On spectral properties of large dilute Wigner random matrices

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We study the spectral norm (maximal eigenvalue $\lambda_{\text{max}}$) of $n \times n$ random real symmetric matrices $H^{(n,\rho)}$ whose elements $H_{ij}^{(n,\rho)}$, $i \leq j$ are given by jointly independent random variables, similarly to the well-known ensemble of Wigner real symmetric matrices.

The difference between $H^{(n,\rho)}$ and the Wigner ensemble is that $H_{ij}^{(n,\rho)}$ is equal to 0 with probability $1 - \rho/n$ (dilute version). The concentration parameter $\rho = \rho_n$ represents the average number of non-zero elements per row in $H^{(n,\rho)}$.

Our results show that in the asymptotic regime when $\rho_n = n^\alpha$, $n \to \infty$, the value $\alpha = 2/3$ is the critical one with respect to the asymptotic behavior of $\lambda_{\text{max}}$. 
I.1. Dilute Wigner random matrices

\[ H_{ij}^{(n,\rho)} = \frac{1}{\sqrt{\rho}} a_{ij} b_{ij}^{(n,\rho)}, \ 1 \leq i \leq j \leq n, \]

where \( \{a_{ij}, i \leq j\} \) are jointly independent r.v. with symmetric probability distribution and

\[ b_{ij}^{(n,\rho)} = \begin{cases} 1, & \text{with probability } \rho/n \\ 0, & \text{with probability } 1 - \rho/n \end{cases} \]

independent r.v. also independent from \( a_{ij} \).

i) If \( \rho = n \), then the matrix

\[ H_{ij}^{(n)} = \frac{1}{\sqrt{n}} a_{ij} \]

represents the Wigner ensemble of real symmetric random matrices;

ii) \( 1 \ll \rho \ll n \), dilute version of Wigner RM;

iii) \( \rho_n = O(1), n \to \infty \), sparse RM.
I.2. Semi-circle law (Wigner law)

a) Normalized eigenvalue counting function (NCF)

$$\sigma_n(\lambda) = \frac{1}{n} \# \{ j : \lambda_j^{(n)} \leq \lambda \}$$

converges as $n \to \infty$ to $\sigma_W(\lambda)$ with the density

$$\frac{d}{d\lambda} \sigma_W(\lambda) = \frac{1}{2\pi v^2} \sqrt{4v^2 - \lambda^2}, \quad |\lambda| \leq 2v,$$

where $v^2 = E a_{ij}^2$ [E. Wigner, 1955].

b) Spectral norm $\lambda_{\text{max}}^{(n)} = \max_k \{|\lambda_k^{(n)}|\}$ converges to $2v$ [S. Geman, 1980; Z. Füredi and J. Komlós, 1981, V. Girko, 1988; Z.-D. Bai and Y. Q. Yin, 1988];

$$\lambda_{\text{max}}^{(n)} \to 2v \text{ as } n \to \infty;$$

in particular,

$$P \left\{ \lambda_{\text{max}}^{(n)} \geq 2v(1 + x) \right\} \to 0, \ x > 0.$$
I.3 Dilution of random matrices

- **Random graphs**: symmetric random matrix

\[ B_{ij} = \begin{cases} 
1, & \text{with probability } \rho/n \\
0, & \text{with probability } 1 - \rho/n 
\end{cases} \]

is the adjacency matrix of random graph \( G_n(P_n) \) with \( n \) vertices and with the edge probability

\[ P_n = \rho/n \]

(P. Edős and A. Rényi, 1959; E. Gilbert, 1959)

- **Theoretical physics**: dilute and sparse disordered systems
  - [Rodgers-Bray, 1988]
  - [Mirlin-Fyodorov, 1991]

- **Neural networks theory**

- etcetera, ...
I.4 Semicircle law in dilute RM

In $H^{(n,\rho)}$ a number of bonds (connections) between cites $i$ and $j$ destroyed, the structure of random matrix is changed.

However, if $\rho_n \to \infty$ as $n \to \infty$, the Wigner (or semicircle) law is still valid,

$$\sigma_{n,\rho_n}(\lambda) \to \sigma_W(\lambda)$$

with $\text{supp}(\sigma'_W) = [-2v, 2v]$

- [Rodgers-Bray, 1988]
- [K., Khoruzhenko, Pastur, Shcherbina, 1992]
- [Cazati-Girko, 1992]
- ...

