Local Universality of Repulsive Particle Systems and Random Matrices

Friedrich Götze
joint with M.Venker, A.Naumov and A.Tikhomirov

Bielefeld University
www.math.uni-bielefeld.de/~goetze

Workshop ”Random Matrices and their Applications”
Télécom Paristech October 9, 2012
Topics

- Local Correlation Statistics for Repulsive Systems: GUE-Limits (G.- M. Venker)
Topics

- Local Correlation Statistics for Repulsive Systems: GUE-Limits (G.-M. Venker)
- Local Correlation Statistics for Repulsive Systems: $\beta$-Ensemble Universality (M. Venker)
Topics

- Local Correlation Statistics for Repulsive Systems: 
  GUE-Limits (G.- M. Venker)

- Local Correlation Statistics for Repulsive Systems: 
  $\beta$-Ensemble Universality (M. Venker)

- Wigner’s Semicircular Law for Martingale Ensembles
  (G.- A. Naumov and A. Tikhomirov)
Repulsive Particle Systems and GUE

Empirical evidence for "repulsive particle systems" in $\mathbb{R}$

- Distances between parked cars. (Abul-Magd (2005))
- Bus stop waiting times in certain cities. (Krbalek, Seba (2000), Beik, Borodin, Deift, Suidan (2006))
- Spacings of zeros of the Riemann Zeta function on the critical line. (Montgomery, Keating, Snaith)
Empirical evidence for "repulsive particle systems" in $\mathbb{R}$ exhibiting GUE-spacing statistics?
Empirical evidence for “repulsive particle systems” in $\mathbb{R}$ exhibiting GUE-spacing statistics?

- Distances between parked cars. (Abul-Magd (2005))
Empirical evidence for "repulsive particle systems" in $\mathbb{R}$ exhibiting GUE-spacing statistics?

- Distances between parked cars. (Abul-Magd (2005))

- Bus stop waiting times in certain cities. (Krbalek, Seba (2000), Beik, Borodin, Deift, Suidan (2006))
Empirical evidence for "repulsive particle systems" in $\mathbb{R}$ exhibiting GUE-spacing statistics?

- Distances between parked cars. (Abul-Magd (2005))
- Bus stop waiting times in certain cities. (Krbalek, Seba (2000), Beik, Borodin, Deift, Suidan (2006))
- Spacings of zeros of the Riemann Zeta function on the critical line. H. Montgomery, J. Keating, N. Snaith
Repulsive Particle Systems and GUE

Empirical evidence for "repulsive particle systems" in $\mathbb{R}$ exhibiting GUE-spacing statistics?

- Distances between parked cars. (Abul-Magd (2005))
- Bus stop waiting times in certain cities. (Krbalek, Seba (2000), Beik, Borodin, Deift, Suidan (2006))
- Spacings of zeros of the Riemann Zeta function on the critical line. H. Montgomery, J. Keating, N. Snaith

All these particle systems show the phenomenon of repulsion.
Class of Models for Repulsive Particle Systems

Suggestive to study repulsion models with relation to matrix ensembles:
Class of Models for Repulsive Particle Systems

Suggestive to study repulsion models with relation to matrix ensembles:
For a "smooth" even function $\varphi > 0$ define a density of particles $x_j, 1 \leq j \leq N$
Class of Models for Repulsive Particle Systems

Suggestive to study repulsion models with relation to matrix ensembles:
For a "smooth" even function $\varphi > 0$ define a density of particles $x_j, 1 \leq j \leq N$ w.r.t. a potential $Q$ by

$$Z_N, Q \prod_{j < k} \varphi(x_k - x_j)e^{-N \sum_{j=1}^N Q(x_j)}.$$
Class of Models for Repulsive Particle Systems

Suggestive to study repulsion models with relation to matrix ensembles:
For a "smooth" even function $\varphi > 0$ define a density of particles $x_j, 1 \leq j \leq N$ w.r.t. a potential $Q$ by

$$
\frac{1}{Z_{N,Q}} \prod_{j<k} \varphi(x_k - x_j) e^{-N \sum_{j=1}^{N} Q(x_j)}.
$$
Class of Models for Repulsive Particle Systems

Suggestive to study repulsion models with relation to matrix ensembles:
For a "smooth" even function \( \varphi > 0 \) define a density of particles \( x_j, 1 \leq j \leq N \) w.r.t. a potential \( Q \) by

\[
\frac{1}{Z_{N,Q}} \prod_{j<k} \varphi(x_k - x_j) e^{-N \sum_{j=1}^{N} Q(x_j)}.
\]

Writing: \( \varphi(t) := t^2 e^{-h(t)} \), the non unitary invariant deformation of

\[
P_{N,Q}(x) := \frac{1}{Z_{N,Q}} \exp \left\{ 2 \sum_{j<k} \log |x_k - x_j| - N \sum_{j=1}^{N} Q(x_j) \right\}, \quad \text{is}
\]
Class of Models for Repulsive Particle Systems

Suggestive to study repulsion models with relation to matrix ensembles:
For a "smooth" even function $\varphi > 0$ define a density of particles $x_j$, $1 \leq j \leq N$ w.r.t. a potential $Q$ by

$$\frac{1}{Z_{N,Q}} \prod_{j<k} \varphi(x_k - x_j)e^{-N\sum_{j=1}^{N} Q(x_j)}.$$

Writing: $\varphi(t) := t^2 e^{-h(t)}$, the non unitary invariant deformation of

$$P_{N,Q}(x) := \frac{1}{Z_{N,Q}} \exp \left\{ 2 \sum_{j<k} \log |x_k - x_j| - N \sum_{j=1}^{N} Q(x_j) \right\},$$

$$P_{N,\varphi}^h(x) := \frac{1}{Z_{N,Q}^h} \exp \left\{ 2 \sum_{j<k} \log |x_k - x_j| - \sum_{j<k} h(x_k - x_j) - N \sum_{j=1}^{N} Q(x_j) \right\}.$$
Assumptions

Assumptions on potential $Q$:

- Symmetric around zero
- Real analytic
- Strongly convex: $\min_{t \in \mathbb{R}} Q''(t) > 0$

Assumptions on $h$:

- Symmetric around zero
- Schwartz function
- Real analytic
Assumptions on potential $Q$:

- symmetric around zero
Assumptions

Assumptions on potential $Q$:

- symmetric around zero
Assumptions on potential $Q$:

- symmetric around zero
- real analytic
Assumptions

Assumptions on potential $Q$:

- symmetric around zero
- real analytic

Strongly convex: $\min_{t \in \mathbb{R}} Q''(t) > 0$
Assumptions

Assumptions on potential $Q$:

- symmetric around zero
- real analytic
- strongly convex: $\min_{t \in \mathbb{R}} Q''(t) > 0$
Assumptions

Assumptions on potential $Q$:

- symmetric around zero
- real analytic
- strongly convex: $\min_{t \in \mathbb{R}} Q''(t) > 0$
Assumptions

Assumptions on potential $Q$:

- symmetric around zero
- real analytic
- strongly convex: $\min_{t \in \mathbb{R}} Q''(t) > 0$

Assumptions on $h$: 

Assumptions

Assumptions on potential $Q$:

- symmetric around zero
- real analytic
- strongly convex: $\min_{t \in \mathbb{R}} Q''(t) > 0$

Assumptions on $h$:

- symmetric around zero
Assumptions

Assumptions on potential $Q$:
- symmetric around zero
- real analytic
- strongly convex: $\min_{t \in \mathbb{R}} Q''(t) > 0$

Assumptions on $h$:
- symmetric around zero
Assumptions

Assumptions on potential $Q$:

- symmetric around zero
- real analytic
- strongly convex: $\min_{t \in \mathbb{R}} Q''(t) > 0$

Assumptions on $h$:

- symmetric around zero
- Schwartz function
Assumptions

Assumptions on potential $Q$:

- symmetric around zero
- real analytic
- strongly convex: $\min_{t \in \mathbb{R}} Q''(t) > 0$

Assumptions on $h$:

- symmetric around zero
- Schwartz function
Assumptions

Assumptions on potential $Q$:

- symmetric around zero
- real analytic
- strongly convex: $\min_{t \in \mathbb{R}} Q''(t) > 0$

Assumptions on $h$:

- symmetric around zero
- Schwartz function
- real analytic
Global Marginal Distributions

\[ \rho_{N,Q}^{h,k}(x_1, \ldots, x_k) := \int_{\mathbb{R}^{N-k}} P_{N,Q}^h(x) dx_{k+1} \cdots dx_N : \]

the \( k \)-th correlation function of \( P_{N,Q}^h \).
Global Marginal Distributions

\[ \rho_{N,Q}^{h,k}(x_1, \ldots, x_k) := \int_{\mathbb{R}^{N-k}} P_{N,Q}^h(x) dx_{k+1} \ldots dx_N : \]
the \( k \)-th correlation function of \( P_{N,Q}^h \).

**Thm** (G.-Venker ’12)

For all \( h \) above exist \( \alpha_h > 0 \) s.th. for all \( Q \) above with \( \min_{t \in \mathbb{R}} Q''(t) > \alpha_h \),
Global Marginal Distributions

\[ \rho_{h,k}^{N,Q}(x_1, \ldots, x_k) := \int_{\mathbb{R}^{N-k}} P_{h,N,Q}^k(x) \, dx_{k+1} \ldots dx_N : \]
the $k$-th correlation function of $P_{h,N,Q}^k$.

**Thm** (G.-Venker ’12)

For all $h$ above exist $\alpha_h > 0$ s.th. for all $Q$ above with $\min_{t \in \mathbb{R}} Q''(t) > \alpha_h$, there exists $\mu_{Q}^h$, p-measure with compact support s.th.
Global Marginal Distributions

\[ \rho_{N,Q}^{h,k}(x_1, \ldots, x_k) := \int_{\mathbb{R}^{N-k}} P_{N,Q}^h(x) dx_{k+1} \ldots dx_N : \]

the \( k \)-th correlation function of \( P_{N,Q}^h \).

**Thm** (G.-Venker ’12)

For all \( h \) above exist \( \alpha_h > 0 \) s.th. for all \( Q \) above with \( \min_{t \in \mathbb{R}} Q''(t) > \alpha_h \),

there exists \( \mu_Q^h \), \( p \)-measure with compact support s.th.

\( (k\)-th correlation measure of \( P_{N,Q}^h ) \Rightarrow (\mu_Q^h)^{\otimes k} \) as \( N \to \infty \),
Global Marginal Distributions

\[ \rho_{N,Q}^{h,k}(x_1, \ldots, x_k) := \int_{\mathbb{R}^{N-k}} P_{N,Q}^h(x) \, dx_{k+1} \ldots dx_{N} : \]

the $k$-th correlation function of $P_{N,Q}^h$.

**Thm** (G.-Venker ’12)

For all $h$ above exist $\alpha_h > 0$ s.th. for all $Q$ above with $\min_{t \in \mathbb{R}} Q''(t) > \alpha_h$,
there exists $\mu_Q^h$, $p$-measure with compact support s.th.

