Limit Theorems for Linear Eigenvalue Statistics of Random Matrices with Independent Entries

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Outline

- Introduction
- Gaussian Ensembles
  - Law of Large Numbers
  - CLT
- Wigner Ensembles
  - Law of Large Numbers
  - CLT (zero excess)
  - CLT (general case)
- Sample Covariance Matrices
  - Law of Large Numbers
  - CLT (zero excess)
  - CLT (general case)
- Concluding remarks
Introduction

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- its eigenvalues $\{\lambda_l^{(n)}\}_{l=1}^n$
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- linear statistics

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\mathcal{N}_n[\varphi] := \sum_{l=1}^n \varphi \left( \lambda_{(n)}^l \right) = \text{Tr} \varphi(M_n)
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- centered linear statistics

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\mathcal{N}_n^\circ[\varphi] = \mathcal{N}_n[\varphi] - \mathbb{E}\{\mathcal{N}_n[\varphi]\}
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\[ N_n^{\circ}[\varphi] = N_n[\varphi] - \mathbf{E}\{N_n[\varphi]\} \]

• We are interested in the limiting laws of $N_n[\varphi]$ as $n \to \infty$.
  possibly after putting a normalization factor in front (LLN and CLT type)
Introduction

LT’s is an active field of the RMT:
Marchenko, P 67; P 72; Girko 70-80; Bai-Silverstein 80-90, Costin-Lebowitz 95; Khorunzhy-Khoruzhenko-P. 96; Spohn 97; Johansson 98; Sinai-Soshnikov 98; Soshnikov 98, 00; Keating-Snaith 00; Cabanal-Duvillard 01; Diaconis-Evans 01; Guionnet 02; Bai-Silverstein 04; Anderson-Zeitouni 05; P. 06; Lytova-P. 09
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- Valid due to a different mechanism (\( \text{Var}\{\mathcal{N}_n[\varphi]\} \) does not grow with \( n \)) and even not always valid P. 06, Lytova-P. 10.
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Applications and links: statistics, strong Szego theorem on asymptotics of Toeplitz determinants, universal conductance fluctuations of small metallic particles (mesoscopics), $1/n^2$- expansion in SM and QFT, telecommunications, quantitative finances, etc.
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- Noblesse obligé (L.P.): Lyapunov (first modern proof of CLT), S. Bernstein (first CLT for dependent r.v.’s), both from Kharkov
Gaussian Ensembles

Generalities

Definition: \( M_n = n^{-1/2} W_n \), \( W_n = \{ W_{jk} \}_{j,k=1}^n \)

\[
P(dW) = Z_n^{-1} e^{-\text{Tr} W^2 / 4w^2} \prod_{1 \leq j \leq k \leq n} dW_{jk}.
\]

Since

\[
\text{Tr} W_n^2 = \sum_{1 \leq j \leq n} W_{jj}^2 + 2 \sum_{1 \leq j \leq k \leq n} W_{jk}^2,
\]

the above implies that \( \{ W_{jk} \}_{1 \leq j \leq k \leq n} \) are independent Gaussian random variables such that

\[
\mathbb{E}\{ W_{jk} \} = 0, \quad \mathbb{E}\{ W_{jk}^2 \} = w^2 (1 + \delta_{jk}).
\]

Gaussian Orthogonal Ensemble (GOE)
**Theorem**

Let $M_n$ be the GOE) and $\mathcal{N}_n[\varphi]$ be a linear eigenvalue statistics of its eigenvalues. Then we have for any bounded and continuous $\varphi : \mathbb{R} \to \mathbb{C}$ with probability 1:

$$\lim_{n \to \infty} \frac{1}{n} \sum_{l=1}^{n} \varphi \left( \lambda_l^{(n)} \right) = \int \varphi(\lambda) N_{sc\ell}(d\lambda),$$

where the measure

$$N_{sc}(d\lambda) = \rho_{sc}(\lambda) d\lambda, \quad \rho_{sc}(\lambda) = (2\pi w^2)^{-1} \sqrt{4w^2 - \lambda^2} 1_{|\lambda| \leq 2w}$$

is known as the Wigner or the semicircle law.

