Geometric properties of random matrices with independent log-concave rows/columns

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Based on joint work with
O. Guédon, A. Litvak, A. Pajor, N. Tomczak-Jaegermann
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$$\mathbb{E}X = 0$$
Isotropicty, the $\psi_\alpha$ condition

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$X$ is $\psi_\alpha$ ($\alpha \in [1, 2]$) with constant $C$ if for all $y \in \mathbb{R}^n$,

$$\|\langle X, y \rangle\|_{\psi_\alpha} \leq C|y|,$$

where

$$\|Y\|_{\psi_\alpha} = \inf\{a > 0 : \mathbb{E}\exp((Y/a)^\alpha) \leq 2\}.$$
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Fact

For every random vector $X$ not supported on any $n-1$ dimensional hyperplane, there exists an affine map $T: \mathbb{R}^n \to \mathbb{R}^n$ such that $TX$ is isotropic.
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Fact

For every random vector $X$ not supported on any $n-1$ dimensional hyperplane, there exists an affine map $T : \mathbb{R}^n \to \mathbb{R}^n$ such that $TX$ is isotropic.

If for a set $K \subseteq \mathbb{R}^n$ the random vector distributed uniformly on $K$ is isotropic, we say that $K$ is isotropic.
A random vector $X$ in $\mathbb{R}^n$ is log-concave if its law $\mu$ satisfies a Brunn-Minkowski type inequality

$$\mu(\theta A + (1 - \theta)B) \geq \mu(A)^{\theta} \mu(B)^{1-\theta}.$$ 

Theorem (Borell)

A random vector not supported on any $(n - 1)$ dimensional hyperplane is log-concave iff it has density of the form $\exp(-V(x))$, where $V : \mathbb{R}^n \to (-\infty, \infty]$ is convex.

Lemma (Borell)

An isotropic log-concave random vector is $\psi_1$ with a universal constant $C$. 
Examples

The following distributions are log-concave:

- Gaussian measures
- Uniform distributions on convex bodies
- Measures with density of the form $C \exp(-\|x\|)$, where $\|x\|$ is a norm.
- Products, affine images and convolutions of the above distributions.
The basic model

Definition

Let $\Gamma$ be an $n \times N$ matrix with columns $X_1, \ldots, X_N$, where $X_i$’s are independent isotropic log-concave random vectors in $\mathbb{R}^n$.

Questions

What is the operator norm of $\Gamma$: $\ell^N$ to $\ell^n$?

When is $\Gamma^T$ close to a multiple of isometry?

How does $\Gamma$ act on sparse vectors?

What is the smallest singular value of $\Gamma$?
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- What is the smallest singular value of $\Gamma$?
Motivations: sampling convex bodies

Problem

Let $K \subseteq \mathbb{R}^n$ be a convex body, s.t. $B_2^n \subseteq K \subseteq R B_2^n$. Assume we have access to an oracle (a black box), which given $x \in \mathbb{R}^n$ tells us whether $x \in K$. How to generate random points uniformly distributed in $K$? How to compute the volume of $K$? This can be done by using Markov chains. Their speed of convergence depends on the position of the convex body. Preprocessing: First put $K$ in the isotropic position (again by randomized algorithms).
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- This can be done by using Markov chains.
- Their speed of convergence depends on the position of the convex body.
- Preprocessing: First put $K$ in the isotropic position (again by randomized algorithms).
Centering the body is not computationally difficult – takes $O(n)$ steps.

The question boils down to:

How to approximate the covariance matrix of $X$ - uniformly distributed on $K$ by the empirical covariance matrix

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Given an isotropic convex body in $\mathbb{R}^n$, how large $N$ should we take so that

$$\left\| \frac{1}{N} \sum_{i=1}^{N} X_i \otimes X_i - Id \right\|_{\ell_2 \to \ell_2} \leq \varepsilon$$

with high probability?
We have

\[ \left\| \frac{1}{N} \sum_{i=1}^{N} X_i \otimes X_i - \text{Id} \right\|_{\ell_2 \to \ell_2} = \sup_{y \in S^{n-1}} \left| \frac{1}{N} \sum_{i=1}^{N} \langle X_i, y \rangle^2 - 1 \right| \]

\[ = \sup_{y \in S^{n-1}} \left| \frac{1}{N} |\Gamma^T y|^2 - 1 \right| \]
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So the (geometric) question is

Let $\Gamma$ be a matrix with independent columns $X_1, \ldots, X_N$ drawn from an isotropic convex body (log-concave measure) in $\mathbb{R}^n$. 

