Analysis of a Combinatorial Optimization Problem in Wireless Communications

Ralf R. Müller

Department of Electronics & Telecommunications
Norwegian University of Science & Technology, Trondheim

joint work with
Benjamin Zaidel, Dongning Guo, Aris Moustakas, Rodrigo de Miguel, Vesna Gardašević, Finn Knudsen

October 9, 2012
The Problem in a Nutshell

Let

\[ E \equiv \frac{1}{K} \min_{x \in \mathcal{X}} x^\dagger Jx \]

with \( x \in \mathbb{C}^K \) and \( J \in \mathbb{C}^{K \times K} \).
The Problem in a Nutshell

Let

\[ E \equiv \frac{1}{K} \min_{x \in \mathcal{X}} x^\dagger Jx \]

with \( x \in \mathbb{C}^K \) and \( J \in \mathbb{C}^{K \times K} \).

- Sphere:

\[ \mathcal{X} = \{ x : x^\dagger x = K \} \implies E = \min \lambda(J) \]
Let

\[ E \equiv \frac{1}{K} \min_{x \in X} x^\dagger Jx \]

with \( x \in \mathbb{C}^K \) and \( J \in \mathbb{C}^{K \times K} \).

- **Sphere:**
  \[ X = \{ x \mid x^\dagger x = K \} \implies E = \min \lambda(J) \]

- **Cube:**
  \[ X = \{ +1, -1 \}^K \implies ??? \]
The Problem in a Nutshell

Let

\[ E \equiv \frac{1}{K} \min_{x \in X} x^\dagger Jx \]

with \( x \in \mathbb{C}^K \) and \( J \in \mathbb{C}^{K \times K} \).

- **Sphere:**
  \[ X = \{ x : x^\dagger x = K \} \implies E = \min \lambda(J) \]

- **Cube:**
  \[ X = \{ +1, -1 \}^K \implies ??? \]

- **Vector precoding:**
  \[ X = (4\mathbb{Z} + 1)^K \implies ??? \]
Let the received vector be given by

\[ y = Ht + n \]

where
- \( t \) is the transmitted vector
- \( n \) is uncorrelated (white) Gaussian noise
- \( H \) is a coupling matrix accounting for crosstalk

Data rate scales linearly with the minimum of the number of antenna elements.
The Appetizer for Engineers

The Gaussian Vector Channel

Let the received vector be given by

\[ y = Ht + n \]

where

- \( t \) is the transmitted vector
- \( n \) is uncorrelated (white) Gaussian noise
- \( H \) is a coupling matrix accounting for crosstalk

Crosstalk can be processed either at receiver or transmitter
If the transmitter is a base-station and the receiver is a hand-held device, processing at the transmitter is preferred.

In a broadcast situation, processing at the transmitter is mandatory.
Processing at Transmitter

- If the transmitter is a base-station and the receiver is a hand-held device, processing at the transmitter is preferred.
- In a broadcast situation, processing at the transmitter is mandatory.

E.g. let the transmitted vector be

\[ t = H^\dagger (HH^\dagger)^{-1} x \]

where \( x = s \) is the data to be sent.
Processing at Transmitter

- If the transmitter is a base-station and the receiver is a hand-held device, processing at the transmitter is preferred.
- In a broadcast situation, processing at the transmitter is mandatory.

E.g. let the transmitted vector be

\[ t = H^\dagger (HH^\dagger)^{-1} x \]

where \( x = s \) is the data to be sent.

Then,

\[ y = H t + n. \]
\[ = HH^\dagger (HH^\dagger)^{-1} x + n. \]
\[ = s + n. \]

No crosstalk anymore due to channel inversion.
Problems of Simple Channel Inversion

Channel inversion implies a significant power amplification, i.e.

\[ E\{t^\dagger t\} = E\left\{x^\dagger \left( HH^\dagger \right)^{-1} x \right\} > E\{x^\dagger x\} \quad \text{even for } E\{HH^\dagger\} = I \]

In particular, let

- the entries of \( H \in \mathbb{C}^{K \times N} \) be i.i.d. with variance \( 1/N \).
- \( \alpha = \frac{K}{N} \leq 1; \)