Wigner 52 and many others.
Gaussian Ensembles

Law of Large Numbers (proof)

It suffices to consider the Normalized Counting Measure of eigenvalues (NCM)

\[ N_n(\Delta) = \#\{\lambda_i^{(n)} \in \Delta\} / n, \ \forall \Delta \subset \mathbb{R} \]

and its Stieltjes transform

\[ g_n(z) = \int \frac{N_n(d\lambda)}{\lambda - z}, \ \text{Im} \ z \neq 0, \]

determining \( N_n \). Use now

(i) Gaussian differentiation formula:

\[ \mathbb{E}\{\xi_l \Phi(\xi)\} = \mathbb{E}\{\xi_l^2\} \mathbb{E}\{\Phi'_l(\xi)\}, \ l = 1, \ldots, p; \]

(ii) Poincaré-Nash-Chernoff inequality:

\[ \text{Var}\{\Phi\} \leq \sum_{l=1}^{p} \mathbb{E}\{\xi_l^2\} \mathbb{E}\{|\Phi'_l|^2\}. \]
Gaussian Ensembles
Law of Large Numbers (proof)

By spectral theorem $g_n(z) = n^{-1} \text{Tr}(M_n - z)^{-1}$, by resolvent identity for $f_n(z) = \mathbb{E}\{g_n(z)\}$

$$f_n(z) = z^{-1} + (zn)^{-1} \sum_{j,k=1}^n \mathbb{E}\{M_{jk}G_{kj}(z)\},$$

by (i) $f_n(z) = z^{-1} + z^{-1} \mathbb{E}\{g_n^2(z)\}$, and by (ii) (Bose-Chatterjee 04; P. 05)

$$\text{Var}\{g_n(z)\} \leq 2w^2 / n^2 |\text{Im} z|^4$$

while $\text{Var}\{g_n(z)\} \leq w^2 / n |\text{Im} z|^4$ for random Schrodinger.

This leads to

$$f_{sc}(z) = z^{-1} + z^{-1} f_{sc}^2(z)$$

for $\lim_{n \to \infty} f_n = f_{sc}$ uniformly on any compact set of $\mathbb{C} \setminus \mathbb{R}$, thus

$$f_{sc}(z) = (\sqrt{z^2 - 4w^2} - z) / 2w^2 \ (\text{Im} f(z) \text{Im} z > 0).$$

Convergence of $g_n$ to $f_{sc}$, hence $N_n$ to $N_{sc}$ by Borel-Cantelli.
Let $M_n$ be the GOE matrix, $\phi : \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable function with a polynomially bounded derivative. Then $\mathcal{N}_n[\phi] = \mathcal{N}_n[\phi] - \mathbb{E}\{\mathcal{N}_n[\phi]\}$ converges in distribution to the Gaussian random variable with zero mean and the variance

$$V_{\text{GOE}}[\phi] = \frac{1}{2\pi^2} \int_{-2w}^{2w} \int_{-2w}^{2w} \left( \frac{\phi(\lambda_1) - \phi(\lambda_2)}{\lambda_1 - \lambda_2} \right)^2 \times \frac{4w^2 - \lambda_1 \lambda_2}{\sqrt{4w^2 - \lambda_1^2} \sqrt{4w^2 - \lambda_2^2}} d\lambda_1 d\lambda_2.$$
Gaussian Ensembles
Central Limit Theorem (proof)