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So the (geometric) question is

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How large should $N$ be so that $N^{-1/2} \Gamma^T : \mathbb{R}^n \to \mathbb{R}^N$ was an almost isometry?
History of the problem

- Kannan, Lovasz, Simonovits (1995) – $N = \mathcal{O}(n^2)$
- Bourgain (1996) – $N = \mathcal{O}(n \log_3 n)$
- Rudelson (1999) – $N = \mathcal{O}(n \log_2 n)$
- Giannopoulos, Hartzoulaki, Tsolomitis (2005) – unconditional bodies: $N = \mathcal{O}(n \log n)$
- Aubrun (2006) – unconditional bodies: $N = \mathcal{O}(n)$
- Paouris (2006) – $N = \mathcal{O}(n \log n)$

For arbitrary isotropic random vectors, if you do not assume any uniform bound on $\langle X_i, y \rangle$, $y \in S_n^{-1}$, you cannot remove the logarithm (the optimal bound $N = \mathcal{O}(n \log \beta n)$ is due to M. Rudelson). Recently $N = \mathcal{O}(n \log \log n)$ was proven under a uniform bound on $(4 + \varepsilon)$-th moments of $\langle X_i, y \rangle$ (R. Vershynin).
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It turns out that to answer KLS it is enough to have good bounds on

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A_m := \sup_{\Gamma z} \left| \sup_{z \in S^{N-1}} |z| \leq m \right|
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for $m \leq N$.

**Theorem (Litvak, Pajor, Tomczak-Jaegermann, R.A.)**

If $N \leq \exp(c \sqrt{n})$ and the vectors $X_i$ are log-concave then for $t > 1$, with probability at least $1 - \exp(-ct \sqrt{n})$,

$$\forall m \leq N \quad A_m \leq Ct \left( \sqrt{n} + \sqrt{m \log \left( \frac{2N}{m} \right)} \right).$$

In particular, with high probability $\|\Gamma\| \leq C(\sqrt{n} + \sqrt{N})$. 
Sketch of the proof

A modification of Bourgain’s approach. One approximates an arbitrary vector $z$ with $|\text{supp } z| \leq m$ by $x_0 + x_1 + \ldots + x_l$ ($l < \log_2 m$), where

$$|\text{supp } x_i| \simeq m/2^i, \quad \|x_i\|_\infty \simeq \sqrt{2^i/m}, \quad i \geq 1$$

$$|\text{supp } x_0| \simeq m/2^l, \quad \|x_0\|_\infty \leq 1$$

and $x_i$ comes from a $2^{-i}$–net in the set of sparse vectors of support at most $m/2^i$. 

Then using the $\psi_1$ condition one shows that with high probability

$$A_{2m} \lesssim \max_i |X_i|^2 + A_m (\sqrt{n} + \sqrt{m \log (2N/m)})$$

Theorem (G. Paouris)

$$P(|X_i| \geq C t \sqrt{n}) \leq \exp(-ct \sqrt{n})$$

Thus $\max_i |X_i| \leq C \sqrt{n}$ with high probability and we can solve the inequality for $A_m$. 

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Imagine we have a vector $x \in \mathbb{R}^N$ ($N$ large), which is supported on a small number of coordinates (say $|\text{supp } x| = m << N$).

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What if we don’t know the support?
Imagine we have a vector $x \in \mathbb{R}^N$ ($N$ large), which is supported on a small number of coordinates (say $|\text{supp } x| = m << N$).

If we knew the support of $x$, to determine $x$ it would be enough to take $m$ measurements along basis vectors.

What if we don’t know the support?

