L^1 - smoothing for the multidimensional Ornstein - Uhlenbeck semigroup

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Discrete cube: $\{-1,1\}^n$ with a normalized counting measure $\mu^{\otimes n}$, where $\mu = \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1$.

Walsh-Fourier system: For $A \subseteq \{1,2,\dots,n\}$ we define $\omega_A : \{-1,1\}^n \to \mathbb{R}$ by

$$\omega_A(x) = \prod_{i\in A} x_i \quad (\text{so that } \omega_\emptyset \equiv 1).$$

Then $(\omega_A)_{A\subseteq\{1,2,\dots,n\}}$ is an orthonormal basis of $L^2(\{-1,1\}^n, \mu^{\otimes n})$.

Semigroup: For $t \geq 0$ and $f : \{-1,1\}^n \to \mathbb{R}$ given by $f = \sum_{A\subseteq\{1,2,\dots,n\}} a_A\omega_A$ we define $T_t f : \{-1,1\}^n \to \mathbb{R}$ by

$$(T_t f)(x) = \sum_{A\subseteq\{1,2,\dots,n\}} e^{-t|A|} \cdot a_A \omega_A .$$

Obviously, $T_{t+s} = T_t \circ T_s$. It is easy to check that T_t is a convolution operator: there exists a probability measure ν_t on $\{-1,1\}^n$ such that $\forall_f \;\; T_t f = f * \nu_t$ (pointwise multiplication as a group action on $\{-1,1\}^n$).

Properties of the semigroup $(T_t)_{t\geqslant 0}$:

- positivity preservation: $f \geqslant 0 \Rightarrow T_t f \geqslant 0$ for every $t \geqslant 0$ and $f:\{-1,1\}^n \to \mathbb{R}$,
- contractivity: $\forall_{t\geqslant 0} \forall_f \quad \|T_t f\|_{L^p(\{-1,1\}^n, \mu^{\otimes n})} \leqslant \|f\|_{L^p(\{-1,1\}^n, \mu^{\otimes n})}$ for every $p \in [1,\infty]$,
- hypercontractivity: (Bonami, Beckner, Gross) $\forall_{p>q>1} \forall_f \forall_{t \geqslant \frac{1}{2}\ln\frac{p-1}{q-1}} \quad \|T_t f\|_{L^p} \leqslant \|f\|_{L^q}$,
- symmetry: $\forall_{f,g} \quad \langle f, T_t g\rangle = \mathbb{E}[f \cdot T_t g] = \int_{\{-1,1\}^n} f(x)\,(T_t g)(x)\,d\mu^{\otimes n}(x) = \mathbb{E}[T_t f \cdot g] = \langle T_t f, g\rangle$,
- mean preservation: $\forall_f \forall_{t \geqslant 0} \quad \mathbb{E}[T_t f] = \mathbb{E}[T_t f \cdot \mathbb{1}] = \mathbb{E}[f \cdot T_t \mathbb{1}] = \mathbb{E}[f \cdot \mathbb{1}] = \mathbb{E}[f]$.

Thus for any $f: \{-1,1\}^n \to [0,\infty)$ with $\mathbb{E}[f] = 1$ and any $t \geqslant 0$ we have (by Markov's inequality)

$$\forall_{u>1} \quad \mu^{\otimes n}\left(\{x \in \{-1,1\}^n : (T_t f)(x) > u\}\right) \leqslant \frac{\mathbb{E}[T_t f]}{u} = 1/u .$$

<u>Conjecture</u> (M. Talagrand, 1989):

For each $t > 0$ there exists a function $\psi_t : [1, \infty) \to [0, \infty)$ with $\lim_{u \to \infty} \psi_t(u) = \infty$ and such that for every natural n and every non-negative function f on $\{-1,1\}^n$ with $\mathbb{E}[f] = 1$ the inequality

$$\mu^{\otimes n}\left(\{x \in \{-1,1\}^n : (T_t f)(x) > u\}\right) \leq \frac{1}{u\,\psi_t(u)}$$

holds true for all $u > 1$.