Proof is again based on the Gaussian differentiation formula and the bound

\[ \text{Var}\{\mathcal{N}_n[\varphi]\} \leq \frac{2w^2}{n} \mathbb{E}\{\text{Tr}\varphi'(M_n)(\varphi'(M_n)^*)\} \leq 2w^2 \left( \sup_{\lambda \in \mathbb{R}} |\varphi'(\lambda)| \right)^2 \]

for \( \mathcal{N}_n[\varphi] = \text{Tr}\varphi(M) \) by Poincaré. We have to prove that

\[ \lim_{n \to \infty} Z_n(x) = \exp \left\{ -x^2 V_{GOE}[\varphi]/2 \right\}, \quad Z_n(x) = \mathbb{E}\left\{ e^{ix\hat{\mathcal{N}}_n[\varphi]} \right\} \]

uniformly in \( x \), varying on a finite interval of \( \mathbb{R} \). Assume first that \( \varphi \) admits the Fourier transform \( \hat{\varphi} \) and \( (1 + |t|)|\hat{\varphi}(t)| \in L^1(\mathbb{R}) \). Then

\[ Z_n(x) = 1 + \int_0^x Z'_n(y)dy, \quad Z'_n(x) = i \int \hat{\varphi}(t) Y_n(x, t) dt, \]

where

\[ Y_n(x, t) = \mathbb{E}\left\{ \hat{u}_n(t)e_n(x) \right\}, \quad e_n(x) = e^{ix\hat{\mathcal{N}}_n[\varphi]}, \quad u_n(t) = \text{Tr}e^{itM}. \]
Gaussian Ensembles

Central Limit Theorem (proof)

Use $U_n(t) = e^{itM_n}$ (instead of $G_n(z) = (M_n - z)^{-1}$) and the Duhamel formula

$$u_n(t) = n + i \int_0^t \sum_{j,k=1}^n M_{jk} U_{jk}(t_1) dt_1,$$

the differentiation formula, the Poincaré, and the Schwarz to obtain

$$Y_n(x, t) + 2w^2 \int_0^t dt_1 \int_0^{t_1} dt_2 \overline{v}_n(t_1 - t_2) Y_n(x, t_2) = xZ_n(x) A_n(t) + r_n(x, t),$$

$$A_n(t) = -2w^2 \int_0^t dt_1 \int e^{it_1 \lambda} \phi'(\lambda) \mathbb{E}\{N_n(d\lambda)\}, \overline{v}_n(t) = \mathbb{E}\{n^{-1} \text{Tr} U(t)\}$$

$$|Y_n| \leq \sqrt{2w} |t| \sup_{\lambda \in \mathbb{R}} |\phi'(\lambda)|,$$

$$\left|(Y_n)'\right| \leq \sqrt{2w(1 + w^2 t^2)^{1/2}}, \left|(Y_n)'_x\right| \leq 2w^2 t \sup_{\lambda \in \mathbb{R}} |\phi'|.$$

Hence, there exists $\{Y_{n_j}\}$ converging uniformly on any compact set of $\mathbb{R}^2$ to $Y$, satisfying
Gaussian Ensembles
Central Limit Theorem (proof)

\[ Y(x, t) + 2w^2 \int_0^t dt_1 \int_0^{t_1} dt_2 \nu(t_1 - t_2) Y(x, t_2) = xZ(x)A(t), \]
\[ A(t) = -2w^2 \int_0^t dt_1 \int e^{it_1\lambda} \phi'(\lambda) N_{sc}(d\lambda), \quad \nu(t) = \int e^{i\lambda t} N_{sc}(d\lambda). \]

This leads (by the Laplace transformation) to

\[ Z(x) = 1 - V_{GOE} \int_0^x yZ(y)dy. \]

The equation is uniquely soluble and yield the result for \((1 + |t|)\phi(t) \in L^1(\mathbb{R})\). General case of \(C^1\) (even Lip 1) test functions is obtained by Poincaré and approximations.