**Answer** (Donoho, Candes, Tao, Romberg) Take measurements in random directions $Y_1, \ldots, Y_n$ and set

$$\hat{x} = \text{argmin} \{ \|y\|_1 : \langle Y_i, y \rangle = \langle Y_i, x \rangle \}$$
A polytope $K \subseteq \mathbb{R}^n$ is called $m$-neighbourly if any set of vertices of $K$ of cardinality at most $m + 1$ is the vertex set of a face.
Definition

A (centrally symmetric) polytope $K \subseteq \mathbb{R}^n$ is called $m$-(symmetric)-neighbourly if any set of vertices of $K$ of cardinality at most $m + 1$ (containing no opposite pairs) is the vertex set of a face.
Compressed sensing and neighbourly polytopes

**Definition**

A (centrally symmetric) polytope \( K \subseteq \mathbb{R}^n \) is called \( m \)-(symmetric)-neighbourly if any set of vertices of \( K \) of cardinality at most \( m + 1 \) (containing no opposite pairs) is the vertex set of a face.

**Theorem (Donoho)**

Let \( \Gamma \) be an \( n \times N \) matrix with columns \( X_1, \ldots, X_N \). The following conditions are equivalent

(i) For any \( x \in \mathbb{R}^N \) with \( |\text{supp } x| \leq m \), \( x \) is the unique solution of the minimization problem

\[
\min \| t \|_1, \quad \Gamma t = \Gamma x.
\]

(ii) The polytope \( K(\Gamma) = \text{conv}(\pm X_1, \ldots, \pm X_N) \) has \( 2N \) vertices and is \( m \)-symmetric-neighbourly.
Compressed sensing and neighbourly polytopes

Definition (Restricted Isometry Property (Candès, Tao))

For an $n \times N$ matrix $\Gamma$ define the isometry constant $\delta_m = \delta_m(\Gamma)$ as the smallest number such that

$$(1 - \delta_m) |x|^2 \leq |\Gamma x|^2 \leq (1 + \delta_m) |x|^2$$

for all $m$-sparse vectors $x \in \mathbb{R}^N$. 

Theorem (Candès)

If $\delta_2^m(\Gamma) < \sqrt{2} - 1$ then for every $m$-sparse $x \in \mathbb{R}^n$, $x$ is the unique solution to

$$\min |t|_1, \Gamma t = \Gamma x.$$ 

In consequence, the polytope $K(\Gamma)$ (resp. $K(\Gamma)^+ = \text{conv}(X_1, \ldots, X_N)$) is $m$-symmetric-neighbourly (resp. $m$-neighbourly).
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For an $n \times N$ matrix $\Gamma$ define the **isometry constant** $\delta_m = \delta_m(\Gamma)$ as the smallest number such that

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for all $m$-sparse vectors $x \in \mathbb{R}^N$.

Theorem (Candes)
If $\delta_{2m}(\Gamma) < \sqrt{2} - 1$ then for every $m$-sparse $x \in \mathbb{R}^n$, $x$ is the unique solution to

$$\min ||t||_1, \quad \Gamma t = \Gamma x.$$ 

In consequence, the polytope $K(\Gamma)$ (resp. $K_+(\Gamma) = \text{conv}(X_1, \ldots, X_N)$) is $m$-symmetric-neighbourly (resp. $m$-neighbourly).
The following matrices satisfy RIP

- Gaussian matrices (Candes, Tao), \( m \sim n / \log(2N/n) \)
- Matrices with rows selected randomly from the Fourier matrix (Candes & Tao, Rudelson & Vershynin), \( m \sim n / \log^4(N) \)
- Matrices with independent subgaussian isotropic rows (Mendelson, Pajor, Tomczak-Jaergermann), \( m \sim n / \log(2N/n) \)
- Matrices with independent log-concave isotropic columns (LPTA), \( m \sim n / \log^2(2N/n) \)
Neighbourly polytopes

**Theorem (LPTA)**

Assume that $X_i$'s are $\psi_r$. Let $\theta \in (0, 1/4)$ and assume that

$N \leq \exp(c\theta^c n^c)$ and $m \log^{2/r}\left(\frac{2N}{\theta m}\right) \leq \theta^2 n$. Then, with probability at least $1 - \exp(-c\theta^c n^c)$