<u>Stronger version</u>: Supposedly, one can take $\psi_t(u) = C_t \sqrt{\log u}$, where C_t is some positive constant (depending only on t). It is easy to check that there is no hope for faster growth of ψ_t.

Equivalently, one may formulate the conjecture on the Cantor group $\{-1,1\}^{\mathbb{N}}$.

The Ornstein - Uhlenbeck semigroup:

γ_d - standard $\mathcal{N}(0, \mathrm{Id}_d)$ Gaussian measure on $\mathbb{R}^d$
G - Gaussian random vector with distribution γ_d

For a (say, bounded) Borel function $f : \mathbb{R}^d \to \mathbb{R}$ and $t \geq 0$ let $U_t f : \mathbb{R}^d \to \mathbb{R}$ be given by

$$(U_t f)(x) = \mathbb{E} f\left(e^{-t} \cdot x + \sqrt{1 - e^{-2t}} \cdot G\right).$$

One easily checks that

$$U_{t+s} = U_t \circ U_s .$$

The semigroup is positivity preserving, contractive in $L^p(\mathbb{R}^d, \gamma_d)$, hypercontractive, symmetric with respect to the standard $L^2(\mathbb{R}^d, \gamma_d)$ structure, and mean preserving. In the Gaussian setting the Walsh functions may be replaced by tensor products of the Hermite polynomials, and in this basis U_t acts as a natural multiplier.

Limit transition

Assume that $f: \mathbb{R}^d \to \mathbb{R}$ is bounded and continuous. Consider $g: \{-1,1\}^{nd} \to \mathbb{R}$ defined by

$$g(x_{1,1},\dots,x_{1,n},x_{2,1},\dots,x_{d,n}) = f\left(\frac{x_{1,1}+x_{1,2}+\dots+x_{1,n}}{\sqrt{n}}, \frac{x_{2,1}+\dots+x_{2,n}}{\sqrt{n}}, \dots, \frac{x_{d,1}+\dots+x_{d,n}}{\sqrt{n}}\right).$$

By the Central Limit Theorem, the distribution of g on $(\{-1,1\}^{nd}, \mu^{\otimes nd})$ tends to the distribution of f on $(\mathbb{R}^d, \gamma_d)$, while the distribution of $T_t g$ on $(\{-1,1\}^{nd}, \mu^{\otimes nd})$ tends to the distribution of $U_t f$ on $(\mathbb{R}^d, \gamma_d)$ as $n \to \infty$, for any fixed $t \geq 0$.
Thus, assuming that Talagrand's conjecture in the discrete cube setting is true, it has an immediate corollary in the Gaussian setting.

Technical remark The case of non-continuous or unbounded Borel f (with $\int_{\mathbb{R}^d} |f| \, d\gamma_d < \infty$) can be easily dealt with. It suffices to express $U_t f$ as $U_{t/2}(U_{t/2} f)$ and use the fact that $U_{t/2} f$ is a continuous and bounded (as we will see) function, which is also non-negative and mean one (if $f \geq 0$ and $\int_{\mathbb{R}^d} f \, d\gamma_d = 1$).

Pointwise estimate

Assume that a non-negative Borel function $f : \mathbb{R}^d \to \mathbb{R}$ satisfies $\int_{\mathbb{R}^d} f \, d\gamma_d = 1$. Then for any $t > 0$ and any $x \in \mathbb{R}^d$ we have

$$(U_t f)(x) \le (1 - e^{-2t})^{-d/2} \cdot e^{x^2/2}.$$

<u>Proof</u>: $(U_t f)(x) = \mathbb{E} f(e^{-t} x + \sqrt{1 - e^{-2t}} \cdot G) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(e^{-t} x + \sqrt{1-e^{-2t}}\, y) e^{-y^2/2} dy.$