The scheme dates back to Khorunzhy-Khoruzhenko-P. 96, where the Stieljtes transform (the resolvent) was used, thus real analytic test functions. Here we use the Fourier transform and obtain \(C^1\) test functions.
\[ M_n = n^{-1/2} W_n, \quad W_n = \left\{ W_{jk}^{(n)} \right\}_{j,k=1}^n \]

with \( W_{jk}^{(n)} = W_{kj}^{(n)} \in \mathbb{R}, \ 1 \leq j \leq k \leq n \) independent and

\[
\mathbf{E}\{ W_{jk}^{(n)} \} = 0, \quad \mathbf{E}\{ (W_{jk}^{(n)})^2 \} = (1 + \delta_{jk}) w^2,
\]

i.e. the two first moments of the entries coincide with those of the GOE or

\[
P(dW_n) = \prod_{1 \leq j \leq k \leq n} F_{jk}^{(n)}(dW_{jk}),
\]

where \( F_{jk}^{(n)} \) has above moments. The GOE corresponds to

\[
F_{jk}^{(n)}(dW) = \frac{1}{(2\pi \sigma_{jk}^2)^{1/2}} e^{-W^2/2\sigma_{jk}^2} dW, \quad \sigma_{jk}^2 = (1 + \delta_{jk}) w^2.
\]
**Theorem**

Let $M_n = n^{-1/2} W_n$ be the Wigner matrix, satisfying the L2 (à la Lindeberg)

$$\lim_{n \to \infty} n^{-2} \sum_{1 \leq j \leq k \leq n} \int_{|w| \geq \tau \sqrt{n}} W^2 F_{jk}^{(n)}(dW), \quad \forall \tau > 0.$$ 

and $N_n$ be the Normalized Counting Measure of its eigenvalues. Then with p.1: $\lim_{n \to \infty} N_n(\Delta) = N_{sc}(\Delta), \ \forall \Delta \subset \mathbb{R}$ (macroscopic universality).

**P. 72; Girko 75.** No Poincaré but the martingale-type bounds $E\{|N_n^\circ(\Delta)|^4\} = O(n^{-2})$. Thus, it suffices to prove that if $M_n$ is the Wigner matrix and $\hat{M}_n$ is the corresponding GOE, then

$$R_n(z) := E\{n^{-1}\text{Tr}(M_n - z)^{-1}\} - E\{n^{-1}\text{Tr}(\hat{M}_n - z)^{-1}\} \to 0, \ n \to \infty$$

uniformly on a compact set of $\mathbb{C} \setminus \mathbb{R}$, cf recent results by *Erdos et al 09* for $\text{Im} z = O(n^{-1})$.
Proof is based on

- **General differentiation formula** (*Khorunzhy-Khoruzhenko-P. 95*):

  If $\mathbb{E}\{|\zeta|^{p+2}\} < \infty$, $p \in \mathbb{N}$, $\Phi : \mathbb{R} \to \mathbb{C}$ of $C^{p+1}$ with bounded derivatives, then

  $$
  \mathbb{E}\{\zeta \Phi(\zeta)\} = \sum_{j=0}^{p} \frac{\kappa_{j+1}}{j!} \mathbb{E}\{\Phi^{(j)}(\zeta)\} + \varepsilon_p,
  $$

  where $\{\kappa_j\}_{j=1}^{\infty}$ are the cumulants of $W_{12}$. Note that the $l = 1$ term is "Gaussian".

- **"Interpolation trick"** (*P. 2000*): use the product space of the Wigner $M_n$ and the GOE $\hat{M}_n$ with the same first and second moments and set

  $$
  M_n(s) = s^{1/2} M_n + (1 - s)^{1/2} \hat{M}_n, \quad 0 \leq s \leq 1,
  $$
Assume first $w_3 := \sup_n \max_{1 \leq j \leq k \leq n} E \left\{ |W_{jk}^{(n)}|^3 \right\} < \infty$ and write

$$R_n(z) = \int_0^1 \frac{d}{ds} E\{n^{-1} \text{Tr} (M_n(s) - z)^{-1}\} ds = \frac{1}{2} \int_0^1 (T_1 - T_2) ds$$

$$T_1 = (n^3 s)^{-1/2} \sum_{1 \leq j, k \leq n} E\{W_{jk}^{(n)} (G^2)_{jk}\},$$

$$T_2 = (n^3 (1 - s))^{-1/2} \sum_{1 \leq j, k \leq n} \hat{E}\{\hat{W}_{jk} (G^2)_{jk}\}.$$ 