What about $\lambda_{\text{max}}^{(n,\rho)} \to ?$ and

$$\mathbb{P}\left\{\lambda_{\text{max}}^{(n,\rho)} > 2v(1 + x_n)\right\} ?$$
II. Critical value for the spectral norm


If \( \rho_n = (\log n)^{1+\beta}, \beta > 0, \) then

\[
P \left\{ \lambda_{\text{max}}^{(n,\rho)} > 2v(1+x) \right\} \to 0, \ x > 0.
\]

If \( \rho_n = (\log n)^{1-\beta'} \) with \( \beta' > 0, \) then

\[
\limsup_{n \to \infty} \lambda_{\text{max}}^{(n,\rho)} = +\infty.
\]

Conclusion: the value \( \rho_n^* = \log n \) is critical for the asymptotic behavior of \( \lambda_{\text{max}}^{(n,\rho_n)}. \)

Relation with the properties of large random graphs: the edge probability

\[
P_n^* = \frac{\log n}{n}
\]

is the critical one (a sharp threshold) with respect to the connectedness of the random graph \( G_n(P_n). \)
III.1 Moments of random matrices

Since the works of E. Wigner, the moments

$$M_{2k}^{(n)} = \mathbb{E} \frac{1}{n} \text{Tr} \left( H^{(n)} \right)^{2k}, \ k = 0, 1, 2, \ldots$$

have been used to study the moments of $\sigma_n(\lambda)$,

$$M_{2k}^{(n)} = \mathbb{E} \frac{1}{n} \sum_{j=1}^{n} \left( \lambda_j^{(n)} \right)^{2k} = \mathbb{E} \int \lambda^{2k} \ d\sigma_n(\lambda).$$

In particular, E. Wigner has shown that

$$M_{2k}^{(n)} \rightarrow v^{2k} \frac{(2k)!}{k!(k + 1)!} = v^{2k} t_k,$$

where $t_k$ are the Catalan numbers.

The key idea of S. Geman [Ann. Probab., 1980] inspired by U. Grenander is that the limiting behavior of $\lambda_{\text{max}}^{(n)}$ can be studied by means of the high moments

$$nM_{2k_n}^{(n)}, \ n, k_n \rightarrow \infty.$$
III.2 High moments of Wigner RM

1) $k_n = O(\log n)$ [Geman, 1980; Bai-Yin, 1988]

\[ M_{2k_n}^{(n)} \leq \left( v^2(1 + \varepsilon) \right)^{k_n} t_{k_n}, \quad k_n = O(\log n) \]

implies that

\[ P \{ \lambda_{\text{max}}^{(n)} > 2v(1 + x) \} \to 0 \ \text{as} \ n \to \infty; \]

2) $k_n = O(n^{1/6})$ [Füredi-Komlós, 1981]

\[ k_n = O(n^{1/2}), \quad k_n = o(n^{2/3}) \]

[Ya. G. Sinai and A. Soshnikov, 1998]

3) $k_n = \chi n^{2/3}, \chi > 0$ [A. Soshnikov, 1999]:

\[ nM_{2k_n}^{(n)} \to \mathcal{L}(\chi) = \mathcal{L}_{\text{GOE}}(\chi), \]

where $\mathcal{L}(\chi)$ does not depend on the details of the probability distribution of $a_{ij}$; as a corollary, one gets

\[ P \left\{ \lambda_{\text{max}}^{(n)} > 2v \left(1 + \frac{y}{n^{2/3}}\right) \right\} \leq \mathcal{G}_\chi(y), \ y > 0. \]

The border spectral scale is $n^{-2/3}$. 
IV.1 Dilute Wigner RM

**Theorem** [K., arXiv-2011, in preparation]

Let the probability law of $a_{ij}$ has a finite support. Then

$$P \left\{ \lambda_{\max}^{(n, \rho_n)} > 2v \left( 1 + \frac{y}{n^{2/3}} \right) \right\} \leq G_\chi(y), \ y > 0$$

for $\rho_n = n^{2/3(1+\gamma)}$ with any given $\gamma > 0$.

**Main technical results:**

**A)** If $\rho_n = n^{2/3(1+\gamma)}$, $\gamma > 0$, then

$$\limsup_{n \to \infty} nM_{2k_n}^{(n, \rho_n)} \leq \mathcal{L}(\chi), \ \ k_n = \chi n^{2/3}.$$

The upper bound $\mathcal{L}$ is universal in the sense that it does not depend on higher moments $V_4, V_6, \ldots$, where $V_{2l} = \mathbf{E}|a_{i,j}|^{2l}, \ l \geq 2$.

