac{1}{2} B^+ - \frac{1}{2} (B^+)^T \]

where \( A^+ \) and \( B^+ \) are lower triangular, and \( A^0 \) is diagonal. Choose \( A^+ = B^+, \ A^0 = yI_n \) to get

\[ A + B = A^+ + yI_n. \]

Assume that \( \{A^+_{jk}\}_{n \geq j > k \geq 1} \) are independent Gaussian, \( \mathbf{E}\{A^+_{jk}\} = 0, \mathbf{E}\{(A^+_{jk})^2\} = 1/n \) to obtain a mean field type model for quasi-free fermions requiring the spectrum of

\[ M_n = (A^+ + yI_n)(A^+ + yI_n)^T. \]

Cf. Cholesky decomposition (linear algebra, numerics)
Theorem

Let $M_n$ be as above. Then its NCM converges weakly with probability 1 to the non-random limit $N$, whose Stieltjes transform $f$ solves uniquely

$$\log(1 + f) = \left(y^2 - z(1 + f)\right)^{-1}, \quad \Re f \cdot \Re z > 0, \quad \Re z \neq 0.$$

We have: $\text{supp } N = [a_-(y), a_+(y)] \subset \mathbb{R}_+$, $N$ is a. c. and if $\rho = N'$, then

(i) $y \neq 0$: $a_-(y) \sim e^{-1}y^4 e^{-1/y^2}$, $y \to 0$, $a_+(y) \sim e(1 + y^2)$, $y \to 0$

$$\rho(\lambda) \sim \text{Const } |a_\pm - \lambda|^{1/2}, \quad |a_\pm - \lambda| \to 0,$$

(ii) $y = 0$: $a_-(0) = 0$, $a_+(0) = e$ and

$$\rho(\lambda) \sim \begin{cases} \text{Const } (e - \lambda)^{1/2}, & \lambda \nearrow e, \\ (\lambda \log^2 \lambda)^{-1}, & \lambda \searrow 0. \end{cases}$$
A short(est) proof of the quarter-circle law for Gaussian vectors is as follows:

(i) Pass to the Stieltjes transform of $N_n$:

$$g_n(z) := \int \frac{N_n(d\lambda)}{\lambda - z} = n^{-1} \text{Tr } G(z), \quad G = (M - z)^{-1}$$

(ii) Use the Poincaré inequality to prove

$$\text{Var}\{g_n(z)\} \leq \text{Const} / n^2 |\text{Im } z|^4$$

thereby reducing the problem to the convergence of $\mathbf{E}\{g_n(z)\}$.

(iii) Use the resolvent identity and the integration by parts to prove

$$f_n := \mathbf{E}\{g_n\} = -\frac{1}{z} + \frac{1}{z} f_n - \frac{1}{zn} \mathbf{E}\{g_n \text{Tr } M_n G\}.$$
(iv) Use again the resolvent identity and (ii) – (iii) to obtain

\[zf_n^2 + zf_n + 1 = C(z)/n, \ C(z) < \infty, \ \Im z \neq 0.\]

(v) Pass to the limit \(n \to \infty\), solve the limiting quadratic equation for \(\Im f(z) \Im z > 0\) and recover \(N\) from the Stieltjes-Frobenius inversion formula.
Consider the technically simpler case $y = 0$. Use again the Stieltjes transform of $N_n$ and the Poincaré

$$\text{Var}\{g_n(z)\} \leq \frac{1}{n^2} |\Im z|^4,$$

reducing the problem to the study of

$$f = \lim_{n \to \infty} f_n, \ f_n := E\{g_n\} = n^{-1} \sum_{j=1}^{n} E\{G_{jj}\}, \ \Im z \neq 0.$$
The resolvent identity, the integration by parts and vanishing of fluctuations of $n^{-1}\text{Tr}...$ imply:

$$
E\{G_{jj}\} \sim -\frac{1}{z} + \frac{1}{z} \frac{j-1}{n} E\{G_{jj}\} - \frac{1}{z} E\{G_{jj}\} \sum_{k=1}^{j-1} E\{n^{-1}\text{Tr}(A^T GA)_{kk}\} 
$$

$$
E\{n^{-1}\text{Tr}(A^T GA)_{jj}\} \sim \frac{1}{n} \sum_{k=j}^{n} E\{G_{kk}\} - \frac{1}{n} \sum_{k=j}^{n} E\{G_{kk}\} E\{n^{-1}\text{Tr}(A^T GA)_{jj}\}.
$$