Apply to $T_1$ the general differentiation formula with $p = 1$ and $\Phi = (G^2)_{jk}$ and to $T_2$ the Gaussian differentiation formula. We have the cancellation, resulting only in $\varepsilon_1$:

$$|\varepsilon_1| \leq \frac{C_1 w_3}{n^{5/2}} \sum_{1 \leq j \leq k \leq n} \sup_{M \in S_n} |D_{jk} (G^2)_{jk}| \leq \frac{C'_1 w_3}{n^{1/2} |\mathcal{S} z|^4}, \quad D_{jk} = \frac{\partial}{\partial M_{jk}}.$$

$S_n$ is the set of $n \times n$ real symmetric matrices.
Wigner Ensembles
Central Limit Theorem (zero excess)

**Theorem**

Let $M_n = n^{-1/2} W_n$, $W_n = \{ W_{jk}^{(n)} \}_{j,k=1}^n$ be the real symmetric Wigner random matrix. Assume that $\mu_4 = \mathbb{E}\{(W_{jk}^{(n)})^4\}$ does not depend on $j, k$ and $n$, $\kappa_4 = \mu_4 - 3\omega^4 = 0$, and the $L^4$:

$$
\lim_{n \to \infty} n^{-2} \sum_{j,k=1}^n \int_{|W| \geq \tau \sqrt{n}} W^4 F_{jk}^{(n)}(dW) = 0, \ \forall \tau > 0,
$$

If $\phi$ possesses the Fourier transform $\hat{\phi}$ and $(1 + |t|^5)|\hat{\phi}(t)| \in L^1(\mathbb{R})$, then $\mathcal{N}_n[\phi]$ converges in distribution to the Gaussian random variable with zero mean the GOE variance (again the macroscopic universality, even a bit more).

Proof by the "interpolation" trick from the GOE. For "Lindeberg-4 " see KKP, 95.
Wigner Ensembles
Central Limit Theorem (general case)

**Theorem**

Let $M_n = n^{-1/2} W_n$ be the real symmetric Wigner random matrix, 
$\mu_4 = \mathbb{E}\{(W_{jk}^{(n)})^4\}$ do not depend on $j, k$ and $n$ and

$$w_6 := \sup_n \max_{1 \leq j, k \leq n} \mathbb{E}\{(W_{jk}^{(n)})^6\} < \infty.$$ 

If $(1 + |t|^5)\hat{\phi}(t) \in L^1(\mathbb{R})$, then 
$\tilde{\mathcal{N}}_n[\varphi] = \mathcal{N}_n[\varphi] - \mathbb{E}\{\mathcal{N}_n[\varphi]\}$ converges in distribution to the Gaussian random variable of zero mean and of variance

$$V_{Wig}[\varphi] = V_{GOE}[\varphi] + \frac{\kappa_4}{2\pi^2 w^8} I_{Wig}^2,$$

$$I_{Wig} = \int_{-2w}^{2w} \varphi(\mu) \frac{2w^2 - \mu^2}{\sqrt{4w^2 - \mu^2}} d\mu.$$
Assume that $\kappa_4 \neq 0$, then:

$I_{Wig} = 0$: the GOE CLT, e.g. for an ODD $\varphi$.