**B)** If $\rho_n = n^{2/3}$ and $k_n = \chi n^{2/3}$, then

$$nM_{2k_n}^{(n, \rho_n)} \geq \ell(\chi) (1 + \chi V_4), \ n \to \infty.$$
IV.2 Critical value for border scale

Our results show that the value $\rho_n = n^{2/3}$ represents a critical value for the spectral properties at the border of the spectrum $2\nu$: 

- if the dilution is *weak*, $\rho_n \gg n^{2/3}$, then one can expect that the local spectral properties of Dilute RM are the same as for the Wigner RM ensembles; these properties should be independent on the details of the probability distribution of $a_{ij}$.

To prove: correlation function of the moments, Moment version of IPR (K. *arXiv*, 2010)

- if the dilution is *moderate*, $\rho_n = O(n^{2/3})$, then the asymptotic behavior of $\lambda_{\text{max}}^{(n)}$ will depend on $V_4 = \mathbf{E}|a_{ij}|^4$. The same can be true for other local spectral characteristics.

- in the case of *strong dilution*, $\rho_n \ll n^{2/3}$, the spectral scale at the border $2\nu$ changes from $\frac{1}{n^{2/3}}$ to $\frac{\phi(n)}{\rho}$, with $\phi(n) = \log n$ (?)
V. Relations with the Wigner RM

The value of $\gamma$ in $\rho_n = n^{2/3(1+\gamma)}$ depends on the moments $V_{2l} = \mathbb{E}|a_{ij}|^{2l}$:

\[ \text{if } V_{12+2\phi} < \infty, \text{ then } \gamma > \varepsilon = \frac{3}{6 + \phi}. \]

Inversely, if $\rho_n = n^{2/3(1+\gamma)}$, then the universal upper bound of $nM_{2k_n}^{(n,\rho_n)}$ exists provided

\[ \phi > \frac{3}{\gamma} - 6. \]

For the Wigner ensemble, we have $\rho_n = n$, $\gamma = 1/2$ and then $\phi > 0$, in accordance with the following generalization of earlier results [A. Soshnikov, 1999];

**Theorem** [K. 2012] *If* $V_{12+2\delta}$ *exists for any* $\delta > 0$, *then for the Wigner RM,*

\[ \lim_{n \to \infty} n M_{2k_n}^{(n)} = \mathcal{L}_{\text{GOE}}(\chi), \quad k_n = \chi n^{2/3}, \]

*where* $\mathcal{L}_{\text{GOE}}$ (*or* $\mathcal{L}_{\text{GUE}}$) *does not depend on the moments of* $V_{2l}$, $l = 2, \ldots, 6$ *and on* $V_{12+2\delta}$. 

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VI.1 Proof of the upper bound


Start point: E. Wigner’s representation of traces

\[ nM_{2k} = \sum_{i_0, \ldots, i_{2k-1}} \mathbb{E} \left\{ H_{i_0, i_1} \cdots H_{i_{2k-1}, i_0} \right\} \]

as a sum over 2k-step trajectories

\[ \mathcal{I}_{2k} = (i_0, i_1, i_2, \ldots, i_{2k-2}, i_{2k-1}, i_0). \]

The family \{\mathcal{I}_{2k}\} can be separated into the classes of equivalence determined by the number \( \mathcal{K} \) of self-intersections of the trajectories.

When \( \mathcal{K} = 0 \), the classes are described by the family \( \mathcal{D}_{2k} \) of the Dyck paths: discrete simple walks of 2k steps in the upper half-plane that start and end at 0. These are equivalent to the rooted half-plane trees. The cardinality \(|\mathcal{D}_{2k}|\) is given by the Catalan number \( t_k \).
VI.2 Technical questions

- Wigner RM, Sinai-Soshnikov approach: the study of simple self-intersections (open ones; $V_4$-direct); vertex of maximal exit degree $\beta$;

- K., Vengerovsky: proper and imported cells at $\beta$; Brocken-Tree-Structure instants;

- K. *Rand. Oper. Stoch. Eqs.*: $V_4$-direct and inverse edges; generalization to the case of $V_{2k}$

- Dilute RM, K. 2012: detailed study of the vertex $\beta$ of maximal exit degree $D$;

  \[ D = d_1 + \ldots + d_L, \quad \bar{d}_L = (d_1, \ldots, d_L). \quad (A) \]

  The following statement improves the tools used by Ya. G. Sinai and A. Soshnikov.

