$L^1$-smoothing for the multidimensional Ornstein-Uhlenbeck semigroup

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joint work with:

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such that $A^\lambda f = f * \lambda$ (pointwise multiplication as a group action)

a convolution operator: there exists a probability measure $\mu$ on $\mathbb{R}^\infty$

Obviously, \[ f \ast \mu = 1 \circ \mu \] It is easy to check that $\mu$

\[ \text{where } \lambda = \frac{1}{\sqrt{n}} e^{-it} \] for $t > 0$

$\phi_{\lambda \mu}$

is an orthogonal basis of $L^2(\mathbb{R}^\infty, \mu)$

Then $\phi(x) = \prod_{k \geq 1} \phi_k(x)$ where $\phi_k \equiv \phi_k(t) = \phi_k(0)$

\[ m(A) = \prod_{k \geq 1} m(A^k) \]

Thus, $A \in \mathcal{A}$ with $\mathcal{A}$

\[ g_k \ast h = g_k \ast h \]

where $h = \frac{1}{\sqrt{n}} e^{-it}$ with a normalized counting measure $\mu^n$, on

\[ \text{Discrete cube: } \{-1, 1\}^n \]
Thus for any $f: (A, |A|) \to (0, \infty)$ with $E_{\mathbb{F}}[f] = 1$ and any $t > 0$ we have (by Hölder's inequality)

$$\frac{\|f_t\|_p}{\|f\|_p} \leq \frac{1}{t}$$

Properties of the semigroup $(T_t)_{t \geq 0}$

- Positivity preservation: $f \geq 0 \Rightarrow Tf \geq 0$
- Monotonicity: $f \geq g \Rightarrow T_tf \geq T_tg$
- Convexity: $\forall x, y \in A$, $\forall u, v \geq 0$ with $u + v = 1$

$$T_t(f \cdot g) = E_{\mathbb{F}}[f \cdot T_tg] = \int_A f(x) T_tg(x) \mathbb{F}(x) \, d\mu(x) = E_{\mathbb{F}}[T_tf \cdot g] = \langle T_tf, g \rangle$$
Conjecture (M. Talagrand, 1988):

For each $t > 0$ there exists a function $\psi_t : [1, \infty) \to [0, \infty)$ with $\lim_{u \to \infty} \psi_t(u) = \infty$ and such that for every natural $n$ and every non-negative function $f$ on $[-1, 1]^n$ with $\mathbb{E}[f] = 1$ the inequality

$$\mu^n \left( \{ x \in [-1, 1]^n : (T_t f)(x) > u \} \right) \leq \frac{1}{u \psi_t(u)}$$

holds true for all $u > 1$.

Stronger version: Supposedly, one can take $\psi_t(u) = C_t \sqrt{\log u}$,

where $C_t$ is some positive constant (depending only on $t$).

It is easy to check that there is no hope for faster growth of $\psi_t$.

Equivalently, one may formulate the conjecture on the Cantor group $[-1, 1]^n$. 
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One easily checks that

\[ U_{t+s} = U_t \circ U_s \]

and in this basis $U_t$ acts as a natural multiplier.

For a (say, bounded) Borel function $f: \mathbb{R}^d \to \mathbb{R}$ and $t \geq 0$ let $U_t f: \mathbb{R}^d \to \mathbb{R}$ be given by

\[ (U_t f)(x) = \mathbb{E}_f(e^{-\frac{t}{2} |x|^2}) \]
non-negative and mean one ($\int f > 0$ and $\int f = 1$).

To achieve this, we will see, as we will see, that if $W_1$ is a continuous and bounded (as we will see) function, which is also

can be easily dealt with. It suffices to express $W_1$ as $W_{1/n}$ ($\log_{n} f$) and use the fact

**Technical remark.** The case of non-continuous or unbounded $f$ (with $\lim_{x \to \infty} f(x) = \infty$)


Thus, assuming that $L_{n}$'s corollary in the Gaussian setting is true,


By the Central Limit Theorem, the distribution of $g$ on $(x_{1},x_{2},\ldots,x_{n})$ tends to the

\[
\left(\frac{X_{1}}{\sqrt{n}},\ldots,\frac{X_{n}}{\sqrt{n}}\right) \Rightarrow \left(\frac{X_{1}}{\sqrt{n}},\ldots,\frac{X_{n}}{\sqrt{n}}\right)
\]

Assume that $f$ is bounded and continuous. Consider $g: \mathbb{R}^{m}$. $f$ is defined by

\[
\text{Limit transition}
\]
In the complement of the ball \( B(0, r) \) with \( r = \log(1 - e^{-\epsilon}) \), \( \log \) is continuous, where \( (x, y) \in \mathbb{C} \). In other words, the set \( \{ x : \log(1 - e^{-\epsilon}) < y \} \times \{ x : \log(1 - e^{-\epsilon}) < y \} \) is contractible, and \( \{ x : \log(1 - e^{-\epsilon}) < y \} \times \{ x : \log(1 - e^{-\epsilon}) < y \} \) is contractible, which ends the proof.