By substituting $y = (z - e^{-t} x)/\sqrt{1 - e^{-2t}}$, with $dy = (1 - e^{-2t})^{-d/2} dz$ and $y^2 = z^2 - x^2 + \frac{e^{-2t}}{1 - e^{-2t}} (z - e^{t} \cdot x)^2$, we arrive at

$$(U_t f)(x) = (1 - e^{-2t})^{-d/2} e^{x^2/2} \cdot (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(z) e^{-\frac{e^{-2t}}{1-e^{-2t}} (z - e^{t} x)^2/2} e^{-z^2/2} dz \le$$

$$\le (1 - e^{-2t})^{-d/2} e^{x^2/2} \cdot \int_{\mathbb{R}^d} f \, d\gamma_d$$ which ends the proof.

<u>Corollary</u>: $\{x \in \mathbb{R}^d : (U_t f)(x) > u\} \subseteq \{x \in \mathbb{R}^d : (1 - e^{-2t})^{-d/2} e^{x^2/2} > u\} = \{x \in \mathbb{R}^d : |x| > \sqrt{2 \log u + d \cdot C_t}\}$,

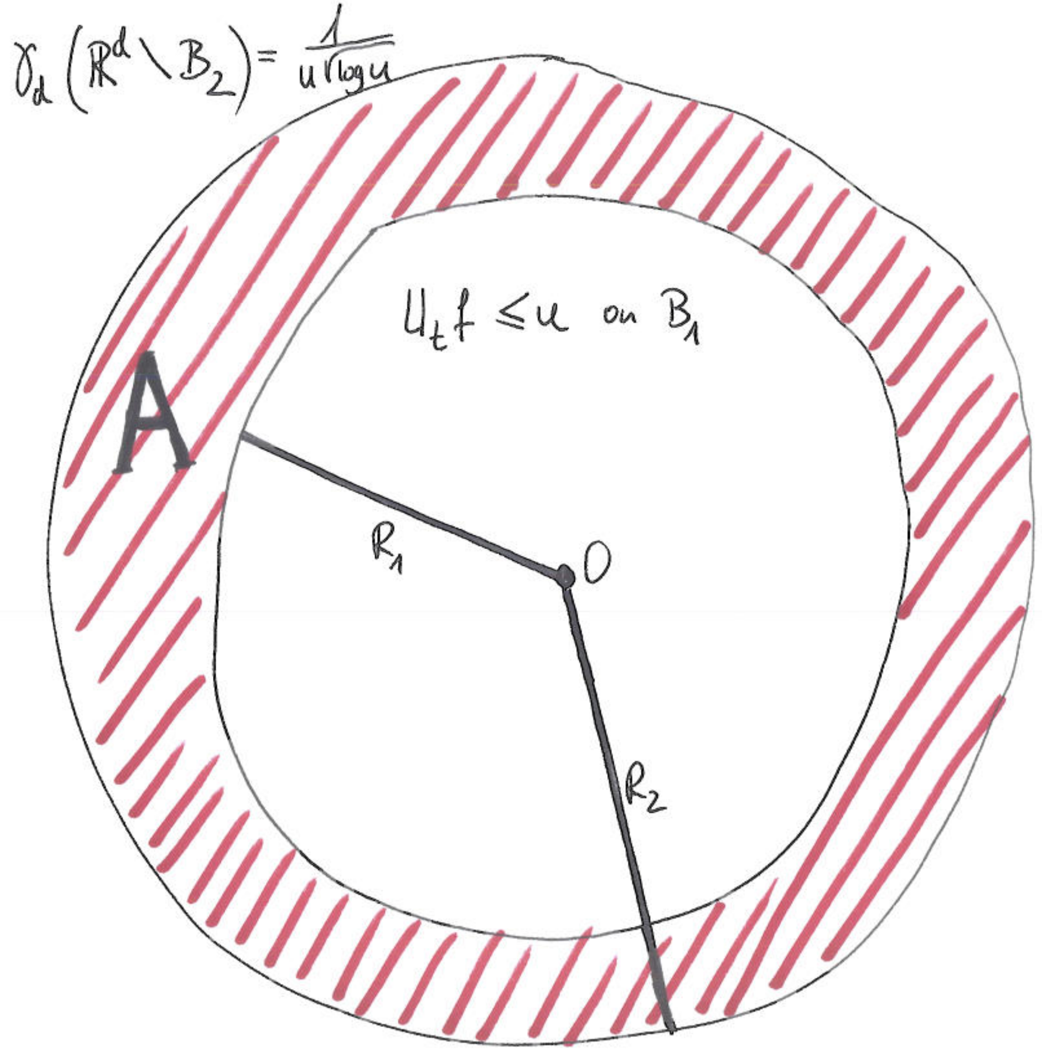
where $C_t = \log(1 - e^{-2t})$. In other words, the set $\{x \in \mathbb{R}^d : (U_t f)(x) > u\}$ is contained in the complement of the ball $B(0, R_1)$ with $R_1 = \sqrt{2 \log u + d \cdot C_t} = \sqrt{2 \log u}\, (1 + o(1))$.