$(k$-th correlation measure of $P_{N,Q}^h) \Rightarrow (\mu_Q^h)^{\otimes k}$ as $N \to \infty$,
i.e. for $g \in C_b(\mathbb{R}^k)$
Global Marginal Distributions

\[ \rho^{h,k}_{N,Q}(x_1, \ldots, x_k) := \int_{\mathbb{R}^{N-k}} P^h_{N,Q}(x) dx_{k+1} \ldots dx_N : \]
the \( k \)-th correlation function of \( P^h_{N,Q} \).

**Thm** (G.-Venker ’12)

For all \( h \) above exist \( \alpha_h > 0 \) s.th. for all \( Q \) above with \( \min_{t \in \mathbb{R}} Q''(t) > \alpha_h \),
there exists \( \mu^h_Q \), p-measure with compact support s.th.
(\( k \)-th correlation measure of \( P^h_{N,Q} \)) \( \Rightarrow (\mu^h_Q)^\otimes k \) as \( N \to \infty \),
i.e. for \( g \in C_b(\mathbb{R}^k) \)

\[
\lim_{N \to \infty} \int_{\mathbb{R}^k} g(t) \rho^{h,k}_{N,Q}(t) dt^k = \int_{\mathbb{R}^k} g(t)(\mu^h_Q)^\otimes k(dt).
\]
Global Marginal Distributions

\[ \rho_{N,Q}^{h,k}(x_1, \ldots, x_k) := \int_{\mathbb{R}^{N-k}} P_{N,Q}^h(x) dx_{k+1} \ldots dx_N : \]

the \( k \)-th correlation function of \( P_{N,Q}^h \).

**Thm** (G.-Venker ‘12)

For all \( h \) above exist \( \alpha_h > 0 \) s.th. for all \( Q \) above with \( \min_{t \in \mathbb{R}} Q''(t) > \alpha_h \),

there exists \( \mu_Q^h \), p-measure with compact support s.th.

(\( k \)-th correlation measure of \( P_{N,Q}^h \)) \( \Rightarrow (\mu_Q^h)^\otimes k \) as \( N \to \infty \),

i.e. for \( g \in C_b(\mathbb{R}^k) \)

\[
\lim_{N \to \infty} \int_{\mathbb{R}^k} g(t) \rho_{N,Q}^{h,k}(t) d^k t = \int_{\mathbb{R}^k} g(t)(\mu_Q^h)^\otimes k (dt).
\]
**Thm** (G.-Venker 2012, arxiv:1205.0671)

Above assumptions on $Q$, $h$ and $\alpha_h > 0$:

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{a \in \text{supp} \{\mu_h \circ \rho_h\}} \det \left[ \sin \left( \pi \left( t_i - t_j \right) \right) \right]_{i, j = 1}^{k} = \left( \frac{\sqrt{2}}{\pi} \right)^k.
\]
Thm (G.-Venker 2012, arxiv:1205.0671)
Above assumptions on $Q$, $h$ and $\alpha_h > 0$:
For $k \geq 1$ and $a \in \text{supp}(\mu^h_Q)$
Local Correlations in the Bulk

**Thm** (G.-Venker 2012, arxiv:1205.0671)

Above assumptions on $Q$, $h$ and $\alpha_h > 0$:

For $k \geq 1$ and $a \in \text{supp}(\mu^h_Q)$,

density $\mu^h_Q(a) > 0$, uniformly on compacts in $t_1, \ldots, t_k$.
Thm (G.-Venker 2012, arxiv:1205.0671)

Above assumptions on $Q, h$ and $\alpha_h > 0$:

For $k \geq 1$ and $a \in \text{supp}(\mu^h_Q)^{\circ}$

density $\mu^h_Q(a) > 0$, uniformly on compacts in $t_1, \ldots, t_k$

\[
\lim_{N \to \infty} \frac{1}{\mu^h_Q(a)^k} \rho^{h,k}_{N,Q} \left( a + \frac{t_1}{N \mu^h_Q(a)}, \ldots, a + \frac{t_k}{N \mu^h_Q(a)} \right)
\]
Local Correlations in the Bulk

**Thm** (G.-Venker 2012, arxiv:1205.0671)

Above assumptions on $Q$, $h$ and $\alpha_h > 0$:

For $k \geq 1$ and $a \in \text{supp}(\mu_h^h)^\circ$

density $\mu_h^h(a) > 0$, uniformly on compacts in $t_1, \ldots, t_k$

$$
\lim_{N \to \infty} \frac{1}{\mu_h^h(a)^k} \rho_{N,Q}^{h,k} \left( a + \frac{t_1}{N\mu_h^h(a)}, \ldots, a + \frac{t_k}{N\mu_h^h(a)} \right)
= \det \left[ \frac{\sin(\pi(t_i - t_j))}{\pi(t_i - t_j)} \right]_{1 \leq i, j \leq k}.
$$
Extensions to $\beta$-Ensembles

Let $\varphi$ be an even, smooth, nonnegative function with $\varphi(0) = 0$ and $\varphi(t) > 0$ for $t \neq 0$. 

Extensions to $\beta$-Ensembles

Let $\varphi$ be an even, smooth, nonnegative function with $\varphi(0) = 0$ and $\varphi(t) > 0$ for $t \neq 0$. Assume that for some $\beta > 0$ and $c > 0$
Extensions to $\beta$-Ensembles

Let $\varphi$ be an even, smooth, nonnegative function with $\varphi(0) = 0$ and $\varphi(t) > 0$ for $t \neq 0$. Assume that for some $\beta > 0$ and $c > 0$

$$\lim_{\varepsilon \to 0} \frac{\varphi(\varepsilon)}{\varepsilon^\beta} = c.$$
Extensions to $\beta$-Ensembles

Let $\varphi$ be an even, smooth, nonnegative function with $\varphi(0) = 0$ and $\varphi(t) > 0$ for $t \neq 0$. Assume that for some $\beta > 0$ and $c > 0$

$$\lim_{\varepsilon \to 0} \frac{\varphi(\varepsilon)}{|\varepsilon|^\beta} = c.$$ 

Let $Q$ be a strictly convex function of sufficient growth at infinity and define $P_{N,Q}^{\varphi,\beta}$ as the probability measure on $\mathbb{R}^N$ with density
Extensions to $\beta$-Ensembles

Let $\varphi$ be an even, smooth, nonnegative function with $\varphi(0) = 0$ and $\varphi(t) > 0$ for $t \neq 0$. Assume that for some $\beta > 0$ and $c > 0$

$$\lim_{\varepsilon \to 0} \frac{\varphi(\varepsilon)}{|\varepsilon|^\beta} = c.$$  

Let $Q$ be a strictly convex function of sufficient growth at infinity and define $P^{\varphi,\beta}_{N,Q}$ as the probability measure on $\mathbb{R}^N$ with density

$$P^{\varphi,\beta}_{N,Q}(x) := \frac{1}{Z_{\varphi,\beta}^N} \prod_{i < j} \varphi(x_i - x_j)e^{-N \sum_{j=1}^N Q(x_j)} dx.$$
Extensions to $\beta$-Ensembles

Let $\varphi$ be an even, smooth, nonnegative function with $\varphi(0) = 0$ and $\varphi(t) > 0$ for $t \neq 0$. Assume that for some $\beta > 0$ and $c > 0$

$$\lim_{\varepsilon \to 0} \frac{\varphi(\varepsilon)}{|\varepsilon|^\beta} = c.$$ 

Let $Q$ be a strictly convex function of sufficient growth at infinity and define $P_{N,Q}^{\varphi,\beta}$ as the probability measure on $\mathbb{R}^N$ with density

$$P_{N,Q}^{\varphi,\beta}(x) := \frac{1}{Z_{N,Q}^{\varphi,\beta}} \prod_{i<j} \varphi(x_i - x_j) e^{-N \sum_{j=1}^N Q(x_j)} \, dx.$$

We conjecture that $P_{N,Q}^{\varphi,\beta}$ has the same bulk local $k$-correlation, say $\rho^k$, as the Gaussian-$\beta$ ensemble

$$P_N^\beta(x) := \frac{1}{Z_N^\beta} \prod_{j<k} |x_k - x_j|^{\beta} e^{-N \sum_{j=1}^N x_j^2}.$$
\( \beta \)-Ensembles

**Theorem (Venker 2012)**

Write \( \varphi(x) := |x|^\beta \exp\{h\} \), \( h \) real analytic and even Schwartz function, \( \alpha^h \geq 0 \) s. th. for all real analytic, strongly convex and even \( Q \) with \( \alpha_Q > \alpha^h \):

The correlation measure \( \rho^h_{Q,\beta} \) converges weakly to a compactly supported p.m. \( \mu^h_{Q,\beta} \). Conditions on \( Q \) as above, there is \( \mu^Q_{\beta} \) of compact support, semicircular for \( Q(x) = x^2 \) and a scaled deformed correlation function \( \mu^h_{Q,\beta}(a) \) k \( \rho_{h,\beta}^k \), \( a \in \text{supp}(\mu^h_{Q,\beta}(a)) \).

For \( h = 0 \) and \( Q = x^2 \), the limit \( \lim_{N \to \infty} \) exists for \( Q(x) = G(x) := x^2 \) and \( h = 0 \) by Valko-Virag (09).
\( \beta \)-Ensembles

**Theorem (Venker 2012)**

Write \( \varphi(x) := |x|^\beta \exp\{h\} \), \( h \) real analytic and even Schwartz function, \( \alpha^h \geq 0 \) s. th. for all real analytic, strongly convex and even \( Q \) with \( \alpha_Q > \alpha^h \): The correlation measure \( \rho_{N,Q,\beta}^{h,1} \) converges weakly to a compactly supported p.m. \( \mu_{Q,\beta}^h \).
Theorem (Venker 2012)

Write \( \varphi(x) := |x|^{\beta} \exp\{h\} \), \( h \) real analytic and even Schwartz function, \( \alpha^h \geq 0 \) s. th. for all real analytic, strongly convex and even \( Q \) with \( \alpha_Q > \alpha^h \):
The correlation measure \( \rho_{N,Q,\beta}^{h,1} \) converges weakly to a compactly supported p.m. \( \mu_{Q,\beta}^h \).

Conditions on \( Q \) as above, there is \( \mu_{Q,\beta} \) of compact support, semicircular for \( Q(x) = x^2 \)
**β-Ensembles**

**Theorem (Venker 2012)**

Write \( \varphi(x) := |x|^{\beta} \exp\{h\} \), \( h \) real analytic and even Schwartz function, \( \alpha^h \geq 0 \) s. th. for all real analytic, strongly convex and even \( Q \) with \( \alpha_Q > \alpha^h \): The correlation measure \( \rho_{N,Q,\beta}^{h,1} \) converges weakly to a compactly supported p.m. \( \mu_{Q,\beta}^h \).

Conditions on \( Q \) as above, there is \( \mu_{Q,\beta} \) of compact support, semicircular for \( Q(x) = x^2 \) and a scaled deformed correlation function
Theorem (Venker 2012)

Write \( \varphi(x) := |x|^\beta \exp\{h\} \), \( h \) real analytic and even Schwartz function, \( \alpha^h \geq 0 \) s. th. for all real analytic, strongly convex and even \( Q \) with \( \alpha_Q > \alpha^h \):

The correlation measure \( \rho_{N,Q,\beta}^{h,1} \) converges weakly to a compactly supported p.m. \( \mu_{Q,\beta}^h \).