View this as the finite-difference scheme for

$$
f(t, z) = \lim_{n \to \infty, j/n \to t} E\{G_{jj}\}.
$$
Then the limit $j/n \to t \in [0, 1]$ yields the equations

$$f(t, z) = -\left( z - \int_0^t h(s, z) ds \right)^{-1},$$

$$h(t, z) = \left( 1 + \int_t^1 f(s, z) ds \right)^{-1},$$

and

$$f(z) = \int_0^1 f(t, z) dt.$$

Denote

$$\varphi(t, z) = \int_t^1 f(s, z) ds, \quad \varphi(0, z) = f(z),$$

to obtain

$$\frac{\partial^2}{\partial t^2} \varphi = \left( \frac{\partial}{\partial t} \varphi \right)^2 (1 + \varphi)^{-1}, \quad \left. \frac{\partial}{\partial t} \varphi \right|_{t=0} = z^{-1}, \quad \varphi(0, z) = f(z),$$

thus

$$\varphi(t, z) = -1 + e^{-C(t-1)}, \quad Ce^{-C} = -z^{-1}.$$
(i) $f$ is not algebraic, cf Anderson-Zeitouni 08, e.g. Silverstein-Dozier case

$$f = \left( y^2 (1 + f)^{-1} - z(1 + f) \right)^{-1}.$$ 

(ii) Most singular hard edge known. Recall the standard hard edge

$$\rho(\lambda) = \text{Const} \lambda^{-1/2} (1 + o(1)), \quad \lambda \downarrow 0,$$

of the quarter-circle law and more general Laguerre-type ensembles.

(iii) Implies an interesting quantum phase transition via the "scaling asymptotics" of $\rho$ for $\lambda \sim y^2 \to 0$.

(iv) The rate of convergence of minimum eigenvalue of $M_n$, eigenvectors, etc.
(v) Matrices $\{ Z_{jk}^+ \}_{j,k=1}^n$ with i.i.d. (but not necessarily Gaussian) entries. Use the "interpolation trick" (a two-term integration by parts) for

$$n^{-1/2}(\sqrt{1-tA^+} + \sqrt{tZ^+}).$$

(vi) More general versions

$$H + n^{-1}Z^+ T(Z^+)^T, \text{ and } (Z_0 + n^{-1/2}Z^+) T(Z_0 + n^{-1/2}Z^+)^T$$

where $Z$ has independent entries and $H$, $T$ and $Z_0$ are given.
Definition

Consider complex random i.i.d. vectors \( \{ \varphi^j_{\alpha} \}_{\alpha,j=1}^{p,k} \), \( p = 1, 2..., k \) is fixed, and \( \varphi^j_{\alpha} \in \mathbb{C}^d \) is

- either \( d^{-1/2} X^j_{\alpha} \), and \( X^j_{\alpha} \) is complex Gaussian vectors with i.i.d. standard components
- or uniformly distributed over the unit sphere.

Set

\[ \Phi_{\alpha} = \varphi^1_{\alpha} \otimes ... \otimes \varphi^k_{\alpha} \]

and consider the \( d^k \times d^k \) random matrix

\[ M_{p,d,k} = \sum_{\alpha=1}^{p} \Phi_{\alpha} \otimes \Phi_{\alpha}. \]

We are interested in the (non-random) limit as \( p \to \infty, d \to \infty, \frac{p}{d^k} = \frac{p}{n} \to c \in (0, \infty) \) of
the Normalized Counting Measure (NCM)

\[ N_{p,d,k} = d^{-k} \sum_{l=1}^{d^k} \delta_{\lambda_l}, \quad n = d^k. \]

It is also of interest the limits of the extreme eigenvalues, local statistics, fluctuations of \( N_{p,d,k} \), etc.


Proved the MP law for the limit \( N \) of the expectation of the NCM and the convergence of extreme eigenvalues to the endpoints of the support of \( N \) by fairly involved combinatorial analysis of moments \( d^{-k} \text{Tr} M_{p,d,k}^m, \ m \in \mathbb{N} \).
Remark. For Gaussian $\varphi$’s $\Phi_\alpha \in (\mathbb{C}^d)^\otimes k$ has just $dk$ independent parameters, while a generic $\Psi \in (\mathbb{C}^d)^\otimes k$ has $d^k$ independent parameters. Nevertheless the MP law and the convergence of extreme eigenvalues hold in this case.