**D-lemma.** Denote by $T^{(u)}(\bar{d}_L)$ the family of Catalan trees of height $u$ that have $L$ vertices of exit degrees $\bar{d}_L$ (A). Then

\[
\sum_{u=1}^{k} e^{\chi u / \sqrt{k}} |T^{(u)}(\bar{d}_L)| \leq L e^{-\eta D} B(\chi) t_k,
\]

where $\eta = \ln(4/3)$ and $B(\chi)$ is related with the Brownian bridge.
VII.1 Tree-type walks with multiple edges

Each plane tree generates, by the chronological run over it, a walk of $2k$ steps such that each edge is passed exactly two times (there and back). The number of these Catalan walks is

$$t_k = \frac{(2k)!}{k!(k+1)!}, \quad k \geq 0.$$ 

Lemma [K., arXiv, 2012] Consider the family of Catalan-type walks of $2k$ steps such that there exists exactly one special edge passed four times. Then its cardinality is given by

$$t^{(2)}_k = \frac{(2k)!}{(k-2)!(k+2)!}, \quad k \geq 2,$$

with obvious equalities $t^{(2)}_0 = t^{(2)}_1 = 0$.

Remark. The cardinality of Catalan walks with one colored edge is obviously equal to

$$t^{(1)}_k = \frac{(2k)!}{(k-1)!(k+1)!}, \quad k \geq 1.$$
VII.2 Bound from below

Relation
\[
t^{(2)}_k = \frac{(2k)!}{(k - 2)! (k + 2)!} = \left(k - \frac{3k}{k + 2}\right) t_k
\]
shows that \( t^{(2)}_k \geq k t_k / 2, k \geq 4 \). This implies the lower bound for the moments of \( H^{(n,\rho_n)} \).

Indeed,
\[
E \left( H^{(n,\rho_n)} \right)^{2k} \geq n t_k V_2^2 + nV_2^{k-2} \cdot \frac{V_4}{\rho} \cdot t^{(2)}_k
\]
\[
\geq n t_k V_2^2 \left(1 + \frac{k V_4}{2 \rho V_2^2}\right).
\]

Therefore, if \( k = \chi n^{2/3} \) and \( \rho = n^{2/3} \), then the estimate from below explicitly contains a non-vanishing term \( \chi V_4 / 2V_2^2 \).

This means that the estimate from above of the moments of the dilute random matrices in the asymptotic regime \( \rho = n^{2/3} \) is crucially different from that in the regime \( \rho = o(n^{2/3}) \).
VI.3 Recurrent relations for $t_k^{(2)}$

The Catalan numbers $t_k$ are determined by recurrence

$$t_k = \sum_{u+v=k-1} t_u t_v, \quad k \geq 1,$$

$t_0 = 1$; it can be obtained with the help of the reduction of the ground step procedure.

A simple reasoning shows that

$$t_k^{(2)} = \sum_{u+v+r+s = k-2} (2u + 1) t_u t_v t_r t_s,$$

for $k \geq 2$. The use of the generating function of $t_k$ leads to the explicit expression for $t_k^{(2)}$.

Several first values of

$$t_k^{(2)} = \frac{(2k)!}{(k-2)! (k+2)!}$$

are as follows,

$$1, 6, 28, 120, 495, \ldots$$

At present time, the N. Sloan’s encyclopedia of integer sequences (OEIS) says nothing about this sequence.
VI.4 More about the sequences $t_k^{(m)}$

Let us denote by $t_k^{(m)}$, $m \geq 1$ the set of even closed tree-type walks of $2k$ steps such that all edges are passed two times (there and back) and there exists one special edge passed $2m$ times.

**Question:** what is the explicit form of $t_k^{(3)}$?

$$t_k^{(m)} = \sum_{u + v_1 + \ldots + v_{2m-1} = k-m} (2u+1)t_u t_{v_1} \cdots t_{v_{2m-1}}.$$

**Answer:** it is not hard to show that

$$t_k^{(3)} = \frac{(2k)!}{(k-3)! (k+3)!}, \quad k \geq m \geq 3.$$

It is natural to assume that for any $m \geq 1$,

$$t_k^{(m)} = \frac{(2k)!}{(k-m)! (k+m)!}, \quad k \geq m.$$
VII.5 Why to study $t_k^{(m)}$?