Conditions on \( Q \) as above, there is \( \mu_{Q,\beta} \) of compact support, semicircular for \( Q(x) = x^2 \) and a scaled deformed correlation function

\[
\frac{1}{\mu_{Q,\beta}^h(a)^k} \rho_{N,Q,\beta}^{h,k} \left( a + \frac{t_1}{N\mu_{Q,\beta}^h(a)}, \ldots, a + \frac{t_k}{N\mu_{Q,\beta}^h(a)} \right),
\]

where \( a \in \text{supp}(\mu_{Q,\beta}^h(a))^\circ \).
Write \( \varphi(x) := |x|^\beta \exp\{h\} \), \( h \) real analytic and even Schwartz function, \( \alpha^h \geq 0 \) s. th. for all real analytic, strongly convex and even \( Q \) with \( \alpha_Q > \alpha^h \): The correlation measure \( \rho_{N,Q,\beta}^{h,1} \) converges weakly to a compactly supported p.m. \( \mu_{Q,\beta}^h \).

Conditions on \( Q \) as above, there is \( \mu_{Q,\beta} \) of compact support, semicircular for \( Q(x) = x^2 \) and a scaled deformed correlation function

\[
\frac{1}{\mu_{Q,\beta}^h(a)^k} \rho_{N,Q,\beta}^{h,k} \left( a + \frac{t_1}{N\mu_{Q,\beta}^h(a)}, \ldots, a + \frac{t_k}{N\mu_{Q,\beta}^h(a)} \right),
\]

where \( a \in \text{supp}(\mu_{Q,\beta}^h(a))^\circ \). For \( h = 0 \) and \( Q = x^2 \),
**β-Ensembles**

**Theorem (Venker 2012)**

Write \( \varphi(x) := |x|^\beta \exp\{h\} \), \( h \) real analytic and even Schwartz function, \( \alpha^h \geq 0 \) s. th. for all real analytic, strongly convex and even \( Q \) with \( \alpha^Q > \alpha^h \): The correlation measure \( \rho_{N,Q,\beta}^{h,1} \) converges weakly to a compactly supported p.m. \( \mu_{Q,\beta}^h \).

Conditions on \( Q \) as above, there is \( \mu_{Q,\beta} \) of compact support, semicircular for \( Q(x) = x^2 \) and a scaled deformed correlation function

\[
\frac{1}{\mu_{Q,\beta}^h(a)^k} \rho_{N,Q,\beta}^{h,k} \left( a + \frac{t_1}{N\mu_{Q,\beta}^h(a)}, \ldots, a + \frac{t_k}{N\mu_{Q,\beta}^h(a)} \right),
\]

where \( a \in \text{supp}(\mu_{Q,\beta}^h(a))^{\circ} \). For \( h = 0 \) and \( Q = x^2 \), the limit \( N \to \infty \) exists for \( Q(x) = G(x) := x^2 \) and \( h = 0 \) by Valko-Virag (09).
Universality of Deformed (Averaged) Local Correlations of $\beta$-Ensembles

Compare local correlations of $P_{N,h^M}$ with those of the Gaussian $\beta$-Ensemble $P_{N,G,\beta}$:
Universality of Deformed (Averaged) Local Correlations of $\beta$-Ensembles

Compare local correlations of $P_{N,h_M}$ with those of the Gaussian $\beta$-Ensemble $P_{N,G,\beta}$:

**Theorem** (Venker 2012, arxiv 1209.317)

$h$ and $Q$ as above. Let $0 < \xi \leq 1/2$ and $s_N := N^{-1+\xi}$. 

Universality of Deformed (Averaged) Local Correlations of $\beta$-Ensembles

Compare local correlations of $P_{N, h^M}$ with those of the Gaussian $\beta$-Ensemble $P_{N, G, \beta}$:

**Theorem (Venker 2012, arxiv 1209.317)**

$h$ and $Q$ as above. Let $0 < \xi \leq 1/2$ and $s_N := N^{-1+\xi}$.

For $k = 1, 2, \ldots$, any $a \in \text{supp}(\mu_{Q, \beta}^h)^\circ$, any $a' \in \text{supp}(\mu_{G, \beta})^\circ$, any smooth function $f : \mathbb{R}^k \rightarrow \mathbb{R}$ with compact support

\[
\lim_{N \to \infty} \int dt^k f(t) \left[ \int a + s_N a - s_N 1^k \mu_{Q, \beta}^h(a) \right] - \int a' + s_N a' - s_N 1^k \mu_{G, \beta}(a') \right] du^2 s_N = 0.
\]

Universality of Deformed (Averaged) Local Correlations of $\beta$-Ensembles

Compare local correlations of $P_{N,h_M}$ with those of the Gaussian $\beta$-Ensemble $P_{N,G,\beta}$:

**Theorem** (Venker 2012, arxiv 1209.317)

$h$ and $Q$ as above. Let $0 < \xi \leq 1/2$ and $s_N := N^{-1+\xi}$.

For $k = 1, 2, \ldots$, any $a \in \text{supp}(\mu_{Q,\beta}^h)$, any $a' \in \text{supp}(\mu_{G,\beta})$, any smooth function $f : \mathbb{R}^k \rightarrow \mathbb{R}$ with compact support

$$
\lim_{N \to \infty} \int dt^k f(t) \left[ \int_{a-s_N}^{a+s_N} \frac{1}{\mu_{Q,\beta}^h(a) \rho_{N,Q,\beta}^{h,k}} \left( u + \frac{t_1}{N\mu_{Q,\beta}^h(a)}, \ldots, u + \frac{t_k}{N\mu_{Q,\beta}^h(a)} \right) \right]
$$
Universality of Deformed (Averaged) Local Correlations of $\beta$-Ensembles

Compare local correlations of $P_{N, h^M}$ with those of the Gaussian $\beta$-Ensemble $P_{N, G, \beta}$:

**Theorem (Venker 2012, arxiv 1209.317)**

$h$ and $Q$ as above. Let $0 < \xi \leq 1/2$ and $s_N := N^{-1+\xi}$.

For $k = 1, 2, \ldots$, any $a \in \text{supp}(\mu_{Q, \beta}^h)^\circ$, any $a' \in \text{supp}(\mu_{G, \beta}^h)^\circ$, any smooth function $f : \mathbb{R}^k \longrightarrow \mathbb{R}$ with compact support

$$\lim_{N \to \infty} \int dt^k f(t) \left[ \int_{a-s_N}^{a+s_N} \frac{1}{\mu_{Q, \beta}^h(a)^k \rho_{N, Q, \beta}^h} \left( u + \frac{t_1}{N\mu_{Q, \beta}^h(a)}, \ldots, u + \frac{t_k}{N\mu_{Q, \beta}^h(a)} \right) \right] \frac{du}{2s_N}$$

$$- \int_{a'-s_N}^{a'+s_N} \frac{1}{\mu_{G, \beta}^h(a')^k \rho_{N, G, \beta}^h} \left( u + \frac{t_1}{N\mu_{G, \beta}^h(a')}, \ldots, u + \frac{t_k}{N\mu_{G, \beta}^h(a')} \right) \frac{du}{2s_N} = 0.$$

Universality of Deformed (Averaged) Local Correlations of $\beta$-Ensembles

Compare local correlations of $P_{N,h^M}$ with those of the Gaussian $\beta$-Ensemble $P_{N,G,\beta}$:

**Theorem (Venker 2012, arxiv 1209.317)**

$h$ and $Q$ as above. Let $0 < \xi \leq 1/2$ and $s_N := N^{-1+\xi}$.

For $k = 1, 2, \ldots$, any $a \in \text{supp}(\mu_{Q,\beta}^h)$, any $a' \in \text{supp}(\mu_{G,\beta})$, any smooth function $f : \mathbb{R}^k \rightarrow \mathbb{R}$ with compact support

$$
\lim_{N \rightarrow \infty} \int dt^k f(t) \left[ \int_{a-s_N}^{a+s_N} \frac{1}{\mu_{Q,\beta}^h(a)} \rho_{N,Q,\beta}^{h,k} \left( u + \frac{t_1}{N\mu_{Q,\beta}^h(a)}, \ldots, u + \frac{t_k}{N\mu_{Q,\beta}^h(a)} \right) \right. \\
- \int_{a'-s_N}^{a'+s_N} \frac{1}{\mu_{G,\beta}(a')} \rho_{N,G,\beta}^{k} \left( u + \frac{t_1}{N\mu_{G,\beta}(a')}, \ldots, u + \frac{t_k}{N\mu_{G,\beta}(a')} \right) \\
\left. \right] \frac{du}{2s_N} = 0.
$$

Proof:  Simple Example

\[ h(x) := -x^2 \text{ and } \gamma > 0 \]

\[ P_{N,\alpha}^\gamma(x) := Z_{N,\alpha,\gamma}^{-1} P_{N,\alpha}^{GUE}(x) \exp\left\{ \gamma \sum_{i<j} (x_i - x_j)^2 \right\}, \]
Proof: Simple Example

\[ h(x) := -x^2 \text{ and } \gamma > 0 \]

\[ P_{N,\alpha}^{\gamma}(x) := Z_{N,\alpha,\gamma}^{-1} P_{N,\alpha}^{\text{GUE}}(x) \exp\{\gamma \sum_{i<j} (x_i - x_j)^2\}, \]

\[ P_{N,\alpha}^{\text{GUE}}(x) := \frac{1}{Z_{N,\alpha}} \prod_{1 \leq i < j \leq N} |x_i - x_j|^2 \exp\{-\alpha N \sum_j x_j^2\}, \]
Proof: Simple Example

$h(x) := -x^2$ and $\gamma > 0$

$$P_{N,\alpha}^\gamma(x) := Z_{N,\alpha,\gamma}^{-1} P_{N,\alpha}^{\text{GUE}}(x) \exp\{\gamma \sum_{i<j}(x_i - x_j)^2\},$$

$$P_{N,\alpha}^{\text{GUE}}(x) := \frac{1}{Z_{N,\alpha}} \prod_{1 \leq i < j \leq N} |x_i - x_j|^2 \exp\{-\alpha N \sum_j x_j^2\},$$

$$\exp\{\gamma \sum_{i<j}(x_i - x_j)^2\} = \exp\{-\gamma N M_2(x)\} \exp\{\gamma M_1(x)^2\},$$
Proof: Simple Example

\[ h(x) := -x^2 \text{ and } \gamma > 0 \]

\[ P_{N,\alpha}^\gamma(x) := Z_{N,\alpha,\gamma}^{-1} P_{N,\alpha}^{GUE}(x) \exp\{\gamma \sum_{i<j} (x_i - x_j)^2\}, \]

\[ P_{N,\alpha}^{GUE}(x) := \frac{1}{Z_{N,\alpha}} \prod_{1 \leq i < j \leq N} |x_i - x_j|^2 \exp\{-\alpha N \sum_j x_j^2\}, \]

\[ \exp\{\gamma \sum_{i<j} (x_i - x_j)^2\} = \exp\{-\gamma N M_2(x)\} \exp\{\gamma M_1(x)^2\}, \text{ where} \]

\[ M_p(x) := N \sum_{j=1}^N x_{pj}, \quad p = 1, 2, \ldots \]

\[ \exp\{\gamma M_1(x)^2\} = c \int_{\mathbb{R}} \exp\{\varepsilon \sqrt{\gamma} M_1(x)^2\} \exp\{-\varepsilon^2 / 4\} \, d\varepsilon, \]
Proof: Simple Example

