Outliers in the Spectrum of Spiked Deformations of Unitarily Invariant Random Matrices

Random Matrices and their Applications

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Definition of a spiked population model (sample covariance setting, Johnstone 2001):
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\[ X \overset{\text{Diag}}{\sim} \begin{pmatrix} \theta_1, \ldots, \theta_1, \ldots, \theta_J, \ldots, \theta_J, 1, \ldots, 1 \end{pmatrix} X^* . \]
Definition of a spiked population model (sample covariance setting, Johnstone 2001):

\[ X \overset{\text{Diag}(\theta_1, \ldots, \theta_{k_1}, \ldots, \theta_{k_J}, 1, \ldots, 1)}{\sim} X^*. \]

Definition of an additive analogue (Péché 2006):
\[ W + \text{Diag}(\theta_1, \ldots, \theta_1, \ldots, \theta_j, \ldots, \theta_j, 0, \ldots, 0). \]
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Two works

Largest eigenvalues of finite rank perturbations of unitarily invariant random matrices.

Theorem (Benaych-Georges and Nadakuditi 2009)

Almost surely,

\[ \lambda_j \to_{N \to +\infty} \begin{cases} G^{-1}_\nu(1/\theta_j) & \text{if } \theta_j > 1/ \lim_{z \downarrow b} G_\nu(z), \\ b & \text{otherwise}, \end{cases} \]

while for each fixed \( j > r \), almost surely, \( \lambda_j \to_{N \to +\infty} b \). Here,

\[ G_\nu : \mathbb{C} \setminus \text{supp}(\nu) \to \mathbb{C}, \quad G_\nu(z) = \int_{\mathbb{R}} \frac{d\nu(t)}{z-t}, \]

is the Cauchy-Stieltjes transform of the limit distribution \( \nu \), and \( b \) is the maximum of its support.
Two works

Eigenvalues of full rank perturbations of Wigner matrices.

**Theorem (Capitaine-Donati-Martin-Féral and F. 2010)**

Let $H(z) := z + \sigma^2 G_{\mu}(z)$, then there are $k_j$ eigenvalues converging almost surely to $H(\theta_j)$ iff $H'(\theta_j) > 0$, where $\mu$ is the limit distribution of the perturbation, and $\sigma^2$ is the variance of the entries of the Wigner matrix.
Model

\[ X_N = U_N^* B_N U_N, \]

- \( B_N = \text{Diag}(\beta_1^{(N)}, \ldots, \beta_N^{(N)}) \),
- \( U_N \) is a random \( N \times N \) unitary matrix distributed according to Haar measure.
Model

\[ X_N = A_N + U_N^* B_N U_N, \]

- \( A_N = \text{Diag}(\theta_1, \ldots, \theta_1, \ldots, \theta_J, \ldots, \theta_J, \alpha_1^{(N)}, \ldots, \alpha_{N-r}^{(N)}) \)
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- \( U_N \) is a random \( N \times N \) unitary matrix distributed according to Haar measure.

Question: Spectrum of \( X_N = A_N + U_N^* B_N U_N \)?
Assumptions

- \( B_N = \text{Diag}(\beta_1^{(N)}, \ldots, \beta_N^{(N)}) \);

\[
\mu_{B_N} := \frac{1}{N} \sum_{i=1}^{N} \delta_{\beta_i^{(N)}} \Rightarrow \nu \in \mathcal{P}_c(\mathbb{R}),
\]

\[
\max_{1 \leq j \leq N} \text{dist}(\beta_j^{(N)}, \text{supp}(\nu)) \to_{N \to \infty} 0.
\]
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- $A_N = \text{Diag}(\underbrace{\theta_1, \ldots, \theta_1}_{k_1}, \ldots, \underbrace{\theta_J, \ldots, \theta_J}_{k_J}, \alpha_1^{(N)}, \ldots, \alpha_{N-r}^{(N)});$  

  $$\mu_{A_N} := \frac{1}{N} \sum_{i=1}^{N} \delta_{\lambda_i(A_N)} \Rightarrow \mu \in \mathcal{P}_c(\mathbb{R}),$$  

  $$\max_{1 \leq j \leq N-r} \text{dist}(\alpha_j^{(N)}, \text{supp}(\mu)) \to N \to \infty 0,$$  

  $$\theta_j \notin \text{supp}(\mu) \text{(the so-called spikes)}.$$
Global behaviour

We will use the usual notation:

$$\mu X_N := \frac{1}{N} \sum_{\lambda \in \text{sp}(X_N)} \delta_\lambda.$$ 

Asymptotic freeness (Voiculescu 91, Speicher 93)