Then, for fixed aspect ratio \( \alpha \)

\[
\lim_{K \to \infty} \frac{x^\dagger \left( HH^\dagger \right)^{-1} x}{x^\dagger x} = \frac{1}{1 - \alpha}
\]

(with probability 1).
Instead of representing the logical "0" by +1, represent it by any element of the set \{...,-7,-3,1,5,...\} = 4Z + 1. Correspondingly, the logical "1" is represented by any element of the set 4Z - 1. Choose that representation that gives the smallest transmit power.
Instead of representing the logical "0" by +1, represent it by any element of the set \{-4, -3, 0, 1, 2, 3\}. Correspondingly, the logical "1" is represented by any element of the set \{-4, -3, 0, 1, 2, 3\}. Choose that representation that gives the smallest transmit power.
Instead of representing the logical "0" by +1, represent it by any element of the set \{\ldots, -7, -3, +1, +5, \ldots \} = 4\mathbb{Z} + 1. Correspondingly, the logical "1" is represented by any element of the set 4\mathbb{Z} - 1.

Choose that representation that gives the smallest transmit power.
Lattice Relaxation of QPSK
Let $B_0$ and $B_1$ denote the sets presenting 0 and 1, resp.

Let $(s_1, s_2, \ldots, s_K) \in \{0, 1\}^K$ denote the data to be transmitted.
Let $B_0$ and $B_1$ denote the sets presenting 0 and 1, resp.

Let $(s_1, s_2, \ldots, s_K) \in \{0, 1\}^K$ denote the data to be transmitted.

Then, the transmitted energy per data symbol is given by

$$E = \frac{1}{K} \min_{x \in \mathcal{X}} x^\dagger J x$$

with

$$\mathcal{X} = B_{s_1} \times B_{s_2} \times \cdots \times B_{s_K}$$

and

$$J = (HH^\dagger)^{-1}.$$
Let $\mathcal{B}_0$ and $\mathcal{B}_1$ denote the sets presenting $0$ and $1$, resp.

Let $(s_1, s_2, \ldots, s_K) \in \{0, 1\}^K$ denote the data to be transmitted.

Then, the transmitted energy per data symbol is given by

$$E = \frac{1}{K} \min_{x \in \mathcal{X}} x^\dagger J x$$

with

$$\mathcal{X} = \mathcal{B}_{s_1} \times \mathcal{B}_{s_2} \times \cdots \times \mathcal{B}_{s_K}$$

and

$$J = (HH^\dagger)^{-1}.$$
Vector precoding is the problem of finding the zero temperature limit of a quadratic energy potential.

The transmitted power is written as a zero temperature limit

\[ E = - \lim_{\beta \to \infty} \frac{1}{\beta K} \log \sum_{x \in x} e^{-\beta \text{tr}(x^\dagger J x)} \]

with \( \frac{1}{\beta} \) denoting temperature.
Zero Temperature Formulation

Vector precoding is the problem of finding the zero temperature limit of a quadratic energy potential.

The transmitted power is written as a zero temperature limit

\[
E = - \lim_{\beta \to \infty} \frac{1}{\beta K} \log \sum_{x \in \mathcal{X}} e^{-\beta \text{tr}(x^\dagger Jx)}
\]

\[
\longrightarrow - \lim_{\beta \to \infty} \lim_{K \to \infty} \frac{1}{\beta K} \log \sum_{x \in \mathcal{X}} e^{-\beta \text{tr}(Jxx^\dagger)}
\]

with \(\frac{1}{\beta}\) denoting temperature.
Let $P$ be any positive semi-definite matrix of bounded rank $n$ and $J$ be unitarily invariant and free of $P$, then

$$\lim_{K \to \infty} \frac{1}{K} \log \mathbb{E}_J e^{-K \text{tr} JP} = - \sum_{a=1}^{n} \lambda_a \int_{0}^{\infty} R_J(-w) dw$$

with

- $\lambda_a$ denoting the positive eigenvalues of $P$ and
- $R_J(w)$ denoting the R-transform of the spectral measure of $J$.