By substituting \( \epsilon = \frac{-1}{kn} \), we arrive at

\[
(1 - e^{-\epsilon}) = \left( \frac{1}{n-\epsilon} \right) + \left( \frac{1}{n+\epsilon} \right)
\]

Thus, for any \( t > 0 \), there exists an \( s \) such that

\[
y(t) = \frac{1}{2} + \frac{1}{2} - \frac{1}{2}t + \frac{1}{2}t.
\]
\[ R_2 - R_1 > R_3 - R_1 \Rightarrow \frac{\log n}{n} > \frac{\log n}{n} \]

Therefore, for the other hand, \( R_1 = 12\log n + e^{-c} \) so that \( \log n - R_1 \geq 2 \).

Hence, for a large enough we have \( R_3 < R_2 \). Let \( R_3 = R_2 - \frac{e^{3\log n}}{n} \), let \( R_2 > R_1 \). Let \( R_2 > 0 \) be such that \( \varphi_a((E R_2, R_2)) = 1 - \frac{\log n}{n} \).

Obviously, for \( d > 2 \), since it is not the case that the primal \( \frac{\log n}{n} \) increases, we obtain the optimal rate.

For \( d = 1 \), we obtain the optimal rate. Hence, for a large enough we have \( \varphi_a((E R_2, R_2)) = 1 - \frac{\log n}{n} \).
\( B_x = B(0, R_x) \leq B_2 = B(0, R_2) \)

\[
R_x = \frac{12}{\ln(2)} \quad \text{and} \quad R_2 = \frac{12}{\ln(2)} \times \frac{\ln(2)}{\text{Vacuum}}
\]

\( A = B_2 \setminus B_1 \)

\( u \leq v \text{ on } B_1 \)

\( u_a(x, B_2) = \frac{1}{u(\ln(2))} \)
No, $(U \cup A)(x) = E \cdot (e^{x} + 1 - e^{x}) = G(x) \in G$. Since $\rho(x) \leq \rho_{0}$, it suffices to show that $\|U_{\rho_{0}} \psi\|_{L^{2}} \leq \frac{K \log \log u}{u^{n}}$ to prove the theorem.

\[ \text{Proof:} \]
\[ C = \left\{ x \in \mathbb{R}^{d} : (U_{\rho_{0}} \psi)(x) > u \right\} \]
\[ \text{We know that } C_{1} = \emptyset \text{.} \]
\[ \text{Hence } \sup_{A} |x| \leq \frac{K \log \log u}{u^{n}} \]

There exists a constant $K = K(d, \epsilon)$ which does not depend on $f$, such that for every $u > 1$.

\[ f_{\rho_{0}} = 1 \]
Hence the distance between $X$ and the arms of symmetry
we see that their distance does not exceed $2r$
Since $X$ and $X'$ belong to the circle of radius $r/2$
also its reflection $X'$ belongs to the intersection
Assume that some point $X$ belongs to the

so that $h > 1 - \frac{r}{2}$

$$h = \frac{r}{2} \left( 2 - \left( \frac{r}{2} \right)^{2} \right)$$

Counterpart of this fact.

Rounding the picture around its axis of symmetry one immediately figures the d-dimensional
with a ring $A$ with radius $r_a$ and $C$ is contained in some strip of width $r_a - r$ up to $r_a + r_a$.
Planar metric observation: Let $R > r_a$. Then the intersection of any circle with radius $R$

\[ n \]

\[ \frac{r}{2} \]
We want to prove that \( d(A) \leq \log y/u \). Recall that \( R = \sqrt{uy} \) and \( R - \tilde{R} = \frac{\log y}{u} \).

Let \( r = 2\sqrt{uy} \), so that \( r^2 = 4uy \).

Then \( d(R \setminus B(0,r)) \geq \frac{(r^2 - r^2)}{\log y} \geq \frac{4uy}{\log y} \).

By the preceding geometric observation there exists a strip (between parallel hyperplanes in \( R^d \)) \( S \) such that \( \pi(B) \subseteq S \). Clearly, \( d(S) \leq \frac{\log y}{u} \).

Note, \( d(A) = d(A \cap B(0,r)) + d(A \setminus B(0,r)) \leq S(S) + d(R \setminus B(0,r)) \leq \frac{\log y}{u} + \log y/u \).

and the proof is finished.