\( h(x) := -x^2 \) and \( \gamma > 0 \)

\[
P_N^{\gamma}(x) := Z^{-1}_{N,\alpha,\gamma} P_{N,\alpha}^{GUE}(x) \exp\{\gamma \sum_{i<j} (x_i - x_j)^2\},
\]

\[
P_{N,\alpha}^{GUE}(x) := \frac{1}{Z_{N,\alpha}} \prod_{1 \leq i < j \leq N} |x_i - x_j|^2 \exp\{-\alpha N \sum_j x_j^2\},
\]

\[
\exp\{\gamma \sum_{i<j} (x_i - x_j)^2\} = \exp\{-\gamma N M_2(x)\} \exp\{\gamma M_1(x)^2\}, \quad \text{where}
\]

\[
M_p(x) := \sum_{j=1}^{N} x_j^p, \quad p = 1, 2,
\]
Proof: Simple Example

\[ h(x) := -x^2 \text{ and } \gamma > 0 \]

\[ P_{N,\alpha}^{\gamma}(x) := Z_{N,\alpha,\gamma}^{-1} P_{N,\alpha}^{GUE}(x) \exp\{\gamma \sum_{i<j}(x_i - x_j)^2\}, \]

\[ P_{N,\alpha}^{GUE}(x) := \frac{1}{Z_{N,\alpha}} \prod_{1 \leq i < j \leq N} |x_i - x_j|^2 \exp\{-\alpha N \sum_j x_j^2\}, \]

\[ \exp\{\gamma \sum_{i<j}(x_i - x_j)^2\} = \exp\{-\gamma NM_2(x)\} \exp\{\gamma M_1(x)^2\}, \quad \text{where} \]

\[ M_p(x) := \sum_{j=1}^N x_j^p, \quad p = 1, 2, \]

\[ \exp\{\gamma M_1(x)^2\} = c \int_{\mathbb{R}} \exp\{\varepsilon \sqrt{\gamma} M_1(x)\} \exp\{-\varepsilon^2/4\} d\varepsilon, \]
Orthogonal Polynomials

\[ P_{N,\alpha}^\gamma(x) = c' \int_{\mathbb{R}} \frac{Z_{N,\epsilon}}{Z_N} P_{N,\epsilon}(x) \exp\left\{-\epsilon^2/4\right\} d\epsilon, \]
Orthogonal Polynomials

\[ P_{N,\alpha}^\gamma(x) = c' \int_{\mathbb{R}} \frac{Z_{N,\varepsilon}}{Z_N} P_{N,\varepsilon}(x) \exp\{-\varepsilon^2/4\} d\varepsilon, \]

\[ P_{N,\varepsilon}(x) := \frac{\Delta(x)^2}{Z_{N,\varepsilon}} \exp\{-\sum_{j=1}^{N} (N(\alpha + \gamma)x_j^2 + \sqrt{\gamma\varepsilon}x_j)\} \]
Orthogonal Polynomials

\[ P_{N,\alpha}^{\gamma}(x) = c' \int_{\mathbb{R}} \frac{Z_{N,\epsilon}}{Z_N} P_{N,\epsilon}(x) \exp\{-\epsilon^2/4\} d\epsilon, \]

\[ P_{N,\epsilon}(x) := \frac{\Delta(x)^2}{Z_{N,\epsilon}} \exp\{-\sum_{j=1}^{N} (N(\alpha + \gamma)x_j^2 + \sqrt{\gamma\epsilon}x_j)\} \]

\[ Z_{N,\epsilon}/Z_N = \sqrt{1 + \frac{\gamma}{\alpha}} \exp\left\{ \frac{\gamma\epsilon^2}{4(\alpha + \gamma)} \right\}. \]
Orthogonal Polynomials

\[ P_{N,\alpha}^{\gamma}(x) = c' \int_{\mathbb{R}} \frac{Z_{N,\varepsilon}}{Z_N} P_{N,\varepsilon}(x) \exp\{-\varepsilon^2/4\} d\varepsilon, \]

\[ P_{N,\varepsilon}(x) := \frac{\Delta(x)^2}{Z_{N,\varepsilon}} \exp\left\{ - \sum_{j=1}^{N} (N(\alpha + \gamma)x_j^2 + \sqrt{\gamma}\varepsilon x_j) \right\} \]

\[ Z_{N,\varepsilon}/Z_N = \sqrt{1 + \frac{\gamma}{\alpha}} \exp\left\{ \frac{\gamma\varepsilon^2}{4(\alpha + \gamma)} \right\}. \]

Orthogonal polynomials w.r.t. the kernel \( \exp\{-N(\alpha + \gamma)t^2 + \varepsilon \sqrt{\gamma}t\} \) are shifted Hermite polynomials.
Orthogonal Polynomials

\[ P_{N,\alpha}^\gamma(x) = c' \int_{\mathbb{R}} \frac{Z_{N,\varepsilon}}{Z_N} P_{N,\varepsilon}(x) \exp\{-\varepsilon^2/4\} d\varepsilon, \]

\[ P_{N,\varepsilon}(x) := \frac{\Delta(x)^2}{Z_{N,\varepsilon}} \exp\left\{ - \sum_{j=1}^{N} (N(\alpha + \gamma)x_j^2 + \sqrt{\gamma}\varepsilon x_j) \right\} \]

\[ \frac{Z_{N,\varepsilon}}{Z_N} = \sqrt{1 + \frac{\gamma}{\alpha}} \exp\left\{ \frac{\gamma\varepsilon^2}{4(\alpha + \gamma)} \right\}. \]

Orthogonal polynomials w.r.t. the kernel \( \exp\{-N(\alpha + \gamma)t^2 + \varepsilon\sqrt{\gamma}t\} \)

are shifted Hermite polynomials.

The ensemble \( P_{N}^\varepsilon \) is determinantal with kernel:
Orthogonal Polynomials

\[ P_{N,\alpha}^\gamma(x) = c' \int_{\mathbb{R}} \frac{Z_{N,\varepsilon}}{Z_N} P_{N,\varepsilon}(x) \exp\{-\varepsilon^2/4\} d\varepsilon, \]

\[ P_{N,\varepsilon}(x) := \frac{\Delta(x)^2}{Z_{N,\varepsilon}} \exp\{-\sum_{j=1}^{N} (N(\alpha + \gamma)x_j^2 + \sqrt{\gamma}\varepsilon x_j)\} \]

\[ Z_{N,\varepsilon}/Z_N = \sqrt{1 + \frac{\gamma}{\alpha}} \exp \left\{ \frac{\gamma \varepsilon^2}{4(\alpha + \gamma)} \right\}. \]

Orthogonal polynomials w.r.t. the kernel \( \exp\{-N(\alpha + \gamma)t^2 + \varepsilon\sqrt{\gamma}t\} \) are shifted Hermite polynomials.

The ensemble \( P_{N}^\varepsilon \) is determinantal with kernel:

\[ K_N^*(t, s) = \exp\left\{ \frac{\omega'^2 \varepsilon^2}{4N} \right\} K_N(t - \frac{\omega'\varepsilon}{2N}, s - \frac{\omega'\varepsilon}{2N}), \quad \omega' := \frac{\sqrt{\gamma}}{\alpha + \gamma} \]
Orthogonal Polynomials

\[ P_{N,\alpha}^\gamma(x) = c' \int_\mathbb{R} \frac{Z_{N,\varepsilon}}{Z_N} P_{N,\varepsilon}(x) \exp\{\varepsilon^2/4\} \, d\varepsilon, \]

\[ P_{N,\varepsilon}(x) := \frac{\Delta(x)^2}{Z_{N,\varepsilon}} \exp\left\{ - \sum_{j=1}^N (N(\alpha + \gamma)x_j^2 + \sqrt{\gamma}\varepsilon x_j) \right\} \]

\[ Z_{N,\varepsilon}/Z_N = \sqrt{1 + \frac{\gamma}{\alpha}} \exp\left\{ \frac{\varepsilon^2}{4(\alpha + \gamma)} \right\}. \]

Orthogonal polynomials w.r.t. the kernel \( \exp\{\varepsilon^2/4\} \) are shifted Hermite polynomials.

The ensemble \( P_{N,\varepsilon}^\gamma \) is determinantal with kernel:

\[ K^*_N(t, s) = \exp\left\{ \frac{\omega'^2 \varepsilon^2}{4N} \right\} K_N(t - \frac{\omega' \varepsilon}{2N}, s - \frac{\omega' \varepsilon}{2N}), \quad \omega' := \frac{\sqrt{\gamma}}{\alpha + \gamma} \]

where \( K_N \) is the kernel of rescaled GUE\( _{\omega} \).
Universality

$\rho_{N}^{\varepsilon,k}$: $k$-th correlation function of $P_{N}^{\varepsilon}$,
Universality

\[ \rho_{N}^{\varepsilon,k} : \text{k-th correlation function of } P_{N}^{\varepsilon}, \]

\sigma \text{ Wigner density on } [-\omega, \omega], \quad \omega := (\alpha + \gamma)^{-1/2}, \]
Universality

\[ \rho_{N}^{\varepsilon,k} : \quad k\text{-th correlation function of } P_{N}^{\varepsilon}, \]

σ Wigner density on \([-\omega, \omega], \quad \omega := (\alpha + \gamma)^{-1/2}, \]

for all \( \varepsilon \in \mathbb{R} \):
Universality

\( \rho_{N}^{\varepsilon, k} : k \)-th correlation function of \( P_{N}^{\varepsilon} \),

\( \sigma \) Wigner density on \([-\omega, \omega] \), \( \omega := (\alpha + \gamma)^{-1/2} \),

for all \( \varepsilon \in \mathbb{R} \): \( \rho_{N}^{1, \varepsilon} \implies \sigma \) and
Universality

$\rho_{N}^{\varepsilon,k}$: $k$-th correlation function of $P_{N}^{\varepsilon}$,

$\sigma$ Wigner density on $[-\omega, \omega]$, $\omega := (\alpha + \gamma)^{-1/2}$,

for all $\varepsilon \in \mathbb{R}$: $\rho_{N}^{1,\varepsilon} \implies \sigma$ and

$$\lim_{N \to \infty} \frac{1}{\sigma(a)^k} \rho_{N}^{\varepsilon,k} \left( a + \frac{t_1}{N\sigma(a)}, \ldots, a + \frac{t_k}{N\sigma(a)} \right) = \det \left( \frac{\sin(\pi(t_i - t_j))}{\pi(t_i - t_j)} \right)_{1 \leq i, j \leq k}.$$
Universality

\( \rho_{\epsilon, k}^N : \) \( k \)-th correlation function of \( P_{\epsilon}^N \),