(Marinari et al. ’94; Guionnet, Maïda ’05)
Examples of R-Transforms

For $H$ with i.i.d. entries:

\[
R_{I}(w) = 1
\]

\[
R_{HH^{†}}(w) = \frac{1}{1 - \alpha w}
\]

\[
R_{(HH^{†})^{-1}}(w) = \frac{1 - \alpha - \sqrt{(1 - \alpha)^2 - 4\alpha w}}{2\alpha w}
\]
The Replica Method

We want

$$E = - \lim_{\beta \to \infty} \frac{1}{\beta} \lim_{K \to \infty} \frac{1}{K} E \log \sum_{x \in \mathcal{X}} e^{-\beta \text{tr}(Jxx^\dagger)}.$$

We know (Harish-Chandra integral)

$$\lim_{K \to \infty} \frac{1}{K} \log E J e^{-K \text{tr}JP} = -n \int_{\mathbb{R}} J(-w) \, dw.$$
We want

\[
\lim_{K \to \infty} \frac{1}{K} \mathbb{E} \log \sum_{x \in \mathcal{X}} e^{-\beta \text{tr}(Jxx^\dagger)}.
\]
The Replica Method

We want

\[
\lim_{K \to \infty} \frac{1}{K} E \log \sum_{x \in \mathcal{X}} e^{-\beta \text{tr}(Jxx^\dagger)}.
\]

We know (Harish-Chandra integral)

\[
\lim_{K \to \infty} \frac{1}{K} \log E e^{-K \text{tr} JP} = - \sum_{a=1}^{n} \int_{0}^{\lambda_a(P)} R_J(-w) dw.
\]

We would like to exchange expectation and logarithm.
The Replica Method

We want

$$\lim_{K \to \infty} \frac{1}{K} E \log \sum_{x \in \mathcal{X}} e^{-\beta \text{tr}(Jxx^\dagger)}.$$ 

We know (Harish-Chandra integral)

$$\lim_{K \to \infty} \frac{1}{K} \log E e^{-K \text{tr} JP} = -\sum_{a=1}^{n} \lambda_a(P) \int_0^R R_J(-w)dw.$$ 

We would like to exchange expectation and logarithm.

$$E \log X = \lim_{n \to 0} \frac{1}{n} \log E X^n$$
The Replica Method cont’d

We want

$$\lim_{K \to \infty} \frac{1}{K} \mathbb{E} \log \sum_{x \in \mathcal{X}} e^{-\beta \text{tr}(Jxx^\dagger)} = \lim_{K \to \infty} \lim_{n \to 0} \frac{1}{nK} \log \mathbb{E} \left( \sum_{x \in \mathcal{X}} e^{-\beta \text{tr}(Jxx^\dagger)} \right)^n$$
We want

\[ \lim_{K \to \infty} \frac{1}{K} \mathbb{E} \log \sum_{x \in \mathcal{X}} e^{-\beta \text{tr}(Jxx^\dagger)} = \lim_{K \to \infty} \lim_{n \to 0} \frac{1}{nK} \log \mathbb{E} \left( \sum_{x \in \mathcal{X}} e^{-\beta \text{tr}(Jxx^\dagger)} \right)^n \]

\[ = \lim_{K \to \infty} \lim_{n \to 0} \frac{1}{nK} \log \mathbb{E} \prod_{a=1}^{n} \sum_{x_a \in \mathcal{X}} e^{-\beta \text{tr}(Jx_a x_a^\dagger)} \]
The Replica Method cont’d

We want

\[
\lim_{K \to \infty} \frac{1}{K} \mathbb{E} \log \sum_{x \in \mathcal{X}} e^{-\beta \text{tr}(Jxx^\dagger)} = \lim_{K \to \infty} \lim_{n \to 0} \frac{1}{nK} \log \mathbb{E} \left( \sum_{x \in \mathcal{X}} e^{-\beta \text{tr}(Jxx^\dagger)} \right)^n \\
= \lim_{K \to \infty} \lim_{n \to 0} \frac{1}{nK} \log \mathbb{E} \prod_{a=1}^{n} \sum_{x_a \in \mathcal{X}} e^{-\beta \text{tr}(Jx_a x_a^\dagger)} \\
= \lim_{K \to \infty} \lim_{n \to 0} \frac{1}{nK} \log \mathbb{E} \sum_{x_1 \in \mathcal{X}} \cdots \sum_{x_n \in \mathcal{X}} e^{-\text{tr} \left( J \beta \sum_{a=1}^{n} x_a x_a^\dagger \right)}
\]
We want