Hence $\gamma_d\left(\{x \in \mathbb{R}^d : (U_t f)(x) > u\}\right) \leq \gamma_d\left(\mathbb{R}^d \setminus B(0,R_1)\right) = \varkappa_d \int_{R_1}^{\infty} r^{d-1} e^{-r^2/2}\,dr \approx \frac{R_1^{d-2}}{e^{R_1^2/2}} \approx \frac{(\log u)^{\frac{d}{2}-1}}{u}$.

For $d=1$ we obtain the optimal rate $\frac{\text{const}}{u\sqrt{\log u}}$, while for $d \geq 2$ the bound seems useless since it is not better that the trivial Markov inequality estimate.

Let $R_2 > 0$ be such that $\gamma_d(B(0,R_2)) = 1 - \frac{1}{u\sqrt{\log u}}$. Obviously, for $d > 2$ and u large enough we have $R_2 > R_1$. Let $R_3 = \sqrt{2\log u} + \frac{d}{4}\frac{\log\log u}{\sqrt{2\log u}}$.
A simple computation shows that $R_3^{d-2} e^{-R_3^2/2} \lesssim \frac{1}{u}(\log u)^{\frac{d-2}{2}}(\log u)^{-d/2} = \frac{1}{u\log u} \ll \frac{1}{u\sqrt{\log u}}$.
Hence, for u large enough we have $R_3 \geq R_2$.
On the other hand, $R_1 = \sqrt{2\log u + d\cdot C_t}$ so that $\sqrt{2\log u} - R_1 \lesssim \frac{1}{\sqrt{\log u}}$.
Therefore

$$R_2 - R_1 \leq R_3 - R_1 \lesssim \frac{\log\log u}{\sqrt{2\log u}}, \text{ for large } u.$$



$B_1 = B(O, R_1) \subseteq B_2 = B(O, R_2)$

$R_1 \approx \sqrt{2\log u}$ and $R_2 \approx \sqrt{2\log u}$, up to $(1+o(1))$ factor

$R_2 - R_1 \lesssim \frac{\log\log u}{\sqrt{\log u}}$

$A = B_2 \setminus B_1$

8

Theorem Let $f:\mathbb{R}^d\to[0,\infty)$ be a Borel function with $\int_{\mathbb{R}^d} f\,d\gamma_d=1$.
There exists a constant $K=K(d,t)$, which does not depend on f, such that for every $u>10$

$$\gamma_d\left(\{x\in\mathbb{R}^d:(U_tf)(x)>u\}\right)\le\frac{K\log\log u}{u\sqrt{\log u}}.$$