$I_{Wig} \neq 0$: a modified CLT, generically and, in particular, for an EVEN $\varphi$ such that

$$
\int_{0}^{2w} \varphi(\mu) \frac{2w^2 - \mu^2}{\sqrt{4w^2 - \mu^2}} d\mu \neq 0.
$$
Proof: by combining the schemes of proof of the CLT for the GOE and the "interpolation" trick, in particular, by proving and using

\textbf{Theorem}

Let $M_n = n^{-1/2} W_n$ be the real symmetric Wigner random matrix and $\mathcal{N}_n[\varphi]$ be the linear eigenvalue statistic of its eigenvalues. Assume that

$$w_6 := \sup_n \max_{1 \leq j, k \leq n} \mathbb{E} \left\{ \left| W_{jk}^{(n)} \right|^6 \right\} < \infty.$$  

Then

$$\text{Var}\{\mathcal{N}_n[\varphi]\} \leq C(w_6) \left( \int (1 + |t|^{5/2}) \tilde{\varphi}(t) |t| dt \right)^{1/2},$$

where $C(w_6)$ depends only on $w_6$.

The bound replaces the Poincaré one in the case of Wigner ensembles.
Wigner Ensembles

CLT (origin of new term in the variance)

\[ Y_n(x, t) = \sum_{a=1}^{3} T_a + \varepsilon_3, \]

where now

\[ T_a = \frac{i}{a! n^{(a+1)/2}} \int_0^t \sum_{j,k=1}^n \kappa_{a+1,jk} \mathbb{E} \{ D_{jk}^a (U_{jk}(s)e_n^o(x)) \} \, ds, \]

\[ D_{jk} = \frac{\partial}{\partial M_{jk}} \]

and

\[ |\varepsilon_3| \leq C(x) w_6^{5/6} (1 + |t|^4) / n^{1/2}. \]

The term \( T_3 \) contains \( U_{jj}(t_1)U_{jj}(t_2)U_{kk}(t_3)U_{kk}(t_4) \) Because of

\[ D_{jk} U_{ab}(t) = i\beta_{jk} \int_0^t ds \left[ U_{aj}(t-s)U_{bk}(s) + U_{bj}(t-s)U_{ak}(s) \right]. \]

These are only combinations of \( U \)'s that contribute.
Universality class w.r.t. to the CLT: the set of random matrices, having the same CLT (variance) for linear eigenvalue statistics.

Universality classes of the Wigner matrices w.r.t. the CLT are indexed by the first two even moments of their off-diagonal entries:

\[ w^2 = \mathbb{E}\{(W_{jk}^{(n)})^2\}, \quad \mu_4 = \mathbb{E}\{(W_{jk}^{(n)})^4\}, \quad 1 \leq j < k \leq n \]

(two dimensional moduli space).

An example of "collective theorem", Linnik 70's.

The Gaussian universality classes: \( \kappa_4 := \mu_4 - 3w^4 = 0. \)

In the conventional probability setting for the CLT of independent random variables \( \{\xi_i^{(n)}\}_{i=1}^n \) the universality classes w.r.t. the CLT of linear statistics are indexed by a single parameter, the variance \( \sigma^2 = \mathbb{E}\{(\xi_i^{(n)})^2\}. \)

All classes are Gaussian.
Sample Covariance Matrices

Generalities

$M_{m,n}$ is a $n \times n$ real symmetric matrix of the form (matrix $\chi^2$)

$$M_{m,n} = n^{-1} A^T_{m,n} A_{m,n},$$

with $A_{m,n} = \{A_{\alpha j}\}_{\alpha,j=1}^{m,n}$ having i.i.d. entries ($m$ observation on $n$ parameters)

$$P(dA_{m,n}) = \prod_{\alpha=1}^{m} \prod_{j=1}^{n} F_{\alpha j}^{(n)} (dA_{\alpha j})$$

such that

$$\mathbf{E}\{A_{\alpha j}\} = 0, \quad \mathbf{E}\{A_{\alpha j}\} = a^2.$$

The case of i.i.d. Gaussian $\{A_{\alpha j}\}_{\alpha,j=1}^{m,n}$ is known since the early 30’s as the (white or null) Wishart Ensemble.
Sample Covariance Matrices