\[
\lim_{K \to \infty} \frac{1}{K} \mathbb{E} \log \sum_{x \in \mathcal{X}} e^{-\beta \text{tr}(Jxx^\dagger)} = \lim_{K \to \infty} \lim_{n \to 0} \frac{1}{nK} \log \mathbb{E} \left( \sum_{x \in \mathcal{X}} e^{-\beta \text{tr}(Jxx^\dagger)} \right)^n
\]

\[
= \lim_{K \to \infty} \lim_{n \to 0} \frac{1}{nK} \log \mathbb{E} \prod_{a=1}^n \sum_{x_a \in \mathcal{X}} e^{-\beta \text{tr}(Jx_a x_a^\dagger)}
\]

\[
= \lim_{K \to \infty} \lim_{n \to 0} \frac{1}{nK} \log \mathbb{E} \sum_{x_1 \in \mathcal{X}} \cdots \sum_{x_n \in \mathcal{X}} e^{-\text{tr} \left( J \beta \sum_{a=1}^n x_a x_a^\dagger \right)} - \text{tr} \left( J \beta \sum_{a=1}^n x_a x_a^\dagger \right)
\]

\[
= \lim_{K \to \infty} \lim_{n \to 0} \frac{1}{nK} \log \mathbb{E} \exp \left[ -K \sum_{a=1}^n \int_0^\infty R_J(-w) dw \beta \lambda_a(Q) \right]
\]

with

\[
Q_{ab} \equiv \frac{1}{K} x_a^\dagger x_b.
\]
Laplace Integration

We find

\[
\lim_{K \to \infty} \frac{1}{K} \mathbb{E} \log \sum_{x \in \mathcal{X}} e^{-\beta \text{tr}(Jxx^\dagger)} = \lim_{K \to \infty} \lim_{n \to 0} \frac{1}{nK} \log \mathbb{E} \exp \left[-K \sum_{a=1}^{n} \beta \lambda_a(Q) \right]
\]

\[
= \lim_{n \to 0} \frac{1}{n} \min Q \left[ \beta \lambda_a(Q) \right]
\]

How to optimize over the \(n \times n\) matrix \(Q\) for \(n \to 0\)?
Laplace Integration

We find

\[
\lim_{K \to \infty} \frac{1}{K} \mathbb{E} \log \sum_{x \in \mathcal{X}} e^{-\beta \text{tr}(Jxx^\dagger)} = \lim_{K \to \infty} \lim_{n \to 0} \frac{1}{nK} \log \mathbb{E} \exp \left[ -K \sum_{a=1}^{n} \beta \lambda_a(Q) \int_{0}^{R} R_J(-w)dw \right] \\
= \lim_{n \to 0} \frac{1}{n} \min_{Q} \left[ - \sum_{a=1}^{n} \beta \lambda_a(Q) \int_{0}^{R} R_J(-w)dw \right]
\]
Laplace Integration

We find

\[
\lim_{K \to \infty} \frac{1}{K} \mathbb{E} \log \sum_{x \in \mathcal{X}} e^{-\beta \text{tr}(Jxx^\dagger)} = \lim_{K \to \infty} \lim_{n \to 0} \frac{1}{nK} \log \mathbb{E} \exp \left[ -K \sum_{a=1}^{n} \int_{0}^{\beta \lambda_a(Q)} R_J(-w) dw \right]
\]

\[
= \lim_{n \to 0} \frac{1}{n} \min_{Q} \left[ -\sum_{a=1}^{n} \int_{0}^{\beta \lambda_a(Q)} R_J(-w) dw \right]
\]

\[
\sim \lim_{n \to 0} \frac{1}{n} \min_{Q} \text{tr} \left[ QR_J(-\beta Q) \right].
\]