$$\delta_m\left(\frac{1}{\sqrt{n}}\mathrm{Γ}\right) \leq \theta.$$

**Corollary (LPTA)**

Let $X_1, \ldots, X_N$ be random vectors drawn from an isotropic $\psi_r$ ($r \in [1, 2]$) convex body in $\mathbb{R}^n$. Then, for $N \leq \exp(cn^c)$, with probability at least $1 - \exp(-cn^c)$, the polytope $K(\mathrm{Γ})$ (resp. $K_+(\mathrm{Γ})$) is $m$-symmetric-neighbourly (resp. $m$-neighbourly) with

$$m = \left\lfloor c \frac{n}{\log^{2/r}(CN/n)} \right\rfloor.$$
Method of proof

We use the same approximation techniques as for the KLS problem to bound

\[ B_m = \sup_{|\text{supp } z| \leq m, \ |z|=1} \left| \left\| \sum_{i \leq N} z_i X_i \right\|^2 - \sum_{i \leq N} z_i^2 |X_i|^2 \right|^{1/2} \]

**Theorem (B. Klartag)**

\[
\mathbb{P} \left( \max_{i \leq N} \left| \frac{|X_i|^2}{n} - 1 \right| \geq \varepsilon \right) \leq C \exp(-c \varepsilon^C n^c).
\]

Thus

\[
\delta_n(n^{-1/2} \Gamma) \leq n^{-1} B_m^2 + \varepsilon
\]

with overwhelming probability.
Smallest singular value

Definition

For an \( n \times n \) matrix \( \Gamma \) let \( s_1(\Gamma) \geq s_2(\Gamma) \geq \ldots \geq s_n(\Gamma) \) be the singular values of \( \Gamma \), i.e. eigenvalues of \( \sqrt{\Gamma \Gamma^T} \). In particular

\[
s_1(\Gamma) = \|A\|, \quad s_n(\Gamma) = \inf_{x \in S^{n-1}} |\Gamma x| = \frac{1}{\|A^{-1}\|}
\]

Theorem (Edelman, Szarek)

Let \( \Gamma \) be an \( n \times n \) random matrix with independent \( \mathcal{N}(0, 1) \) entries. Let \( s_n \) denote the smallest singular values of \( \Gamma \). Then, for every \( \varepsilon > 0 \),

\[
\mathbb{P}(s_n(\Gamma) \leq \varepsilon n^{-1/2}) \leq C\varepsilon,
\]

where \( C \) is a universal constant.
Theorem (Rudelson, Vershynin)

Let $\Gamma$ be a random matrix with independent entries $X_{ij}$, satisfying $\mathbb{E} X_{ij} = 0$, $\mathbb{E} X_{ij}^2 = 1$, $\|X_{ij}\|_{\psi_2} \leq B$. Then for any $\varepsilon \in (0, 1)$,

$$\mathbb{P}(s_n(\Gamma) \leq \varepsilon n^{-1/2}) \leq C\varepsilon + c^n,$$

where $C > 0$, $c \in (0, 1)$ depend only on $B$.

Theorem (Guédon, Litvak, Pajor, Tomczak-Jaegermann, R.A.)

Let $\Gamma$ be an $n \times n$ random matrix with independent isotropic log-concave rows. Then, for any $\varepsilon \in (0, 1)$,

$$\mathbb{P}(s_n(\Gamma) \leq \varepsilon n^{-1/2}) \leq C\varepsilon + C \exp(-cn^c)$$

and

$$\mathbb{P}(s_n(\Gamma) \leq \varepsilon n^{-1/2}) \leq C\varepsilon^{n/(n+2)} \log^C(2/\varepsilon).$$
Corollary

For any $\delta \in (0, 1)$ there exists $C_\delta$ such that for any $n$ and $\varepsilon \in (0, 1)$,

$$
P(s_n(\Gamma) \leq \varepsilon n^{-1/2}) \leq C_\delta \varepsilon^{1-\delta}.
$$

Definition

For an $n \times n$ matrix $\Gamma$ define the **condition number** $\kappa(\Gamma)$ as

$$
\kappa(\Gamma) = \|\Gamma\| \cdot \|\Gamma^{-1}\| = \frac{s_1(\Gamma)}{s_n(\Gamma)}.
$$

Corollary

If $\Gamma$ has independent isotropic log-concave columns, then for any $\delta > 0$, $t > 0$,

$$
P(\kappa(\Gamma) \geq nt) \leq \frac{C_\delta}{t^{1-\delta}}.
$$
Thank you