\( \sigma \) Wigner density on \( [\omega, \omega] \), \( \omega := (\alpha + \gamma)^{-1/2} \),

for all \( \epsilon \in \mathbb{R} : \rho_{\epsilon}^1 \Rightarrow \sigma \) and

\[
\lim_{N \to \infty} \frac{1}{\sigma(a)^k} \rho_{\epsilon}^N (a + \frac{t_1}{N\sigma(a)}, \ldots, a + \frac{t_k}{N\sigma(a)}) = \det \left( \frac{\sin(\pi(t_i - t_j))}{\pi(t_i - t_j)} \right)_{1 \leq i, j \leq k},
\]

locally uniformly in \( t_1, \ldots t_k \), and \( a \) in compact subsets of \( (-\omega, \omega) \).
Universality

$\rho^{\varepsilon,k}_N$: $k$-th correlation function of $P^{\varepsilon}_N$, 

$\sigma$ Wigner density on $[-\omega, \omega]$, \quad $\omega := (\alpha + \gamma)^{-1/2}$,

for all $\varepsilon \in \mathbb{R}$: \quad $\rho^{1,\varepsilon}_N \rightarrow \sigma$ and

$$
\lim_{N \rightarrow \infty} \frac{1}{\sigma(a)k} \rho^{\varepsilon,k}_N \left( a + \frac{t_1}{N \sigma(a)}, \ldots, a + \frac{t_k}{N \sigma(a)} \right) = \det \left( \frac{\sin(\pi(t_i - t_j))}{\pi(t_i - t_j)} \right)_{1 \leq i,j \leq k},
$$

locally uniformly in $t_1, \ldots t_k$, and $a$ in compact subsets of $(-\omega, \omega)$.

Thm (Venker ’11)

$\rho^{k,\gamma}_N, \alpha$, $k$th correlation function of $P^{\gamma}_{N,\alpha}$:
Universality

$\rho_{N}^{\varepsilon,k}$: $k$-th correlation function of $P_{N}^{\varepsilon}$,

$\sigma$ Wigner density on $[-\omega, \omega]$, \quad \omega := (\alpha + \gamma)^{-1/2}$, 

for all $\varepsilon \in \mathbb{R}$: $\rho_{N}^{1,\varepsilon} \implies \sigma$ and

$$\lim_{N \to \infty} \frac{1}{\sigma(a)k} \rho_{N}^{\varepsilon,k} \left( a + \frac{t_{1}}{N\sigma(a)}, \ldots, a + \frac{t_{k}}{N\sigma(a)} \right) = \det \left( \frac{\sin(\pi(t_{i} - t_{j}))}{\pi(t_{i} - t_{j})} \right)_{1 \leq i, j \leq k},$$

locally uniformly in $t_{1}, \ldots t_{k}$, and $a$ in compact subsets of $(-\omega, \omega)$.

**Thm** (Venker ’11)

$\rho_{N,\alpha}^{k,\gamma}$, $k$th correlation function of $P_{N,\alpha}^{\gamma}$:

$$\lim_{N \to \infty} \frac{1}{\sigma(a)k} \rho_{N,\alpha}^{\gamma,k} \left( a + \frac{t_{1}}{N\sigma(a)}, \ldots, a + \frac{t_{k}}{N\sigma(a)} \right) = \det \left( \frac{\sin(\pi(t_{i} - t_{j}))}{\pi(t_{i} - t_{j})} \right)_{1 \leq i, j \leq k}$$
Sketch of Proof: Recentering

Hoeffding type decomposition of interaction

\[ \sum_{i<j} h(x_i - x_j) = \sum_{i<j} \tilde{h}(x_i - x_j) + N \sum_i \int h(x_i - s) d\mu^h_Q(s) + \text{const}. \]
Sketch of Proof: Recentering

Hoeffding type decomposition of interaction

\[ \sum_{i<j} h(x_i - x_j) = \sum_{i<j} \tilde{h}(x_i - x_j) + N \sum_i \int h(x_i - s) d\mu_h^Q(s) + \text{const.} \]

into centered fluctuation (w.r.t to \( \mu_h^Q \)) and additional potential \( h \ast \mu_h^Q \)
Sketch of Proof: Recentering

- Hoeffding type decomposition of interaction

\[ \sum_{i<j} h(x_i - x_j) = \sum_{i<j} \tilde{h}(x_i - x_j) + N \sum_i \int h(x_i - s) d\mu^h_Q(s) + \text{const.} \]

into centered fluctuation (w.r.t to \(\mu^h_Q\)) and additional potential \(h \ast \mu^h_Q\)

- Ensemble \(P^h_{N,Q}\):

\[ P^h_{N,Q}(x) := \frac{1}{Z^h_{N,Q}} \Delta(x)^2 \exp\{-N \sum_{j=1}^N Q(x_j)\} \exp\{-\sum_{j<k} h(x_k - x_j)\} \]
Sketch of Proof: Recentering

Hoeffding type decomposition of interaction

\[ \sum_{i<j} h(x_i - x_j) = \sum_{i<j} \tilde{h}(x_i - x_j) + N \int h(x_i - s) d\mu_Q^h(s) + \text{const.} \]

into centered fluctuation (w.r.t to \( \mu_Q^h \)) and additional potential \( h * \mu_Q^h \)

Ensemble \( P_{N,Q}^h \):

\[
P_{N,Q}^h(x) := \frac{1}{Z_{N,Q}^h} \Delta(x)^2 \exp\left\{-N \sum_{j=1}^{N} Q(x_j)\right\} \exp\left\{-\sum_{j<k} h(x_k - x_j)\right\}
\]

\[
= \frac{1}{Z_{N,Q}^h} \Delta(x)^2 \exp\left\{-N \sum_{j=1}^{N} V(x_j)\right\} \exp\left\{-\sum_{j<k} \tilde{h}(x_k - x_j)\right\},
\]

Claim: Correlation-Fct. of \( P_{N,Q}^h \) equivalent to \( P_{N,V} \) as \( N \to \infty \), where

\[
P_{N,V}(x) := \frac{1}{Z_N^V} \Delta(x)^2 \exp\left\{-N N \sum_{j=1}^{N} V(x_j)\right\} \exp\left\{-\sum_{j<k} \tilde{h}(x_k - x_j)\right\},
\]
Sketch of Proof: Recentering

- Hoeffding type decomposition of interaction
\[
\sum_{i<j} h(x_i - x_j) = \sum_{i<j} \tilde{h}(x_i - x_j) + N \sum_i \int h(x_i - s) d\mu_Q^h(s) + \text{const.}
\]
into centered fluctuation (w.r.t to \(\mu_Q^h\)) and additional potential \(h \ast \mu_Q^h\)

Ensemble \(P_{N,Q}^h\):
\[
P_{N,Q}^h(x) := \frac{1}{Z_{N,Q}^h} \Delta(x)^2 \exp\left\{-N \sum_{j=1}^N Q(x_j)\right\} \exp\left\{-\sum_{j<k} h(x_k - x_j)\right\}
\]
\[
= \frac{1}{Z_{N,Q}^h} \Delta(x)^2 \exp\left\{-N \sum_{j=1}^N V(x_j)\right\} \exp\left\{-\sum_{j<k} \tilde{h}(x_k - x_j)\right\},
\]
\[
V(t) := Q(t) + \int h(t - s) d\mu_Q^h(s)
\]

- Claim: Correlation-Fct. of \(P_{N,Q}^h\) equivalent to \(P_{N,V}\) as \(N \to \infty\), where
\[
P_{N,V}(x) := \frac{1}{Z_{N,V}} \Delta(x)^2 \exp\left\{-N \sum_{j=1}^N V(x_j)\right\}
\]
Equilibrium Measure

For $\nu \in \mathcal{M}^1(\mathbb{R})$ ($Q, h$ as above) consider potential

$$V_{\nu}(Q) := Q + \int h(t-s)d\nu(s).$$

Equilibrium measure for potential $V_{\nu}$ (like $V_{\nu}, Q$ above) is the unique solution, say $\mu = T(V, Q)$, to the minimization problem

$$\min_{\mu \in \mathcal{M}^1(\mathbb{R})} \int V(t)d\mu(t) + \int \int \log|t-s|-1d\mu(t)d\mu(s).$$

By Schauder let $\mu = \mu_h Q$ be a fixed point of $\nu \rightarrow T(V_{\mu}, Q)$, i.e.

$$\mu = T(V_{\mu}, Q),$$

with continuous density $\mu_h Q(x)$ and compact support.
Equilibrium Measure

For $\nu \in \mathcal{M}^1(\mathbb{R})$ ($Q, h$ as above) consider potential

$$V_{\nu, Q}(t) := Q(t) + \int h(t - s)d\nu(s).$$
Equilibrium Measure

For $\nu \in \mathcal{M}^1(\mathbb{R})$ ($Q, h$ as above) consider potential

$$V_{\nu,Q}(t) := Q(t) + \int h(t - s) d\nu(s).$$

Equilibrium measure for potential $V$ (like $V_{\nu,Q}$ above) is the unique solution, say $\mu = T(V)$,
Equilibrium Measure

For $\nu \in M^1(\mathbb{R})$ ($Q, h$ as above) consider potential

$$V_{\nu, Q}(t) := Q(t) + \int h(t - s) d\nu(s).$$

Equilibrium measure for potential $V$ (like $V_{\nu, Q}$ above) is the unique solution, say $\mu = T(V)$, to the minimization problem

$$\min_{\mu \in M^1(\mathbb{R})} \int V(t) d\mu(t) + \int \int \log |t - s|^{-1} d\mu(t) d\mu(s).$$
Equilibrium Measure

For $\nu \in \mathcal{M}^1(\mathbb{R})$ ($Q, h$ as above) consider potential

$$V_{\nu,Q}(t) := Q(t) + \int h(t - s)d\nu(s).$$

Equilibrium measure for potential $V$ (like $V_{\nu,Q}$ above) is the unique solution, say $\mu = T(V)$, to the minimization problem

$$\min_{\mu \in \mathcal{M}^1(\mathbb{R})} \int V(t)d\mu(t) + \int \int \log |t - s|^{-1} d\mu(t)d\mu(s).$$

By Schauder let $\mu = \mu^h_Q$ be a fixed point of $\nu \rightarrow T(V_{\nu,Q})$, i.e.
Equilibrium Measure

For $\nu \in \mathcal{M}^1(\mathbb{R})$ ($Q$, $h$ as above) consider potential

$$V_{\nu,Q}(t) := Q(t) + \int h(t - s) d\nu(s).$$

Equilibrium measure for potential $V$ (like $V_{\nu,Q}$ above) is the unique solution, say $\mu = T(V)$, to the minimization problem

$$\min_{\mu \in \mathcal{M}^1(\mathbb{R})} \int V(t) d\mu(t) + \int \int \log |t - s|^{-1} d\mu(t) d\mu(s).$$

By Schauder let $\mu = \mu^h_Q$ be a fixed point of $\nu \mapsto T(V_{\nu,q})$, i.e.