How to optimize over the \( n \times n \) matrix \( Q \) for \( n \to 0 \)?
We try the following ansatz

\[
Q \equiv \begin{bmatrix}
q + \frac{\chi}{\beta} & q & \cdots & q & q \\
q & q + \frac{\chi}{\beta} & \ddots & q & q \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
q & q & \cdots & q + \frac{\chi}{\beta} & q \\
q & q & \cdots & q & q + \frac{\chi}{\beta}
\end{bmatrix}
\]

with some macroscopic parameters \( q \) and \( \chi \).
Replica Symmetry (RS)

We try the following ansatz

\[
Q \equiv \begin{bmatrix}
q + \frac{x}{\beta} & q & \cdots & q & q \\
q & q + \frac{x}{\beta} & \ddots & q & q \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
q & q & \ddots & q + \frac{x}{\beta} & q \\
q & q & \cdots & q & q + \frac{x}{\beta}
\end{bmatrix}
\]

with some macroscopic parameters \( q \) and \( \chi \).

This is a critical step. In some cases, the structure of \( Q \) is more complicated. We try this structure first.
RS Solution

Let $q$ and $\chi$ be the simultaneous solutions to

$$q = \mathbb{E} \argmin_{s,z \in \mathcal{B}_s} \left| \sum_{x} z \sqrt{2q R'_J(-\chi)} - 2x R_J(-\chi) \right|$$

$$\chi = \frac{1}{\sqrt{2q R'_J(-\chi)}} \mathbb{E} \Re \left\{ z^* \argmin_{x \in \mathcal{B}_s} \left| z \sqrt{2q R'_J(-\chi)} - 2x R_J(-\chi) \right| \right\}$$

where $z$ is a unit variance zero-mean Gaussian random variable.

Then, replica symmetry (RS) implies

$$E = \frac{1}{K} \min_{x \in \mathcal{X}} x^\dagger J x \rightarrow q \frac{\partial}{\partial \chi} \chi R_J(-\chi)$$

as $K \rightarrow \infty$. 
Complex Lattice Precoding

\[
\frac{16}{\pi} \left(1 - \alpha\right)^{\frac{1}{\alpha} - 1}
\]
1-Step Replica Symmetry Breaking

\[
Q \equiv \begin{pmatrix}
\underbrace{q + p + \frac{\chi}{\beta}} & q + p & q & q & \cdots & q & q \\
q + p & q + p + \frac{\chi}{\beta} & q & q & \cdots & q & q \\
q & q & q + p + \frac{\chi}{\beta} & q + p & \vdots & q & q \\
q & q & q + p & q + p + \frac{\chi}{\beta} & \vdots && \\
\vdots & \vdots & \vdots & \vdots & q & q & q \\
q & q & q & \cdots & q & q + p + \frac{\chi}{\beta} & q + p \\
q & q & q & \cdots & q & q + p & q + p + \frac{\chi}{\beta}
\end{pmatrix}
\]

with the macroscopic parameters \( q, p \) and \( \chi \) and the blocksize \( \frac{\mu}{\beta} \).
1-Step Replica Symmetry Breaking

\[ E = \frac{1}{K} \min_{x \in \mathcal{X}} x^\dagger Jx \]

\[ \rightarrow \left( q + p + \frac{\chi}{\mu} \right) R_J(-\chi - \mu p) - \frac{\chi}{\mu} R_J(-\chi) - q(\mu p + \chi) R'_J(-\chi - \mu p) \]

The macroscopic parameters \( q, p, \chi \) and \( \mu \) are given by 4 coupled non-linear equations (omitted here).
Complex Lattice Precoding

\[ \frac{16}{\pi} \left(1 - \alpha\right)^{\frac{1}{\alpha} - 1} \]

- RS Solution
- Lower Bound
- 1RSB Solution
- Simulation Results: K=8
- Simulation Results: K=16
- Simulation Results: K=32

\( E \) [dB]

\( \alpha \)
**The Meaning of Replica Symmetry Breaking**

- Replica symmetry means that all vectors close to the optimum have the same inner products, i.e. they differ only in few components.
The Meaning of Replica Symmetry Breaking

- Replica symmetry means that all vectors close to the optimum have the same inner products, i.e. they differ only in few components.