Under these assumptions,

$$\mu X_N \xrightarrow{a.s.} \frac{\mu \boxplus \nu}{N \to +\infty}.$$
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\]

Asymptotic freeness (Voiculescu 91, Speicher 93)

Under these assumptions,

\[
\mu_{\lambda_N} \xrightarrow{\text{a.s.}} \mu \square \nu.
\]

What is this \( \square \) operation?
Free convolution of measures

Given $\tau \in \mathcal{P}_c(\mathbb{R})$, one defines:

**Stieltjes transform**

$$G_{\tau}(z) = \int_{\mathbb{R}} \frac{d\tau(t)}{z - t}, \quad z \notin \mathbb{R}.$$
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**Definition**

One calls free convolution of $\mu$ and $\nu$ the probability measure $\mu \boxplus \nu \in \mathcal{P}_c(\mathbb{R})$ characterized by:

$$R_{\mu \boxplus \nu}(z) = R_\mu(z) + R_\nu(z).$$
Subordination

Theorem (Voiculescu 93, Biane 98)

There is a unique analytic map $\omega : \mathbb{C}^+ \to \mathbb{C}^+$ such that:

$$\forall z \in \mathbb{C}^+, \ G_{\mu \boxplus \nu}(z) = G_{\mu}(\omega(z)).$$
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Lemma

The map $\omega$ has an extension to $\mathbb{C}$ so that:

(a) $\omega$ is continuous on $\mathbb{C}^+ \cup \mathbb{R}$;
(b) $\omega(\{\infty\} \cup \mathbb{R} \setminus \text{supp}(\mu \boxplus \nu)) \subseteq \{\infty\} \cup \mathbb{R} \setminus \text{supp}(\mu)$;
(c) $\forall z \in \mathbb{C} \setminus \mathbb{R}, \ \omega(z) = \omega(\overline{z})$;
(d) $\omega$ is meromorphic on $\mathbb{C} \setminus \text{supp}(\mu \boxplus \nu)$. 

A definition

Definition

For each $j \in \{1, \ldots, J\}$, define $O_j$ the set of solutions in $\mathbb{R} \setminus \text{supp}(\mu \boxplus \nu)$ of the equation

$$\omega(\rho) = \theta_j,$$

and

$$O = \bigcup_{1 \leq j \leq J} O_j.$$
Question

Where are precisely located the eigenvalues of 
\( X_N = A_N + U_N^* B_N U_N \)?
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**Theorem (Collins-Male 2011)**

If $r = 0$ (no spikes), then almost surely,

$$\forall \eta > 0, \exists N_0 \in \mathbb{N}, \forall N \geq N_0, \text{sp}(X_N) \subseteq K_\eta,$$

where $K_\eta := \{x \in \mathbb{R} \mid d(x, \text{supp}(\mu \boxplus \nu)) \leq \eta\}$. 
Main result

In the general case, one proves:

**Theorem**

The following results hold almost surely:

- for each $\rho \in O_j$, for all small enough $\varepsilon > 0$, for all large enough $N$,

$$\text{card}\{\text{sp}(X_N) \cap ]\rho - \varepsilon; \rho + \varepsilon[\} = k_j;$$

- for almost all $\eta > 0$, for all small enough $\varepsilon > 0$, for large enough $N$,

$$\text{sp}(X_N) \cap \mathbb{C} \setminus K_\eta \subset \bigcup_{\rho \in O \cap \mathbb{C} \setminus K_\eta} ]\rho - \varepsilon; \rho + \varepsilon[.$$
Remark

Actually, our result holds for

\[ \tilde{X}_N = \tilde{A}_N + \tilde{B}_N, \]

where \( \tilde{A}_N \) and \( \tilde{B}_N \) are independent random Hermitian matrices, provided the distribution of \( \tilde{B}_N \) is invariant by conjugation by unitary matrices.
In the particular case of a finite rank deformation $A_N$, one recovers the result of Benaych-Georges and Nadakuditi (BGN 2009) on the convergence of the largest eigenvalues:

**Theorem (Benaych-Georges and Nadakuditi 2009)**

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\lambda_j \rightarrow_{N \rightarrow +\infty} \begin{cases} 
G^{-1}_\nu(1/\theta_j) & \text{if } \theta_j > 1/\lim_{z \downarrow b} G_\nu(z), \\
b & \text{otherwise,}
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while for each fixed $j > r$, almost surely, $\lambda_j \rightarrow_{N \rightarrow +\infty} b$. 
Comments

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while for each fixed $j > r$, almost surely, $\lambda_j \to_{N \to +\infty} b$.