We show below that the MP law is valid for the limit with probability 1 of $N_{p,d,k}$ in the above and more general cases (vectors with independent but not necessarily Gaussian components as well as for vectors with log-concave distribution).
The approach used above for the quarter-circle law and its "triangular" analog does not apply to the tensor product version, i.e. $k > 1$ (unlike the case $k = 1$). We use an extension of the Marchenko-P. and Girko approach. Its version for $k = 1$ is given by Pajor-P. It is applicable not necessarily Gaussian $\varphi_\alpha$'s and any $1 \leq k < \infty$.

(i) Observe that

$$M = \sum_{\alpha=1}^{p} L_\alpha, \quad L_\alpha = (\cdot, \varphi_\alpha)\varphi_\alpha.$$ 

(ii) Use either martingale differences (or Poincaré for Gaussian) to prove

$$\text{Var}\{g_n(z)\} = o(1), \quad \exists z \neq 0, \quad n \to \infty, \quad p \to \infty, \quad p/n \in [0, \infty).$$

(iii) Use the resolvent identity to write

$$g_n := n^{-1}\text{Tr}G = -z^{-1} + (zn)^{-1} \sum_{\alpha=1}^{p} (G\varphi_\alpha, \varphi_\alpha).$$
(iv) Use the rank one perturbation formulas:

\[ G = G_\alpha - \frac{G_\alpha L_\alpha G_\alpha}{1 + (G_\alpha \varphi_\alpha, \varphi_\alpha)}, \quad G_\alpha = G|_{\varphi_\alpha = 0} \]

implying

\[ (G \varphi_\alpha, \varphi_\alpha) = \frac{(G_\alpha \varphi_\alpha, \varphi_\alpha)}{1 + (G_\alpha \varphi_\alpha, \varphi_\alpha)}. \]

to rewrite (iii) as

\[ g_n = -z^{-1} + (zn)^{-1} \sum_{\alpha=1}^{p} \frac{(G_\alpha \varphi_\alpha, \varphi_\alpha)}{1 + (G_\alpha \varphi_\alpha, \varphi_\alpha)}. \]
(v) Use the independence of $G_\alpha$ and $\varphi_\alpha$ and to obtain:

$$E_\alpha\{(G_\alpha\varphi_\alpha, \varphi_\alpha)\} = n^{-1}\text{Tr}G_\alpha, \quad \text{Var}\{(G_\alpha\varphi_\alpha, \varphi_\alpha)\} \leq \text{Const}/n|\Im z|^2.$$

(iv) Use (ii) and (v) to replace $(G_\alpha\varphi_\alpha, \varphi_\alpha)$ in (iv) by its expectation $f_{\alpha n} := E\{n^{-1}\text{Tr}G_\alpha\}$.

(v) Use the rank one perturbation formula of (iv) to find that $f_{\alpha n} = f_n + O(1/n)$ and get the "pre"-limiting quadratic equation

$$f_n = -\frac{1}{z} + \frac{c}{z} \frac{f_n}{1 + f_n} + o(1), \quad \Im z \neq 0, \quad c = p/n$$

equivalent to the above.
Tensor Product Version of Sample Covariance Matrices

Basic Relations

For any $n \times n$ matrix $A$ we need random vectors $\varphi \in \mathbb{C}^n$ possessing

(i) isotropy

$\mathbb{E}\{(A\varphi, \varphi)\} = n^{-1} \text{Tr } A$;

(ii) vanishing of fluctuations of $(A\varphi, \varphi)$ ("good" vectors)

$\text{Var}\{(A\varphi, \varphi)\} = \|A\| \delta_n$, $\delta_n = O(1)$, $n \to \infty$.

Lemma

Let $\varphi \in \mathbb{C}^d$ be a random vector as above and $A$ is $d^k \times d^k$ matrix. If $\varphi^1, \ldots, \varphi^k$ are $k$ independent copies of $\varphi$ then the random vector $\Phi = \varphi^1 \otimes \ldots \otimes \varphi^k$ also possesses the above properties in which $n = d^k$ and $\delta_n$ is replaced by $C_k \delta_d$, where $C_k$ depends only on $k$.

Proof is based on the martingale-differences.
Study the extreme eigenvalues, both for $c > 1$ (both edges are standard soft) and $c = 1$ (lower edge is standard soft). Have likely different rates of convergence (depending on $k$).

Example: for Gaussian vectors

$$\text{Var} \{ g_n \} \leq \frac{C(z)k}{n^{1+1/k}}, \quad 0 < C(z) < \infty, \quad \text{Im} \ z \neq 0,$$

thus, different scaling of fluctuations of linear eigenvalue statistics (CLT), etc.