Law of Large Numbers

**Theorem**

Let $M_{m,n}$ be the sample covariance matrix such that $\tau > 0$

$$\lim_{n \to \infty, m \to \infty, m/n \to c \in (0, \infty)} \frac{1}{mn} \sum_{\alpha=1}^{m} \sum_{j=1}^{n} \int_{|y| > \tau \sqrt{n}} y^2 F_{\alpha j}^{(n)}(dy) \to 0, \forall \tau > 0$$

Then its Normalized Counting Measures $N_n$ converges with probability 1 to the non-random measure: $N_W(d\lambda) = \rho_W(\lambda)d\lambda$

$$\rho_W(\lambda) = (1 - c) + \delta_0 + \sqrt{((\lambda - a_-)(a_+ - \lambda))_+} / 2\pi a^2 \lambda,$$

where $a_\pm = a^2 (1 \pm \sqrt{c})^2$ (macroscopic universality again)

Marchenko, P. 67; Girko 70's.

Proof: Wishart by the resolvent identity, Gaussian differentiation formula, and Poincaré. General case as for the Wigner (i.e. the interpolation again).
Theorem

Let $M_{m,n}$ be the Wishart random matrix. If $\varphi$ is $C^1$, then $\mathcal{N}_n[\varphi]$ converges in distribution as $m, n \to \infty$, $m/n \to c > 0$ to the Gaussian random variable with zero mean and the variance

$$V_{\text{Wish}}[\varphi] = \frac{1}{2\pi^2} \int_{a_-}^{a_+} \int_{a_-}^{a_+} \left( \frac{\varphi(\lambda_1) - \varphi(\lambda_2)}{\lambda_1 - \lambda_2} \right)^2 \frac{4a^4c - (\lambda_1 - \bar{a})(\lambda_2 - \bar{a})}{\sqrt{4a^4c - (\lambda_1 - \bar{a})^2} \sqrt{4a^4c - (\lambda_2 - \bar{a})^2}} d\lambda_1 d\lambda_2,$$

where $\bar{a} = 1/2(a_- + a_+) = a^2(c + 1)$.

Proof: By mimicking the proof for the GOE, i.e. by the Gaussian differentiation formula and Poincaré.
Sample Covariance Matrices

CLT (4th cumulant is zero)

**Theorem**

Let $M_{m,n}$ be the sample covariance matrix such that:

(i) $w_5 := \sup_{m,n} \max_{1 \leq \alpha \leq m, 1 \leq j \leq n} \mathbb{E}\left\{ |A_{\alpha j}|^5 \right\} < \infty$

(ii) $\mu_4 = \mathbb{E}\left\{ |A_{\alpha j}|^4 \right\}$ do not depend on $\alpha, j, m,$ and $n,$ and

$$\kappa_4 := \mu_4 - 3a^4 = 0.$$

If $(1 + |t|^5)|\widehat{\phi}(t)| \in L^1(\mathbb{R}),$ then $\tilde{N}_n[\varphi]$ converges in distribution as $m, n \to \infty,$ $m/n \to c > 0$ to the Gaussian random variable with zero mean and the variance $V_{Wish}[\varphi].$

Proof: by interpolation from Wishart.

*Bai, Silverstein, 04: Stieltjes transform, real analytic test functions, direct and rather long proof.*
Sample Covariance Matrices

CLT (general case)

Theorem

Let $M_{m,n}$ be the sample covariance matrix such that:

(i) $w_6 := \sup_{m,n} \max_{1 \leq \alpha \leq m, 1 \leq j \leq n} \mathbb{E}\left\{|A_{\alpha j}|^6\right\} < \infty$

(ii) $\mu_4 = \mathbb{E}\left\{|A_{\alpha j}|^4\right\}$ do not depend on $\alpha$, $j$, $m$, and $n$.