**Selfconsistency:** $\mu = T(V_{\mu,Q})$, 
Equilibrium Measure

For $\nu \in \mathcal{M}^1(\mathbb{R})$ ($Q$, $h$ as above) consider potential

\[ V_{\nu,Q}(t) := Q(t) + \int h(t - s)d\nu(s). \]

Equilibrium measure for potential $V$ (like $V_{\nu,Q}$ above) is the unique solution, say $\mu = T(V)$, to the minimization problem

\[
\min_{\mu \in \mathcal{M}^1(\mathbb{R})} \int V(t)d\mu(t) + \int \int \log |t - s|^{-1} d\mu(t)d\mu(s).
\]

By Schauder let $\mu = \mu^h_Q$ be a fixed point of $\nu \to T(V_{\nu,Q})$, i.e.

**Selfconsistency:** $\mu = T(V_{\mu,Q})$,

with continuous density $\mu^h_Q(x)$ and compact support.
Fourier Representation of $\tilde{h}$

Fourier representation, $\hat{h}$ real, $\mu = \mu^h_Q$,

$$- \sum_{l \neq k} \tilde{h}(x_l - x_k) = - \int \hat{h}(t) \left| \sum_j e^{i x_j t} - \langle e^{i x_j} \rangle_\mu \right|^2 dt,$$
Fourier Representation of $\tilde{h}$

- Fourier representation, $\hat{h}$ real, $\mu = \mu_Q^h$,

$$- \sum_{l \neq k} \tilde{h}(x_l - x_k) = - \int \hat{h}(t) \left| \sum_j e^{ix_j t} - \langle e^{ix_j \cdot} \rangle_\mu \right|^2 dt,$$

- Let e.g. $S(t) = \sum_j \sin(t x_j)$.
Fourier Representation of $\tilde{h}$

- Fourier representation, $\hat{h}$ real, $\mu = \mu_Q^h$,

$$- \sum_{l \neq k} \tilde{h}(x_l - x_k) = - \int \hat{h}(t) \left| \sum_j e^{i x_j t} - \langle e^{i x_j} \rangle_{\mu} \right|^2 dt,$$

- Let e.g. $S(t) = \sum_j \sin(t x_j)$. If $g(t) = -\hat{h} \geq 0$
Fourier Representation of $\tilde{h}$

- Fourier representation, $\hat{h}$ real, $\mu = \mu^h_Q$,

$$- \sum_{l \neq k} \tilde{h}(x_l - x_k) = - \int \hat{h}(t) \left| \sum_j e^{ix_j t} - \langle e^{ix_j \cdot} \rangle \mu \right|^2 dt,$$

- Let e.g. $S(t) = \sum_j \sin(t x_j)$. If $g(t) = -\hat{h} \geq 0$

$$\exp \left\{ \frac{1}{2} \int_0^\infty g(t) S(t)^2 dt \right\} = \mathbb{E} \exp \left\{ \int_0^\infty g^{1/2}(t) S(t) dB_t \right\}$$

$$=: \mathbb{E}_\omega \exp \left\{ \sum_j f(x_j, \omega) \right\},$$
Fourier Representation of $\tilde{h}$

- Fourier representation, $\widehat{h}$ real, $\mu = \mu_{\mathcal{Q}}$

\[-\sum_{l \neq k} \tilde{h}(x_l - x_k) = - \int \hat{h}(t) \left| \sum_{j} e^{i x_j t} - \langle e^{i x_j \cdot} \rangle_{\mu} \right|^2 dt,\]

- Let e.g. $S(t) = \sum_{j} \sin(t x_j)$. If $g(t) = -\hat{h} \geq 0$

\[\exp \left\{ \frac{1}{2} \int_{0}^{\infty} g(t) S(t)^2 dt \right\} = \mathbb{E} \exp \left\{ \int_{0}^{\infty} g^{1/2}(t) S(t) dB_t \right\} =: \mathbb{E}_\omega \exp \left\{ \sum_{j} f(x_j, \omega) \right\},\]

and one may linearize $-\sum_{l \neq k} \tilde{h}(x_l - x_k)$.
Fourier Representation of $\tilde{h}$

- Fourier representation, $\tilde{h}$ real, $\mu = \mu_Q^h$,
  \[ - \sum_{l \neq k} \tilde{h}(x_l - x_k) = - \int \tilde{h}(t) \left| \sum_j e^{i x_j t} - \langle e^{i x_j} \rangle_\mu \right|^2 dt, \]

Let e.g. $S(t) = \sum_j \sin(t x_j)$. If $g(t) = -\tilde{h} \geq 0$

\[ \exp \left\{ \frac{1}{2} \int_0^\infty g(t) S(t)^2 dt \right\} = E \exp \left\{ \int_0^\infty g^{1/2}(t) S(t) dB_t \right\} \]
\[ =: E_\omega \exp \left\{ \sum_j f(x_j, \omega) \right\}, \]

and one may linearize $- \sum_{l \neq k} \tilde{h}(x_l - x_k)$.

- Need real $f$: extend limit results to $g(t) = -\tilde{h}(t) < 0$: 

\[ \text{Local/Global Universality} \]
Fourier Representation of $\tilde{h}$

- Fourier representation, $\hat{\tilde{h}}$ real, $\mu = \mu^h_Q$,

$$- \sum_{l \neq k} \tilde{h}(x_l - x_k) = - \int \hat{\tilde{h}}(t) \left| \sum_j e^{ix_j t} - \langle e^{ix_j} \rangle_\mu \right|^2 dt,$$

Let e.g. $S(t) = \sum_j \sin(t x_j)$. If $g(t) = -\hat{h} \geq 0$

$$\exp \left\{ \frac{1}{2} \int_0^\infty g(t) S(t)^2 dt \right\} = E \exp \left\{ \int_0^\infty g^{1/2}(t) S(t) dB_t \right\}$$

$$=: E_\omega \exp \left\{ \sum_j f(x_j, \omega) \right\},$$

and one may linearize $- \sum_{l \neq k} \tilde{h}(x_l - x_k)$.

- Need real $f$: extend limit results to $g(t) = -\hat{h}(t) < 0$:

to family: $g_z := g_+ + zg_- \geq 0$, $z \geq 0$
Fourier Representation of $\tilde{h}$

- Fourier representation, $\hat{h}$ real, $\mu = \mu^h_Q$,

$$- \sum_{l \neq k} \tilde{h}(x_l - x_k) = - \int \hat{h}(t) \left| \sum_j e^{ix_j t} - \langle e^{ix_j} \rangle \mu \right|^2 dt,$$

- Let e.g. $S(t) = \sum_j \sin(t x_j)$. If $g(t) = -\hat{h} \geq 0$

$$\exp \left\{ \frac{1}{2} \int_0^\infty g(t) S(t)^2 dt \right\} = \mathbb{E} \exp \left\{ \int_0^\infty g^{1/2}(t) S(t) dB_t \right\} =: \mathbb{E}_\omega \exp \left\{ \sum_j f(x_j, \omega) \right\},$$

and one may linearize $- \sum_{l \neq k} \tilde{h}(x_l - x_k)$.

- Need real $f$: extend limit results to $g(t) = -\hat{h}(t) < 0$:

  to family: $g_z := g_+ + zg_- \geq 0$, $z \geq 0$

  Analytical extension of limits to $\Re (z) < 0$: get $g_{-1} = -\hat{h}$
Concentration Inequalities

For equivalence use concentration inequalities:

For $f$ Lipschitz, some $c_f, C > 0$,
Concentration Inequalities

For equivalence use concentration inequalities:

For $f$ Lipschitz, some $c_f, C > 0,$

$$P_{N,Q} \left( \left| \sum_{j=1}^{N} f(x_j) - \mathbb{E}_{N,Q} \sum_{j=1}^{N} f(x_j) \right| > \eta \right) \leq Ce^{-c_f \eta^2}.$$
Concentration Inequalities

- For equivalence use concentration inequalities:

  For $f$ Lipschitz, some $c_f, C > 0$,
  
  $$ P_{N,Q} \left( \left| \sum_{j=1}^{N} f(x_j) - \mathbb{E}_{N,Q} \sum_{j=1}^{N} f(x_j) \right| > \eta \right) \leq Ce^{-c_f \eta^2}. $$

- Truncation of $x_j$ to range $[-L, L]$: s.th. $c_f$ is bounded a.s.
  
  for $f(x_j) = f(x_j, \omega)$ Gaussian process in $C_b^k[-L, L]$. 
Concentration Inequalities

- For equivalence use concentration inequalities:

  For $f$ Lipschitz, some $c_f, C > 0$,

  $$P_{N,Q} \left( \left| \sum_{j=1}^{N} f(x_j) - \mathbb{E}_{N,Q} \sum_{j=1}^{N} f(x_j) \right| > \eta \right) \leq Ce^{-c_f\eta^2}.$$ 

- Truncation of $x_j$ to range $[-L, L]$: s.th. $c_f$ is bounded a.s.
  for $f(x_j) = f(x_j, \omega)$ Gaussian process in $C^b_b[-L, L]$.

- Approximate $P^h_{N,Q}$ by a mixture of asymptotic unitary invariant ensembles w.r.t. the Wiener measure.
Concentration Inequalities

For equivalence use concentration inequalities:

For $f$ Lipschitz, some $c_f, C > 0$,

$$P_{N,Q} \left( \left| \sum_{j=1}^{N} f(x_j) - \mathbb{E}_{N,Q} \sum_{j=1}^{N} f(x_j) \right| > \eta \right) \leq Ce^{-c_f \eta^2}.$$ 

Truncation of $x_j$ to range $[-L, L]$: s.th. $c_f$ is bounded a.s. for $f(x_j) = f(x_j, \omega)$ Gaussian process in $C^k_b[-L, L]$.

Approximate $P^h_{N,Q}$ by a mixture of asymptotic unitary invariant ensembles w.r.t. the Wiener measure. Show

$$P_{N,V,f} = \frac{1}{Z_{N,V,f}} \prod_{i<j} |x_i - x_j|^2 \exp\left\{ - \sum_{j=1}^{N} \left( NV(x_j) + f(x_j, \omega) \right) \right\} \rightarrow (\mu^h_{Q})^\otimes k \quad (\mu^h_{Q} \text{ equilibrium measure of } V):$$
Concentration Inequalities

- For equivalence use concentration inequalities:

For $f$ Lipschitz, some $c_f, C > 0$,

$$P_{N,Q} \left( \left| \sum_{j=1}^{N} f(x_j) - \mathbb{E}_{N,Q} \sum_{j=1}^{N} f(x_j) \right| > \eta \right) \leq C e^{-c_f \eta^2}.$$ 

- Truncation of $x_j$ to range $[-L, L]$: s.th. $c_f$ is bounded a.s.

for $f(x_j) = f(x_j, \omega)$ Gaussian process in $C_b^k[-L, L]$.