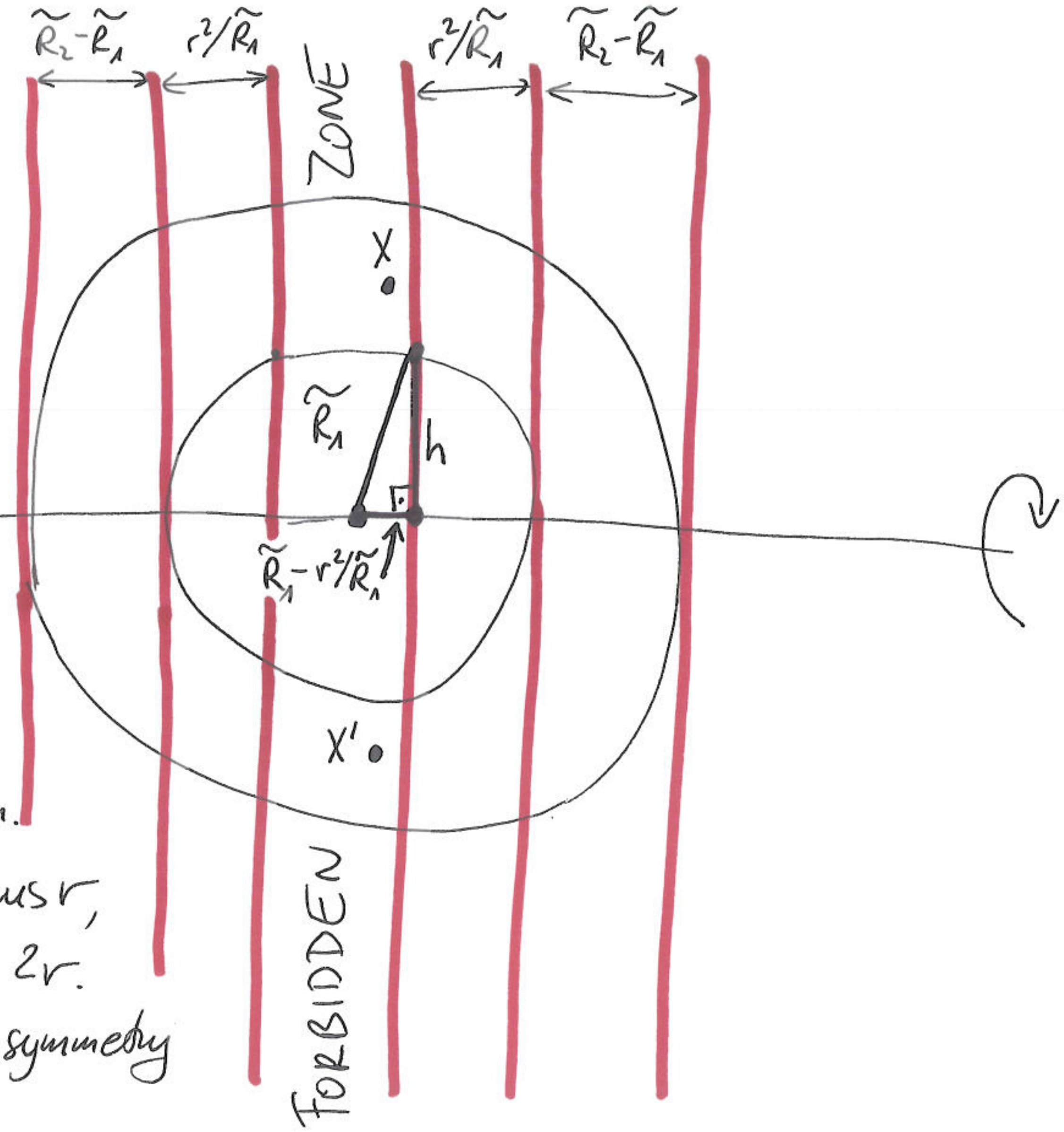
Proof: Let $C=\{x\in\mathbb{R}^d:(U_tf)(x)>u\}$. We know that $C\cap B_1=\emptyset$. Hence

$$\gamma_d(C)=\gamma_d(C\setminus B_2)+\gamma_d(C\cap A)\le\gamma_d(\mathbb{R}^d\setminus B_2)+\frac{\int_{C\cap A}U_tf\,d\gamma_d}{u}\le\frac{1}{u\sqrt{\log u}}+\frac{\int_A U_tf\,d\gamma_d}{u}.$$