- Replica symmetry breaking means that even vectors arbitrarily close to the optimum, may differ in a large portion of its components.
The Meaning of Replica Symmetry Breaking

- Replica symmetry means that all vectors close to the optimum have the same inner products, i.e. they differ only in few components.
  - If there are multiple local extrema, many of those are quite close to each other.

- Replica symmetry breaking means that even vectors arbitrarily close to the optimum, may differ in a large portion of its components.
The Meaning of Replica Symmetry Breaking

- Replica symmetry means that all vectors close to the optimum have the same inner products, i.e. they differ only in few components.
  - If there are multiple local extrema, many of those are quite close to each other.
  - The problem can often be well approximated by iterative algorithms like belief propagation.
- Replica symmetry breaking means that even vectors arbitrarily close to the optimum, may differ in a large portion of its components.
The Meaning of Replica Symmetry Breaking

- Replica symmetry means that all vectors close to the optimum have the same inner products, i.e. they differ only in few components.
  - If there are multiple local extrema, many of those are quite close to each other.
  - The problem can often be well approximated by iterative algorithms like belief propagation.

- Replica symmetry breaking means that even vectors arbitrarily close to the optimum, may differ in a large portion of its components.
  - There are local extrema at very different positions.
The Meaning of Replica Symmetry Breaking

- **Replica symmetry** means that all vectors close to the optimum have the same inner products, i.e. they differ only in few components.
  - If there are multiple local extrema, many of those are quite close to each other.
  - The problem can often be well approximated by iterative algorithms like belief propagation.

- **Replica symmetry breaking** means that even vectors arbitrarily close to the optimum, may differ in a large portion of its components.
  - There are local extrema at very different positions.
  - Belief propagation is often significantly suboptimum.
The Meaning of Replica Symmetry Breaking

- **Replica symmetry** means that all vectors close to the optimum have the same inner products, i.e. they differ only in **few** components.
  - If there are multiple local extrema, many of those are quite **close** to each other.
  - The problem can often be well approximated by iterative algorithms like **belief propagation**.

- **Replica symmetry breaking** means that even vectors arbitrarily close to the optimum, may differ in a **large** portion of its components.
  - There are local extrema at **very different** positions.
  - Belief propagation is often **significantly suboptimum**.

**RS (breaking) ranks the difficulty of approximating an NP-hard problem, in practice.**
Belief Propagation vs. Exhaustive Search

The graph compares the performance of Belief Propagation and Exhaustive Search for different values of $\alpha$. The x-axis represents $\alpha$, and the y-axis represents $E$ in dB. The graph shows two lines: one for RS Solution and another for 1RSB Solution. Additionally, there are markers for Exhaustive search: K=32 and Belief propagation: K=32. The RS Solution line is shown in blue, and the 1RSB Solution line is in red. The Exhaustive search and Belief propagation markers are indicated with stars and circles, respectively.
Convex Relaxation

... allows for convex programming.
Energy Penalty Comparison (QPSK)

- 1RSB Solution
- Discrete Lattice Simulation Results: K=32
- CR–QPSK – RS Solution
- CR–QPSK – Simulation Results: K=32

Graph shows the energy penalty comparison for QPSK with different solutions and simulation results.
Outer Single User Coding

\[ I(s_k, y_k) = h(y_k) - h(y_k | s_k) \]

Spectral efficiency (per transmit antenna) is given by

\[ C = \frac{1}{N} \sum_{k=1}^{K} I(s_k, y_k) \]
Conditional Channel Output Distribution $P(y|s)$

We characterize the joint distribution of all relevant variables in the channel.

$$p(s, x, y) = p(s)p(x|s)p(y|x, s)$$
Conditional Channel Output Distribution $P(y|s)$

We characterize the joint distribution of all relevant variables in the channel.

$$p(s, x, y) = p(s)p(x|s)p(y|x, s)$$

- The precoded signal $x$ is a deterministic function of data $s$. Thus, we have a Markov chain $s \rightarrow x \rightarrow y$. 
Conditional Channel Output Distribution $P(y|s)$