- Approximate $P_{N,Q}^h$ by a mixture of asymptotic unitary invariant ensembles w.r.t. the Wiener measure. Show

$$P_{N,V,f} = \frac{1}{Z_{N,V,f}} \prod_{i<j} |x_i - x_j|^2 \exp\{- \sum_{j=1}^{N} (N V(x_j) + f(x_j, \omega))\}$$

$$\longrightarrow (\mu_Q^h \otimes^k (\mu_Q^h \text{ equilibrium measure of } V):$$

Concentration Inequalities

- For equivalence use concentration inequalities:
  
  For $f$ Lipschitz, some $c_f, C > 0$,
  
  $$
P_{N,Q} \left( \left| \sum_{j=1}^{N} f(x_j) - \mathbb{E}_{N,Q} \sum_{j=1}^{N} f(x_j) \right| > \eta \right) \leq Ce^{-c_f\eta^2}.
  $$

- Truncation of $x_j$ to range $[-L, L]$: s.th. $c_f$ is bounded a.s.
  
  for $f(x_j) = f(x_j, \omega)$ Gaussian process in $C^k_b[-L, L]$.

- Approximate $P^h_{N,Q}$ by a mixture of asymptotic unitary invariant ensembles w.r.t. the Wiener measure. Show
  
  $$
P_{N,V,f} = \frac{1}{Z_{N,V,f}} \prod_{i<j} |x_i - x_j|^2 \exp\left\{ - \sum_{j=1}^{N} \left( N V(x_j) + f(x_j, \omega) \right) \right\}
  \quad \Rightarrow \quad \left( \mu^h_Q \right)^{\otimes k}
  $$

  (Johansson '98): Independence of $f$ in the global limits.
Asymptotic Approximations

- Compare $E_{N,Q} \sum_{j=1}^{N} f(x_j)$ with $N \int f \, d\mu_Q$:

$(\text{Kriecherbauer, Shcherbina '10})$.

Uniform convergence of $\rho_{N,Q}$ to the density of $\mu_Q$ on compacts in $\text{supp}(\mu_Q)$. (Totik '00).

Local bulk universality of correlation functions of ensembles of the form $P_{N,Q}$, $f$. (Levin, Lubinsky '08).

For $\beta \neq 2$ M. Venker used relaxation flow techniques of Bourgade, Erdős, B. Schlein, and H.-T. Yau (2011, 2012).
Asymptotic Approximations

Compare $\mathbb{E}_{N,Q} \sum_{j=1}^{N} f(x_j)$ with $N \int f \, d\mu_Q:
\int f d\rho_{N,Q} = \int f d\mu_Q + O\left(\frac{1}{N^2}\right)$ for real analytic $Q, f$.
(Kriecherbauer, Shcherbina ’10).
Asymptotic Approximations

- Compare $\mathbb{E}_{N,Q} \sum_{j=1}^{N} f(x_j)$ with $N \int f \, d\mu_Q$:
  \[
  \int f \, d\rho_{N,Q} = \int f \, d\mu_Q + \mathcal{O}\left(\frac{1}{N^2}\right)
  \]
  for real analytic $Q, f$. (Kriecherbauer, Shcherbina ’10).

- Uniform convergence of $\rho_{N,Q,f}$ to the density of $\mu_Q$ on compacts in $\text{supp}(\mu_Q)^\circ$. (Totik ’00).

For $\beta \neq 2$ M. Venker used relaxation flow techniques of Bourgade, Erdős, B. Schlein, and H.-T. Yau (2011, 2012).
Asymptotic Approximations

- Compare $\mathbb{E}_{N,Q} \sum_{j=1}^{N} f(x_j)$ with $N \int f \, d\mu_Q$:
  $$\int f \, d\rho_{N,Q} = \int f \, d\mu_Q + O \left( \frac{1}{N^2} \right)$$
  for real analytic $Q, f$.
  (Kriecherbauer, Shcherbina ’10).

- Uniform convergence of $\rho_{N,Q,f}$ to the density of $\mu_Q$
on compacts in $\text{supp}(\mu_Q)^{\circ}$. (Totik ’00).

- Local bulk universality of correlation functions of ensembles of the form $P_{N,Q,f}$. (Levin, Lubinsky ’08).
Asymptotic Approximations

- Compare $\mathbb{E}_{N,Q} \sum_{j=1}^{N} f(x_j)$ with $N \int f \, d\mu_Q$:
  $$\int f \, d\rho_{N,Q} = \int f \, d\mu_Q + \mathcal{O}\left(\frac{1}{N^2}\right)$$
  for real analytic $Q, f$.
  (Kriecherbauer, Shcherbina ’10).

- Uniform convergence of $\rho_{N,Q,f}$ to the density of $\mu_Q$ on compacts in $\text{supp}(\mu_Q)^\circ$. (Totik ’00).

- Local bulk universality of correlation functions of ensembles of the form $P_{N,Q,f}$. (Levin, Lubinsky ’08).

- For $\beta \neq 2$ M. Venker used relaxation flow techniques of Bourgade, Erdős, B. Schlein, and H.-T. Yau (2011,2012).
Semicircle Law

Let $X_{jk}, 1 \leq j \leq k < \infty$ triangular array, s.th. $E X_{jk} = 0$ and $E X_{jk}^2 = \sigma_{jk}^2$, $X_{jk} = X_{kj}$ for $1 \leq j < k < \infty$. 
Semicircle Law

Let $X_{jk}, 1 \leq j \leq k < \infty$ triangular array, s.th.

$E X_{jk} = 0$ and $E X_{jk}^2 = \sigma_{jk}^2$, $X_{jk} = X_{kj}$ for $1 \leq j < k < \infty$. Let

$$X_n := \{X_{jk}\}_{j,k=1}^n.$$

$\lambda_1 \leq \ldots \leq \lambda_n$ eigenvalues of $n^{-1/2}X_n$ with spectral distr.
Semicircle Law

Let $X_{jk}, 1 \leq j \leq k < \infty$ triangular array, s.th.

$E X_{jk} = 0$ and $E X_{jk}^2 = \sigma_{jk}^2$, \quad $X_{jk} = X_{kj}$ for $1 \leq j < k < \infty$. Let

$$X_n := \{X_{jk}\}_{j,k=1}^n.$$

$\lambda_1 \leq \ldots \leq \lambda_n$ eigenvalues of $n^{-1/2}X_n$ with spectral distr.

$$\mathcal{F}^{X_n}(x) = \frac{1}{n} \sum_{i=1}^n \mathbb{1}(\lambda_i \leq x),$$
Semicircle Law

Let $X_{jk}, 1 \leq j \leq k < \infty$ triangular array, s.th.

$E X_{jk} = 0$ and $E X_{jk}^2 = \sigma^2_{jk}, \quad X_{jk} = X_{kj}$ for $1 \leq j < k < \infty$. Let

$$X_n := \{X_{jk}\}_{j,k=1}^n.$$ 

$\lambda_1 \leq \ldots \leq \lambda_n$ eigenvalues of $n^{-1/2}X_n$ with spectral distr.

$$F^{X_n}(x) = \frac{1}{n} \sum_{i=1}^n \mathbb{1}(\lambda_i \leq x),$$

Let $F^{X_n}(x) := E F^{X_n}(x)$ and $G(x)$ d.f. of standard semicircle law.
Semicircle Law

Let $X_{jk}, 1 \leq j \leq k < \infty$ triangular array, s.th. $E X_{jk} = 0$ and $E X_{jk}^2 = \sigma_{jk}^2$, $X_{jk} = X_{kj}$ for $1 \leq j < k < \infty$. Let

$$X_n := \{X_{jk}\}_{j,k=1}^n.$$  

$\lambda_1 \leq \ldots \leq \lambda_n$ eigenvalues of $n^{-1/2}X_n$ with spectral distr.

$$\mathcal{F}^{X_n}(x) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}(\lambda_i \leq x),$$

Let $F^{X_n}(x) := E \mathcal{F}^{X_n}(x)$ and $G(x)$ d.f. of standard semicircle law.

$\sigma$-algebras

$$\mathbb{F}^{(i,j)} := \sigma\{X_{kl} : 1 \leq k \leq l \leq n, (k,l) \neq (i,j)\}, \quad 1 \leq i \leq j \leq n.$$
Semicircle Law for Martingale Ensembles

For any $\tau > 0$ matricial Lindeberg:

$$L_n(\tau) := \frac{1}{n^2} \sum_{i,j=1}^n \mathbb{E}|X_{ij}|_2^{-1} \left( |X_{ij}| \geq \tau \sqrt{n} \right) \to 0 \text{ as } n \to \infty.$$ (1)

In addition:

$$\mathbb{E}(X_{ij}|F(i,j)) = 0; \quad \text{and} \quad \frac{1}{n^2} \sum_{i,j=1}^n \mathbb{E} \left| \mathbb{E}(X_{2ij}|F(i,j)) - \sigma_{ij}^2 \right| \to 0 \text{ as } n \to \infty; \quad (3)$$

For all $1 \leq i \leq n$: average column-variances $B_i^2 := \frac{1}{n} \sum_{j=1}^n \sigma_{ij}^2$. 
Semicircle Law for Martingale Ensembles

For any $\tau > 0$ matricial Lindeberg:

$$L_n(\tau) := \frac{1}{n^2} \sum_{i,j=1}^{n} \mathbb{E} |X_{ij}|^2 \mathbb{1}(|X_{ij}| \geq \tau \sqrt{n}) \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty. \quad (1)$$

In addition:

$$\mathbb{E}(X_{ij}|F(i,j)) = 0; \quad \text{and} \quad \frac{1}{n^2} \sum_{i,j=1}^{n} \mathbb{E} |\mathbb{E}(X_{ij}^2|F(i,j)) - \sigma_{ij}^2| \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty; \quad (2)$$
Semicircle Law for Martingale Ensembles

For any $\tau > 0$ matricial Lindeberg:

$$L_n(\tau) := \frac{1}{n^2} \sum_{i,j=1}^{n} \mathbb{E} |X_{ij}|^2 \mathbb{1}(|X_{ij}| \geq \tau \sqrt{n}) \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty. \quad (1)$$

In addition:

$$\mathbb{E}(X_{ij} | \mathcal{F}^{(i,j)}) = 0; \quad \text{and} \quad (2)$$
Semicircle Law for Martingale Ensembles

For any \(\tau > 0\) matricial Lindeberg:

\[
L_n(\tau) := \frac{1}{n^2} \sum_{i,j=1}^{n} \mathbb{E} |X_{ij}|^2 \mathbb{1}(|X_{ij}| \geq \tau \sqrt{n}) \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty. \quad (1)
\]

In addition:

\[
\mathbb{E}(X_{ij}|\mathcal{F}^{(i,j)}) = 0; \quad \text{and} \quad (2)
\]

\[
\frac{1}{n^2} \sum_{i,j=1}^{n} \mathbb{E} |\mathbb{E}(X_{ij}^2|\mathcal{F}^{(i,j)}) - \sigma_{ij}^2| \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty; \quad (3)
\]
Semicircle Law for Martingale Ensembles

For any $\tau > 0$ matricial Lindeberg:

$$L_n(\tau) := \frac{1}{n^2} \sum_{i,j=1}^{n} \mathbb{E} |X_{ij}|^2 \mathbb{1}(|X_{ij}| \geq \tau \sqrt{n}) \to 0 \quad \text{as} \quad n \to \infty. \quad (1)$$

In addition:

$$\mathbb{E}(X_{ij}|\mathcal{F}(i,j)) = 0; \quad \text{and} \quad (2)$$

$$\frac{1}{n^2} \sum_{i,j=1}^{n} \mathbb{E} \left| \mathbb{E}(X_{ij}^2|\mathcal{F}(i,j)) - \sigma_{ij}^2 \right| \to 0 \quad \text{as} \quad n \to \infty; \quad (3)$$

For all $1 \leq i \leq n$: average column-variances

$$B_i^2 := \frac{1}{n} \sum_{j=1}^{n} \sigma_{ij}^2.$$
Martingale Wigner Law

\[ \sum_{i=1}^{n} |B_{2i} - 1| \to 0 \text{ as } n \to \infty; \]

\[ \max_{1 \leq i \leq n} B_i \leq C < \infty, \] absolute constant.

\[ \text{Theorem (G.-Naumov-Tikhomirov 2012)} \]

Let \( X_n \) satisfy conditions (1)–(5). Then

\[ \sup_x |F_{X_n}(x) - G(x)| \to 0 \text{ as } n \to \infty. \]

Previous results with \( \sigma_{ij} = \text{const.} \):


extensions \( \sigma_{ij} \):

Erdős, Yau and Yin (2010) used \( \max_{i,j} \sigma_{ij}^2 \).