Since
$$\int_A U_tf\,d\gamma_d=\int_{\mathbb{R}^d}\mathbb{1}_A\cdot(U_tf)\,d\gamma_d=\int_{\mathbb{R}^d}f\cdot(U_t\mathbb{1}_A)\,d\gamma_d\le\|U_t\mathbb{1}_A\|_\infty,$$
it suffices to show that $\|U_t\mathbb{1}_A\|_\infty\lesssim\frac{\log\log u}{\sqrt{\log u}}$ to prove the theorem.

Now, $$(U_t\mathbb{1}_A)(x)=\mathbb{E}\,\mathbb{1}_A\left(e^{-t}x+\sqrt{1-e^{-2t}}\cdot G\right)=\mathbb{P}\left(G\in\frac{A-e^{-t}x}{\sqrt{1-e^{-2t}}}\right)=\gamma_d(\widetilde{A}),$$
where $\widetilde{A}$ is a rescaled ring A, with radia $\widetilde{R}_1=R_1/\sqrt{1-e^{-2t}}$ and $\widetilde{R}_2=R_2/\sqrt{1-e^{-2t}}$.

<u>Planimetric observation:</u> Let $r < \tilde{R}_1 < \tilde{R}_2$. Then the intersection of <u>any</u> circle with radius r with a ring $\tilde{A}$ with radia $\tilde{R}_1$ and $\tilde{R}_2$ is contained in some strip of width $\tilde{R}_2 - \tilde{R}_1 + r^2/\tilde{R}_1$.
Rotating the picture around its axis of symmetry one immediately infers the d-dimensional counterpart of this fact.



$$h^2 = \tilde{R}_1^2 - (\tilde{R}_1 - r^2/\tilde{R}_1)^2 =$$

$$= r^2\left(2 - \left(\frac{r}{\tilde{R}_1}\right)^2\right) > r^2,$$

so that $h > r$.

Assume that some point X belongs to the intersection. Because of the symmetry also its reflection X' belongs to the intersection. Since X and X' belong to the circle of radius r, we see that their distance does not exceed $2r$. Hence the distance between X and the axis of symmetry is not greater than $r < h$.

We want to prove that $\gamma_d(\tilde{A}) \lesssim \frac{\log\log u}{\sqrt{\log u}}$. Recall that $\tilde{R}_1 \approx \sqrt{\log u}$ and $\tilde{R}_2 - \tilde{R}_1 \approx \frac{\log\log u}{\sqrt{\log u}}$.

Let $r = 2\sqrt{\log\log u}$, so that $r^2/2 = 2\log\log u$. Then

$$\gamma_d(\mathbb{R}^d \setminus B(0,r)) \approx r^{d-2} e^{-r^2/2} \approx \frac{(\log\log u)^{\frac{d-2}{2}}}{(\log u)^2} \lesssim \frac{\log\log u}{\sqrt{\log u}}.$$

Obviously, $r < \tilde{R}_1$ for u large enough.

By the preceding geometric observation there exists a strip (between parallel hyperplanes in $\mathbb{R}^d$) S of width $\tilde{R}_2 - \tilde{R}_1 + r^2/\tilde{R}_1 \approx \frac{\log\log u}{\sqrt{\log u}}$ such that $\tilde{A} \cap B(0,r) \subseteq S$. Clearly, $\gamma_d(S) \lesssim \frac{\log\log u}{\sqrt{\log u}}$.

Now,

$$\gamma_d(\tilde{A}) = \gamma_d(\tilde{A} \cap B(0,r)) + \gamma_d(\tilde{A} \setminus B(0,r)) \leq \gamma_d(S) + \gamma_d(\mathbb{R}^d \setminus B(0,r)) \lesssim \frac{\log\log u}{\sqrt{\log u}} + \frac{\log\log u}{\sqrt{\log u}}$$

and the proof is finished.