Indeed, in that case, $\mu = \delta_0$ and $\omega(z) = \frac{1}{G_{\nu}(z)}$. 
In the case of a full rank deformation of a GUE, one recovers the result of Capitaine, Donati-Martin, Féral and F. (CDFF 2010).

**Theorem (Capitaine-Donati-Martin-Féral and F. 2010)**

Let $H(z) := z + \sigma^2 G_{\mu}(z)$, then there are $k_j$ eigenvalues converging almost surely to $H(\theta_j)$ iff $H'(\theta_j) > 0$. 
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Let $H(z) := z + \sigma^2 G_\mu(z)$, then there are $k_j$ eigenvalues converging almost surely to $H(\theta_j)$ iff $H'(\theta_j) > 0$.

Indeed, in that case, $\nu$ is semicircular, $\omega$ is invertible with inverse $H$. 
This result illustrates that the free probabilistic interpretation of outliers, discovered in (CDFF 2010) generalizing the one in (BGN 2009), is a general principle.
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Remark

It is noteworthy that, in this situation, a simple spike may create several outliers.
Sketch of proof-1

We use the following decomposition:

\[ A_N = A_N' + A_N'', \]

\[ A_N' = \text{Diag}(\alpha, \ldots, \alpha, \alpha_1^{(N)}, \ldots, \alpha_{N-r}^{(N)}), \]

\[ A_N'' = tP\Theta P, \]

where \( P \) is the \( r \times N \) matrix defined by

\[ P = (I_r|0_{r \times (N-r)}), \]

\( \Theta \) is the \( r \times r \) matrix

\[ \Theta = \text{Diag}(\theta_1 - \alpha, \ldots, \theta_1 - \alpha, \ldots, \theta_J - \alpha, \ldots, \theta_J - \alpha), \]

and \( \alpha \in \text{supp}(\mu). \)
Sketch of proof-2

\[
\det(\lambda I_N - X_N) = \det(\lambda I_N - (A'_N + U^*_N B_N U_N)) \det(I_N - R_N(\lambda)^t P\Theta P),
\]
where

\[
R_N(\lambda) = (\lambda I_N - (A'_N + U^*_N B_N U_N))^{-1}. \tag{2}
\]

Using that, for rectangular matrices \( X \in M_{N,r}(\mathbb{C}) \), \( Y \in M_{r,N}(\mathbb{C}) \), one has \( \det(I_N - XY) = \det(I_r - YX) \), one obtains:

\[
\det(\lambda I_N - X_N)) = \det(\lambda I_N - (A'_N + U^*_N B_N U_N)) \det(I_r - PR_N(\lambda)^t P\Theta).
\]

Hence, the outliers of \( X_N \) are precisely the zeros of \( \det(M_N) \) outside the support of \( \mu \boxplus \nu \), where

\[
M_N := I_r - PR_N^t P\Theta. \tag{3}
\]
Key point

Using Hurwitz’s theorem, the zeros of det($M_N$) will cluster towards those of det($M$), where $M$ is the almost sure uniform limit of $M_N$. 
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- By concentration arguments, $M_N - I_r - P\mathbb{E}(R_N)^tP\Theta$ tends to 0 as $N$ goes to infinity.
Using Hurwitz’s theorem, the zeros of \( \text{det}(M_N) \) will cluster towards those of \( \text{det}(M) \), where \( M \) is the almost sure uniform limit of \( M_N \).

- By concentration arguments, \( M_N - I_r - P\mathbb{E}(R_N)^tP\Theta \) tends to 0 as \( N \) goes to infinity.
- It is known that \( \mathbb{E}(R_N) \) is diagonal (Kargin 2011). Actually, it is a polynomial in \( A'_N \).
  In particular, \( P\mathbb{E}(R_N)^tP \) is a scalar matrix.
Define $\omega_N$ so that:

$$P\mathbb{E}(R_N)^t P = \frac{1}{\omega_N - \alpha} I_r.$$
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Then $(\omega_N)_{N \in \mathbb{N}}$ is a normal sequence of analytic functions, whose limit points $l$ shall satisfy the subordination equation:

$$G_{\mu \boxplus \nu}(z) = G_{\mu}(l(z)),$$

which has the subordination map $\omega$ as a unique solution.
So $M_N$ almost surely uniformly converges to:

$$M := I_r - \frac{1}{\omega - \alpha} \Theta.$$  \hfill (4)
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And $z$ such that $\text{det}(M(z)) = 0$ are precisely solutions of $\omega(z) = \theta_j$ for some $j$, concluding the proof.
Thank you for your attention!