If $(1 + |t|^4) |\hat{\phi}(t)| \in L^1(\mathbb{R})$, then $\mathcal{N}^\circ_n[\varphi]$ converges in distribution as $m, n \to \infty$, $m/n \to c > 0$ to the Gaussian random variable with zero mean and the variance

$$V_{\text{Wish}}[\varphi] + \frac{\kappa_4}{4c\pi^2 a^8} \left( \int_{a_-}^{a_+} \varphi(\mu) \frac{\mu - \bar{a}}{\sqrt{4a^4 c - (\mu - \bar{a})^2}} d\mu \right)^2.$$

Proof: by the same scheme as in the Wigner case, i.e., by combining the schemes of proof of the CLT for the Wishart case and the "interpolation" trick.
Multivariate statistics

\( \varphi : \mathbb{R}^p \rightarrow \mathbb{R} \) symmetric and

\[
\mathcal{U}_{p,n}[\varphi] = \sum_{0 \leq l_1 < \ldots < l_p \leq n} \varphi(\lambda_{l_1}^{(n)}, \ldots, \lambda_{l_p}^{(n)}),
\]

\[
\mathcal{N}_{p,n}[\varphi] = \sum_{l_1=\ldots=l_p=1}^{n} \varphi(\lambda_{l_1}^{(n)}, \ldots, \lambda_{l_p}^{(n)}).
\]

We have:

1. with probability 1 (LLN):

\[
\lim_{n \to \infty} n^{-p} \mathcal{U}_{p,n}[\varphi] = \lim_{n \to \infty} n^{-p} \mathcal{N}_{p,n}[\varphi]
\]

\[
= \int p \text{ times } \int \varphi(\lambda_1, \ldots, \lambda_p) \mathcal{N}_{scl}(d\lambda_1) \ldots \mathcal{N}_{scl}(d\lambda_p);
\]
2. in distribution

\[
\lim_{n \to \infty} n^{-p+1} \mathcal{U}_{p,n}[\varphi] = \lim_{n \to \infty} n^{-p+1} \mathcal{N}_{p,n}[\varphi] \\
= \lim_{n \to \infty} \mathcal{N}_{1,n}[\varphi^*],
\]

where

\[
\varphi^*(\lambda) = \int (p - 1) \times \int \varphi(\lambda, \lambda_2, ..., \lambda_p) \\
\times \mathcal{N}_{scl}(d\lambda_2) \cdots \mathcal{N}_{scl}(d\lambda_p),
\]

i.e., the CLT.
Both assertions are valid in the cases, where there are corresponding results for \( p = 1 \).
Theorem

Let $U_n$ be a $n \times n$ unitary random matrix, whose probability law is the normalized Haar measure on $U(n)$, and $A_n$ be a $n \times n$ matrix such that

$$\lim_{n \to \infty} n^{-1} \text{Tr} A_n^* A_n = 1.$$  

Then $\text{Tr} U_n A_n$ converges in distribution to the standard complex Gaussian variable: $\gamma = \gamma_1 + i \gamma_2$, $\mathbb{E}\{\gamma_1\} = \mathbb{E}\{\gamma_1\} = 0$, $\mathbb{E}\{\gamma_1^2\} = \mathbb{E}\{\gamma_2^2\} = 1/2$.

E. Borel 05 ($A_n = \{\delta_{j1}\delta_{k1}\}_{j,k=1}^n$), Diaconis et al 03; Snyady-Stolz 06.

On the other hand, by using analogs of the differentiation formula and the Poincaré type inequality for $U(n)$ and $O(n)$ (P.-Vasilchuk 06) and the above scheme, a short and simple proof of the assertion can be obtained. Analogous assertions are valid for $O(n)$ and $Sp(n)$. 