Note that (4)-(5) is implied by

\[ \max_{1 \leq i \leq n} |B_{2i} - 1| \to 0 \text{ as } n \to \infty. \]

Martingale Wigner Law

\[ \frac{1}{n} \sum_{i=1}^{n} |B_i^2 - 1| \to 0 \quad \text{as} \quad n \to \infty; \]
Martingale Wigner Law

\[ \frac{1}{n} \sum_{i=1}^{n} |B_i^2 - 1| \to 0 \quad \text{as} \quad n \to \infty; \]  

(4)

\[ \max_{1 \leq i \leq n} B_i \leq C < \infty, \quad \text{absolute constant.} \]  

(5)
Martingale Wigner Law

\[ \frac{1}{n} \sum_{i=1}^{n} |B_i^2 - 1| \to 0 \quad \text{as} \quad n \to \infty; \quad (4) \]

\[ \max_{1 \leq i \leq n} B_i \leq C < \infty, \quad \text{absolute constant.} \quad (5) \]

**Theorem (G.-Naumov-Tikhomirov 2012)**

Let \( X_n \) satisfy conditions (1)–(5). Then

\[ \sup_{x} |F_{X_n}^{x}(x) - G(x)| \to 0 \quad \text{as} \quad n \to \infty. \]
Martingale Wigner Law

\[ \frac{1}{n} \sum_{i=1}^{n} |B_i^2 - 1| \to 0 \quad \text{as} \quad n \to \infty; \quad (4) \]

\[ \max_{1 \leq i \leq n} B_i \leq C < \infty, \quad \text{absolute constant.} \quad (5) \]

**Theorem (G.-Naumov-Tikhomirov 2012)**

Let \( X_n \) satisfy conditions (1)–(5). Then

\[ \sup_x |F_{X_n}(x) - G(x)| \to 0 \quad \text{as} \quad n \to \infty. \]

Previous results with \( \sigma_{ij} = \text{const.} \):
Martingale Wigner Law

\[
\frac{1}{n} \sum_{i=1}^{n} |B_i^2 - 1| \to 0 \quad \text{as} \quad n \to \infty; \quad (4)
\]

\[
\max_{1 \leq i \leq n} B_i \leq C < \infty, \quad \text{absolute constant.} \quad (5)
\]

Theorem (G.-Naumov-Tikhomirov 2012)

Let \( X_n \) satisfy conditions (1)–(5). Then

\[
\sup_x |F_{X_n}(x) - G(x)| \to 0 \quad \text{as} \quad n \to \infty.
\]

Martingale Wigner Law

\[
\frac{1}{n} \sum_{i=1}^{n} |B_i^2 - 1| \to 0 \quad \text{as} \quad n \to \infty; \quad (4)
\]

\[
\max_{1 \leq i \leq n} B_i \leq C < \infty, \quad \text{absolute constant.} \quad (5)
\]

**Theorem (G.-Naumov-Tikhomirov 2012)**

Let $X_n$ satisfy conditions (1)–(5). Then

\[
\sup_x |F_{X_n}(x) - G(x)| \to 0 \quad \text{as} \quad n \to \infty.
\]

Previous results with $\sigma_{ij} = \text{const.}$: Wigner (1958), Arnold (1971), Pastur (1973), G.-Tikhomirov (2006). Extensions $\sigma_{i,j}$: Erdős, Yau and Yin (2010) used $\max_{i,j} \sigma_{ij}^2$. 
Martingale Wigner Law

\[ \frac{1}{n} \sum_{i=1}^{n} |B_i^2 - 1| \to 0 \quad \text{as} \quad n \to \infty; \quad (4) \]

\[ \max_{1 \leq i \leq n} B_i \leq C < \infty, \quad \text{absolute constant.} \quad (5) \]

**Theorem (G.-Naumov-Tikhomirov 2012)**

Let \( X_n \) satisfy conditions (1)–(5). Then

\[ \sup_x |F_{X_n}(x) - G(x)| \to 0 \quad \text{as} \quad n \to \infty. \]

Previous results with \( \sigma_{ij} = \text{const.} \): Wigner (1958), Arnold (1971), Pastur (1973), G.-Tikhomirov (2006). Extensions \( \sigma_{i,j} \): Erdős, Yau and Yin (2010) used \( \max_{i,j} \sigma_{ij}^2 \).

Note that (4)-(5) is implied by
Martingale Wigner Law

\[ \frac{1}{n} \sum_{i=1}^{n} |B_i^2 - 1| \to 0 \quad \text{as} \quad n \to \infty; \quad (4) \]

\[ \max_{1 \leq i \leq n} B_i \leq C < \infty, \quad \text{absolute constant.} \quad (5) \]

**Theorem (G.-Naumov-Tikhomirov 2012)**

Let \( X_n \) satisfy conditions (1)–(5). Then

\[ \sup_x |F_{X_n}(x) - G(x)| \to 0 \quad \text{as} \quad n \to \infty. \]

Previous results with \( \sigma_{ij} = \text{const.} \):  
Wigner (1958), Arnold (1971), Pastur (1973),  
Extensions \( \sigma_{i,j} \):  
Erdős, Yau and Yin (2010) used \( \max_{i,j} \sigma_{ij}^2 \).

Note that (4)-(5) is implied by

\[ \max_{1 \leq i \leq n} |B_i^2 - 1| \to 0 \quad \text{as} \quad n \to \infty. \]
Martingale Wigner Law

\[ \frac{1}{n} \sum_{i=1}^{n} |B_i^2 - 1| \to 0 \quad \text{as} \quad n \to \infty; \quad (4) \]

\[ \max_{1 \leq i \leq n} B_i \leq C < \infty, \quad \text{absolute constant.} \quad (5) \]

**Theorem (G.-Naumov-Tikhomirov 2012)**

Let \( X_n \) satisfy conditions (1)–(5). Then

\[ \sup_x |F^{X_n}(x) - G(x)| \to 0 \quad \text{as} \quad n \to \infty. \]

Previous results with \( \sigma_{ij} = \text{const.} \): Wigner (1958), Arnold (1971), Pastur (1973), G.-Tikhomirov (2006). Extensions \( \sigma_{i,j} \): Erdős, Yau and Yin (2010) used \( \max_{i,j} \sigma_{ij}^2 \).

Note that (4)-(5) is implied by

\[ \max_{1 \leq i \leq n} |B_i^2 - 1| \to 0 \quad \text{as} \quad n \to \infty. \]

Marcenko-Pastur laws for martingale ensembles: G.-Tikhomirov (2004/6), Adamczak (2011)
Counterexamples I

\[ X_n = \begin{pmatrix} A & B \\ B^T & D \end{pmatrix}, \quad n = 2m, \quad m = 500 \text{ even} \]

A: \( m \times m \) symmetric \( N(0, 1) \),

B: \( m \times m \): i.i.d. \( N(0, 1) \).

D: \( m \times m \): \( N(0, 1) \) diagonal matrix
Counterexamples I

\[ X_n = \begin{pmatrix} A & B \\ B^T & D \end{pmatrix}, \quad n = 2m, \quad m = 500 \quad \text{even} \]

A: \( m \times m \) symmetric \( N(0, 1) \),
B: \( m \times m \): i.i.d. \( N(0, 1) \).
D: \( m \times m \): \( N(0, 1) \) diagonal matrix

(4) does not hold: Simulated density of \( F^{X_n} \):
Counterexamples I

\[ X_n = \begin{pmatrix} A & B \\ B^T & D \end{pmatrix}, \quad n = 2m, \quad m = 500 \quad \text{even} \]

A: \( m \times m \) symmetric \( \mathcal{N}(0, 1) \),

B: \( m \times m \): i.i.d. \( \mathcal{N}(0, 1) \).

D: \( m \times m \): \( \mathcal{N}(0, 1) \) diagonal matrix

(4) does not hold: Simulated density of \( F^{X_n} \):
Counterexamples II

\[ X_n = \begin{pmatrix} A & B \\ B^T & D \end{pmatrix}, \quad n = 4000, \quad m = 1000 \text{ even} \]

A: \(1000 \times 1000\) symmetric \(N(0, 10)\)
B: \(1000 \times 3000\): i.i.d. \(N(0, 1)\).
D: \(3000 \times 3000\): \(N(0, 10)\)
Counterexamples II

\[ X_n = \begin{pmatrix} A & B \\ B^T & D \end{pmatrix}, \quad n = 4000, \quad m = 1000 \quad \text{even} \]

A: 1000 \times 1000 \quad \text{symmetric } N(0, 10)

B: 1000 \times 3000: \quad \text{i.i.d. } N(0, 1).

D: 3000 \times 3000: \quad N(0, 10)

(4) does not hold: Simulated density of \( F^{X_n} \):
Counterexamples II

\[ X_n = \begin{pmatrix} A & B \\ B^T & D \end{pmatrix}, \quad n = 4000, \quad m = 1000 \text{ even} \]

A: 1000 × 1000  symmetric \( N(0, 10) \)

B: 1000 × 3000: i.i.d. \( N(0, 1) \).

D: 3000 × 3000: \( N(0, 10) \)

(4) does not hold: Simulated density of \( F^{X_n} \):
Counterexamples II

\[ X_n = \begin{pmatrix} A & B \\ B^T & D \end{pmatrix}, \quad n = 4000, \quad m = 1000 \text{ even} \]

\[ \begin{align*}
A &: 1000 \times 1000 \quad \text{symmetric } N(0, 10) \\
B &: 1000 \times 3000: \quad \text{i.i.d. } N(0, 1). \\
D &: 3000 \times 3000: \quad N(0, 10)
\end{align*} \]

(4) does not hold: Simulated density of \( F_{X_n} \):

Can be proved via asymptotic freeness of blocks or Lenczewski (arxiv 2012).
Steps of Proof

- Lindeberg-type universality:
  Replacing $X_{ij}$ by Gaussian $Y_{ij}$ using Stieltjes-transforms and conditional moments

- Graph summation using moment methods for non identical Gaussian entries
Thank You!