We characterize the joint distribution of all relevant variables in the channel.

$$p(s, x, y) = p(s)p(x|s)p(y|x)$$

- The precoded signal $x$ is a deterministic function of data $s$. Thus, we have a Markov chain $s \rightarrow x \rightarrow y$. 
Conditional Channel Output Distribution \( P(y|s) \)

We characterize the joint distribution of all relevant variables in the channel.

\[
p(s, x, y) = p(s)p(x|s)p(y|x)
\]

- The precoded signal \( x \) is a deterministic function of data \( s \). Thus, we have a Markov chain \( s \rightarrow x \rightarrow y \).
- The channel \( p(y|x) \) is an additive Gaussian noise channel.
Conditional Channel Output Distribution $P(y|s)$

We characterize the joint distribution of all relevant variables in the channel.

$$p(s, x, y) = p(s)p(x|s)p(y|x)$$

- The precoded signal $x$ is a deterministic function of data $s$. Thus, we have a Markov chain $s \rightarrow x \rightarrow y$.
- The channel $p(y|x)$ is an additive Gaussian noise channel.
- The channel $p(x|s)$ is found by the replica method via

$$P(x, s) = \lim_{h \to 0} \frac{\partial}{\partial h} \lim_{K \to \infty} \lim_{\beta \to \infty} \mathbb{E} \frac{1}{\beta K} \log \sum_{x \in \mathcal{X}} e^{-\beta x^\dagger Jx + \beta h \sum_{k=1}^{K} 1\{(x_k, s_k) = (x, s)\}}$$
Spectral Efficiency (Optimum $\alpha$)

$C \ [\text{bit/sec/Hz/Tx Antenna}]$

$E_b/N_0 \ [\text{dB}]$

- **DPC**
- **Linear Precoding – QPSK Input**
- **Lattice Precoding – QPSK Input**
- **Convex Precoding – QPSK Input**
- **GTHP – QPSK**
Inverting Singular Channels

What happens if the channel is rank-deficient, e.g. $K > N$?

Can we precode without interference?
Inverting Singular Channels

What happens if the channel is rank-deficient, e.g. \( K > N \)?

Can we precode without interference?

- The precoder produces

\[
\lim_{\epsilon \to 0} \arg\min_{x \in \mathcal{X}} \frac{x^\dagger (HH^\dagger + \epsilon I)^{-1} x}{K}
\]

- The received signal becomes

\[
y = \lim_{\epsilon \to 0} HH^\dagger (HH^\dagger + \epsilon I)^{-1} x + n.
\]

If the energy is finite, there is no interference.
The probability that a random \( N \) dimensional subspace in \( K \) real dimensions intersects the 1. \( K \)-tant is (Wendel '62)

\[
P(K, N) = 2^{1-K} \sum_{\ell=0}^{N-1} \binom{K-1}{\ell}
\]

As \( K, N \) to infinity, we get

\[
P(K, N) = \begin{cases} 
1 & K < 2N \\
1/2 & K = 2N \\
0 & K > 2N 
\end{cases}
\]
The probability that a random $N$ dimensional subspace in $K$ complex dimensions intersects the 1. $K$-tant is

$$P(K, N) = 2^{1-2K} \sum_{\ell=0}^{2N-1} \binom{2K-1}{\ell}$$

As $K, N$ to infinity, we get

$$P(K, N) = \begin{cases} 1 & K < 2N \\ 1/2 & K = 2N \\ 0 & K > 2N \end{cases}$$
Overloaded Convex Precoding
Wanted

More general versions of the Harish-Chandra integral, e.g.

- no unitary invariance, only freeness required

\[ \lim_{K \to \infty} \frac{1}{K} \log \mathbb{E}_{A,B} e^{-K \text{tr} APBP} = f\{R_A(\cdot), R_B(\cdot), \ldots \} \]

- or other more complicated exponents
Negative Entropy

\[ S = \chi R_J(-\chi) - \int_0^{\chi} R_J(-w)dw \]

The closer the entropy is to zero, the better the RSB approximation.