In the regime $\rho = n^{2/3}$, the estimate from below of the moments involves the terms

$$n \, M_{2k}^{(n,\rho)} \geq n t_k \, V_2^2 \left(1 + \frac{k \, V_4}{2\rho \, V_2^2} + \frac{k \, V_6}{6\rho^2 \, V_2^3} + \ldots\right)$$

for sufficiently large values of $k$ because

$$t_k^{(3)} = \frac{(2k)!}{(k-3)! \, (k+3)!} = t_k \left(k - 8 - \frac{36k + 48}{k^2 + 5k + 6}\right).$$

Expression of the form $\frac{k \, V_6}{\rho^2 \, V_2^3}$ means that the terms with $V_6$ should disappear from the limiting expression for $n \, M_{2k}^{(n,\rho)}$. The same could be true for the terms with $V_8, V_{10}, \ldots$.

**Conjecture.** The limiting expression for $n \, M_{2k}^{(n,\rho)}$ with $\rho_n = n^{2/3}$ contains the Wigner-GOE part (Wigner-GUE part for the case of Hermitian matrices) and the terms that involve $V_4$, but not $V_{2k}$, $k \geq 3$. 

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VIII.1 Beyond the threshold $n^{2/3}$

Let us try to imagine the picture for the strong dilution regime $\rho_n \ll n^{2/3}$. One can expect the following phenomena in the walks:

- the walks that have self-intersections of degree $\kappa = 3$ disappear from the limiting $n M_{2k}^{(n,\rho)}$;

- the walks that have simple self-intersections with broken tree structure disappear from the limiting $n M_{2k}^{(n,\rho)}$;

  **Consequence:** the difference between real symmetric and hermitian cases vanishes;

- if our $V_4$-conjecture is true, then the walks that have multiple edges $V_{2l}$ with $l \geq 3$ disappear from the limiting expression for $n M_{2k}^{(n,\rho)}$.

One could assume that the leading contribution to $n M_{2k}^{(n,\rho)}$ is given by the tree-type walks with simple self-intersections only ($\kappa = 2$) that have 2- and 4-multiple edges.
VIII.2 Basic walks for moments

Instead of the Catalan walks of $2k$ steps, where each edge is passed two times (there and back), the walks with 2- and 4-multiple edges could play the role of the basic walks. So, the number of such basic walks is given by the number $T_k = T_k^{(2,4)}$ of these $(2, 4)$-Catalan walks.

We can write that $T_k = R_k^{(0)}(\rho)$, where

$$R_k^{(0)} = a \sum_{u=0}^{k-1} R_{k-1-u}^{(0)} R_u^{(0)} + \frac{b}{k} \sum_{u=0}^{k-2} R_{k-2-u}^{(1)} R_u^{(1)}$$

with $a = V_2 = v^2$ and $b = \chi V_4$.

This recurrent relation resembles the one for the semicircle moments $v^{2k} t_k$, but is in fact (much) more complicated.

Finally, to find the limiting expression for $\mathcal{L}_{\text{DRM}}$, one could try with

$$\lim_{n,k \to \infty} n R_k^{(0)}(\rho), \quad \rho = \chi k.$$
VIII.3 Equations for $R_k^{(m)}$

For $k \geq 1$ and $m \geq 1$, we have

$$R_k^{(m)} = R_k^{(m-1)} + a \sum_{u=0}^{k-1} R_{k-1-u}^{(0)} R_u^{(m)}$$

$$+ \frac{b}{k} \sum_{u=0}^{k-2} R_{k-2-u}^{(1)} R_u^{(m+1)},$$

where $a = v^2 = V_2$ and $b = \chi V_4$.

In other terms,

$$R_k^{(m)} = \sum_{r=0}^{k} (r+1)(r+2) \cdots (r+m) S(k, r);$$

the numbers $S(k, r)$, $1 \leq r \leq k$ are uniquely determined by recurrence

$$S(k, r) = a \sum_{u=0}^{k-r} \sum_{v=0}^{u} S(u, v) S(k-u-1, r-1)$$

$$+ \frac{b}{k} \sum_{u=0}^{k-r} (r-1) \sum_{v=0}^{u} (v+1) S(u, v) S(k-u-2, r-2).$$